Common fixed point theorems in generalized and G-metric spaces

This chapter is concerned with some common fixed point theorems for weakly compatible maps and generalized \((\psi, \varphi, \varphi)\)-weakly contractive mappings in generalized metric spaces along with E.A. property and (CLR) property. Also, some common fixed point theorems for \(\varphi\)-weakly expansive mappings in G-metric spaces are proved.

This chapter is divided into seven sections. The first section is concerned with notions of generalized and G-metric spaces which are central to this chapter. In Section 3.2, the results of Chen [23] using weakly compatible maps are generalized. Sections 3.3 and 3.4 are concerned with common fixed point theorems for a pair of weakly compatible maps along with E.A. and (CLR) properties. In Section 3.5, some common fixed point theorems for generalized \((\psi, \varphi, \varphi)\)-weakly contractive mappings in generalized metric spaces are proved. Section 3.6 is concerned with common fixed point theorems for a pair of weakly compatible maps along with E.A. and (CLR) properties for generalized \((\psi, \varphi, \varphi)\)-weakly contractive mappings. In last section, some common fixed point theorems for \(\varphi\)-weakly expansive mappings in G-metric spaces are proved.

**Introduction**

Branciari [19] introduced the notion of generalized metric spaces as follows:

**Definition 3.1.1.** Let \(X\) be a nonempty set and \(d : X \times X \to [0, \infty)\) be a mapping such that for all \(x, y \in X\) and for all distinct points \(u, v \in X\) each of them different from \(x\) and \(y\), one has the following:

(i) \(d(x, y) = 0\) if and only if \(x = y\),

(ii) \(d(x, y) = d(y, x)\),

(iii) \(d(x, y) \leq d(x, u) + d(u, v) + d(v, y)\) (rectangular inequality)

Then \((X, d)\) is called a generalized metric space (or shortly g.m.s.).
Definition 3.1.2. Let $(X, d)$ be a g.m.s. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$. We say that $\{x_n\}$ is

(i) g.m.s convergent to $x$, if and only if, $d(x_n, x) \to 0$ as $n \to \infty$.

(ii) g.m.s Cauchy sequence, if and only if, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $n > m > n_0$.

(iii) complete g.m.s, if every g.m.s Cauchy sequence is g.m.s convergent in $X$.

Let $\Psi$ be the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

($\psi 1$) $\psi$ is continuous and monotone non-decreasing,

($\psi 2$) $\psi(t) = 0$ if and only if $t = 0$.

Let $\Phi$ be the set of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

($\varphi 1$) $\varphi$ is continuous,

($\varphi 2$) $\varphi(t) = 0$ if and only if $t = 0$.

Let $\Gamma$ be the set of functions $\gamma : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

($\gamma 1$) $\gamma$ is lower semi-continuous,

($\gamma 2$) $\gamma(t) = 0$ if and only if $t = 0$.

Definition 3.1.3.[42] A mapping $T : X \to X$ is said to be $(\psi, \varphi, \gamma)$-weak contraction if there exist three maps $\psi, \varphi, \gamma : [0, \infty) \to [0, \infty)$ such that

$$\psi(d(Tx, Ty)) \leq \varphi(d(x, y)) - \gamma(d(x, y)),$$

where

\[
M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, x_n) + d(x_n, y), d(x_n, y) + d(x, x_n)\}.
\]

The notion of generalized $(\psi, \varphi, \gamma)$-weak contraction is introduced as follows:

Definition 3.1.4. A mapping $T : X \to X$ is said to be generalized $(\psi, \varphi, \gamma)$-weak contraction if there exist three maps $\psi, \varphi, \gamma : [0, \infty) \to [0, \infty)$ such that

$$\psi(d(Tx, Ty)) \leq \varphi(M(x, y)) - \gamma(M(x, y)),$$

where

\[
M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, x_n) + d(x_n, y) + d(x_n, y) + d(x, x_n)}{2}\}.
\]

The notion of generalized $(\psi, \varphi, \gamma)$-weak contraction is introduced as follows:

Definition 3.1.4. A mapping $T : X \to X$ is said to be generalized $(\psi, \varphi, \gamma)$-weak contraction if there exist three maps $\psi, \varphi, \gamma : [0, \infty) \to [0, \infty)$ such that

$$\psi(d(Tx, Ty)) \leq \varphi(M(x, y)) - \gamma(M(x, y)),$$

where

\[
M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, x_n) + d(x_n, y) + d(x_n, y) + d(x, x_n)}{2}\}.
\]

(i) $\psi$ is continuous and monotone non-decreasing,

(ii) $\varphi$ is continuous,

(iii) $\gamma$ is lower semi-continuous,

(iv) $\psi(t) = 0 = \varphi(t) = \gamma(t)$, if and only if, $t = 0$. 
Definition 3.1.5. A mapping $g : X \to X$ is said to be generalized $(\varphi, \omega, \nu)$ weak contraction with respect to $f : X \to X$ if there exists three maps $\varphi, \omega, \nu : [0, +\infty) \to [0, +\infty)$ such that

$$\psi(d(gx, gy)) \leq \varphi(N(fx, fy)) - \omega(N(fx, fy)),$$

where

$$N(fx, fy) = \max\{d(fx, fy), d(fx, gx), d(fy, gy), \varphi(d(fx, gx)) + \nu(d(fy, gy))\}$$

and

(i) $\psi$ is continuous and monotone non-decreasing,
(ii) $\omega$ is continuous,
(iii) $\nu$ is lower semi-continuous,
(iv) $\psi(t) = 0 = \omega(t) = \nu(t)$, if and only if, $t = 0$.

Mustafa and Sims [68] have shown that most of the results concerning Dhage’s D-metric space are invalid. Therefore, they introduced an improved version of the generalized metric space structure, and called it as G-metric spaces.

Mustafa and Sims [69] introduced the concept of G-metric spaces as follows:

Definition 3.1.6. Let $X$ be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following:

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y$ in $X$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z$ in $X$ with $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a$ in $X$ (rectangle inequality).

Then the function $G$ is called a generalized metric or, more specifically, a G-metric on $X$ and the pair $(X, G)$ is called a G-metric space.

Let $(X, G)$ be a G-metric space. Then for $x_0 \in X$, $r > 0$, the G-ball with center $x_0$ and radius $r$ is

$$B_G(x_0, r) = \{y \in X; G(x_0, y, y) < r\}$$

Let $(X, G)$ be a G-metric space. Then a sequence $\{x_n\}$ is said to be

(i) G-convergent to $x$ if $\lim_{n, m \to \infty} G(x, x_n, x_m) = 0$; i.e., for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ (set of natural numbers) such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \cdot N$.

(ii) G-Cauchy if for each $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \cdot N$ that is if $G(x_n, x_m, x_l) \cdot 0$ as $n, m, l \cdot \cdot \cdot$.

(iii) G-complete (or a complete G-metric space) if every G-Cauchy sequence in $(X, G)$ is G-convergent in $(X, G)$. 
A G-metric space \((X, G)\) is called a symmetric G-metric space if \(G(x, y, y) = G(y, x, x)\) for all \(x, y \in X\).

**Proposition 3.1.7.** [71] Let \((X, G)\) be a G-metric space. Then, for any \(x, y, z, a \in X\) it follows that:

(i) if \(G(x, y, z) = 0\), then \(x = y = z\),

(ii) \(G(x, y, z) \cdot G(x, x, y) + G(x, x, z)\),

(iii) \(G(x, y, y) \cdot 2G(y, x, x)\),

(iv) \(G(x, y, z) \cdot G(x, a, z) + G(a, y, z)\),

(v) \(G(x, y, z) \cdot \frac{2}{3} (G(x, y, a) + G(x, a, z) + G(a, y, z))\),

(vi) \(G(x, y, z) \cdot (G(x, a, a) + G(y, a, a) + G(z, a, a))\).

Chen [23] introduced the notions of \(W\) function and \(S\) function as follows:

**Definition 3.1.8.** A function \(\psi : \mathbb{R}^+ \to \mathbb{R}^+\) is called a \(W\) function if it satisfies the following conditions:

(i) \(\psi(t) < t\) for all \(t > 0\) and \(\psi(0) = 0\),

(ii) \(\lim_{n \to \infty} \inf t_n \psi t_n < t\) for all \(t > 0\).

**Definition 3.1.9.** A function \(\phi : \mathbb{R}^+^3 \to \mathbb{R}^+\) is called a \(S\) function if it satisfies the following conditions:

(i) \(\phi\) is a strictly increasing, continuous function in each coordinate,

(ii) for all \(t > 0\), \(\phi(t, t, t) < t\), \(\phi(t, 0, 0) < t\), \(\phi(0, t, 0) < t\), and \(\phi(0, 0, t) < t\).

**Weakly compatible maps in generalized metric spaces**

In this section, we generalize the results of Chen [23] for pairs of weakly compatible maps.

Chen [23] proved the following fixed point theorems for compatible maps:

**Theorem 3.2.1.** Let \((X, d)\) be Hausdorff and complete g.m.s. and let \(\psi : \mathbb{R}^+ \to \mathbb{R}^+\) be a \(W\) function. Let \(S, T, F, G\) be self maps on \(X\) satisfying the following:

\(3.1\) \(TX \subseteq FX\) and \(SX \subseteq GX\);

\(3.2\) \(d(Sx, Ty) \leq \phi(\max\{d(Fx, Gy), d(Fx, Sx), d(Gy, Ty)\})\) for all \(x, y \in X\);

\(3.3\) pairs \(\{S, F\}\) and \(\{T, G\}\) are compatible.

If \(F\) or \(G\) is continuous, then \(S, T, F\) and \(G\) have a unique common fixed point in \(X\).
**Theorem 3.2.2.** Let \((X, d)\) be Hausdorff and complete g.m.s. and let \(\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) be a \(\mathcal{S}\) function. Let \(S, T, F, G\) be self maps on \(X\) satisfying (3.1), (3.3) and the following:

\[(3.4) \ d(Sx, Ty) \leq \phi \left( \max \{d(Fx, Gy), d(Fx, Sx), d(Gy, Ty)\} \right) \text{ for all } x, y \in X.\]

If one of \(SX, TX, FX\) and \(GX\) is a complete subset of \(X\) then \(S, T, F\) and \(G\) have a unique common fixed point in \(X\).

Now, our concern is to extend the Theorems 3.2.1 and 3.2.2 using weakly compatible maps.

**Theorem 3.2.3.** Let \((X, d)\) be Hausdorff and complete g.m.s. and let \(\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) be a \(\mathcal{W}\) function. Let \(S, T, F, G\) be self maps on \(X\) satisfying (3.1), (3.2) and the following:

\[(3.5) \ \text{pairs } \{S, F\} \text{ and } \{T, G\} \text{ are weakly compatible.}\]

If one of \(SX, TX, FX\) and \(GX\) is a complete subset of \(X\) then \(S, T, F\) and \(G\) have a unique common fixed point in \(X\).

**Proof.** Let \(x_0 \in X\). Define the sequence \(\{x_n\}\) recursively as follows:

\[Gx_{2n+1} = Sx_{2n} = z_{2n}, \quad Fx_{2n+2} = Tx_{2n+1} = z_{2n+1}.\]

By Theorem 2.1 [23], \(\{z_n\}\) is g.m.s. Cauchy sequence in \(X\). Now suppose that \(FX\) is complete subset of \(X\) then the subsequence \(z_{2n+1} = Fx_{2n+2}\) must get a limit in \(FX\). Call it to be \(u\) and assume that \(v \in F^{-1}u\). Then \(Fv = u\). As \(\{z_n\}\) is a Cauchy sequence containing a convergent subsequence \(\{z_{2n+1}\}\), therefore the sequence \(\{z_n\}\) also converges implying thereby the convergence of \(\{z_{2n}\}\) being a subsequence of the convergent sequence \(\{z_n\}\).

On taking \(x = v\) and \(y = x_{2n+1}\) in (3.2) one gets

\[d(Sv, Tx_{2n+1}) \leq \phi \left( \max \{d(Fv, Gx_{2n+1}), d(Fv, Sv), d(Gx_{2n+1}, Tx_{2n+1})\} \right).\]

Letting \(n \to \infty\), we have

\[d(Sv, u) \leq \phi \left( \max \{d(u, u), d(u, Sv), d(u, u)\} \right) = \phi(d(u, Sv)) < d(Sv, u), \text{ a contradiction.}\]

Therefore, \(Sv = u = Fv\), implies that the pair \((S, F)\) has a point of coincidence.

As \(SX \subseteq GX\), \(Sv = u\) implies that \(u \in GX\). Let \(w \in G^{-1}u\), then \(Gw = u\).

From (3.2), we have

\[d(z_{2n}, Tw) = d(Sx_{2n}, Tw) \leq \phi \left( \max \{d(Fx_{2n}, Gw), d(Fx_{2n}, Sx_{2n}), d(Gw, Tw)\} \right).\]

Letting \(n \to \infty\), we have

\[d(u, Tw) \leq \phi \left( \max \{d(u, u), d(u, u), d(u, Tw)\} \right) = \phi(d(u, Tw)) < d(u, Tw), \text{ a contradiction.}\]

Therefore, \(u = Tw\).
Thus we have shown that \( u = Sv = Fv = Tw = Gw \), which amounts to say that both pairs have point of coincidence. If one assumes \( GX \) to be complete, then an analogous argument establishes this claim.

The remaining two cases pertain essentially to the previous cases. Indeed if \( SX \) is complete, then \( u \in SX \subseteq GX \) and if \( TX \) is complete, then \( u \in TX \subseteq FX \). Thus \( S \) and \( F \), \( T \) and \( G \) have a point of coincidence.

Since the pairs \((S, F)\) and \((T, G)\) are weakly compatible at \( v \) and \( w \) respectively, then
\[
Su = S(Fv) = F(Sv) = Fu \quad \text{and} \quad Tu = T(Gw) = G(Tw) = Gu.
\]

If \( Su \neq u \), then
\[
d(Su, u) = d(Su, Tw) \leq \phi(\max \{d(Fu, Gw), d(Fu, Su), d(Gw, Tw)\})
\]
\[
= \phi(\max \{d(Su, u), 0, 0\})
\]
\[
= \phi(d(Su, u)) < d(Su, u), \quad \text{a contradiction.}
\]

Therefore, \( Su = u \).

Similarly, we can show that \( Tu = u \). Thus \( u \) is the common fixed point of \( S, T, F \) and \( G \).

Finally, we prove that \( S, T, F \) and \( G \) have a unique common fixed point.

Let \( t \) be another common fixed point of \( S, T, F \) and \( G \).

From (3.2), we have
\[
d(t, u) = d(St, Tu) \leq \phi(\max \{d(Ft, Gu), d(Ft, St), d(Gu, Tu)\})
\]
\[
= \phi(\max \{d(t, u), d(t, t), d(u, u)\})
\]
\[
< d(t, u), \quad \text{a contradiction unless} \ d(t, u) = 0, \quad \text{that is,} \ t = u.
\]

Hence \( u \) is the unique common fixed point of \( S, T, F \) and \( G \) in \( X \).

We give the following example to illustrate Theorem 3.2.3.

**Example 3.2.4.** Let \( X = \{a_1, a_2, a_3, a_4, a_5\} \), where \( a_1, a_2, a_3, a_4, a_5 \) are positive constants.

We define \( d : X \times X \rightarrow [0, \infty) \) by

\[
(1) \ d(x, x) = 0, \quad \text{for all} \ x \in X,
\]
\[
(2) \ d(x, y) = d(y, x), \quad \text{for all} \ x, y \in X,
\]
\[
(3) \ d(a_1, a_2) = 3k,
\]
\[
(4) \ d(a_1, a_3) = d(a_2, a_3) = k,
\]
\[
(5) \ d(a_1, a_4) = d(a_2, a_4) = d(a_3, a_4) = 2k,
\]
\[
(6) \ d(a_1, a_5) = d(a_2, a_5) = d(a_3, a_5) = d(a_4, a_5) = 2k,
\]

where \( k > 0 \) is a constant.
If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, $\phi(t) = (2/3)t$, then $\phi$ is a $W$ function.

We next define $S$, $T$, $F$, $G : X \to X$ by

\begin{align*}
Sx &= a_3 \text{ if } x \neq a_4, \\
a_5 \text{ if } x = a_4, \\
a_3 \text{ if } x \neq a_4, \\
Tx &= a_3 \text{ if } x = a_4, \\
Gx &= \text{ the identity mapping}, \\
a_3 \text{ if } x = a_4, \\
Fx &= a_1 \text{ if } x \neq a_1, a_2, a_5, a_2 \text{ if } x = a_4.
\end{align*}

Then all the conditions of Theorem 3.2.3 are satisfied and $a_3$ is a unique common fixed point of $S$, $T$, $F$ and $G$.

**Theorem 3.2.5.** Let $(X, d)$ be Hausdorff and complete g.m.s. and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a $W$ function. Let $S$, $T$, $F$, $G$ be self maps on $X$ satisfying (3.1), (3.4) and (3.5). If one of $Sx$, $Tx$, $Fx$ and $Gx$ is a complete subset of $X$ then $S$, $T$, $F$ and $G$ have a unique common fixed point in $X$.

**Proof.** Given that $x_0 \in X$. Define the sequence $\{x_n\}$ recursively as follows:

\[ Gx_{2n+1} = Sx_{2n} = z_{2n}, \quad Fx_{2n+2} = Tx_{2n+1} = z_{2n+1}. \]

Then due to Theorem 2.2 [23], $\{z_n\}$ is g.m.s. Cauchy sequence in $X$. Now suppose that $Fx$ is complete subset of $X$ then the subsequence $z_{2n+1} = Fx_{2n+2}$ must get a limit in $Fx$. Call it to be $u$ and assume that $v \in F^{-1}u$. Then $Fv = u$. As $\{z_n\}$ is a Cauchy sequence containing a convergent subsequence $\{z_{2n+1}\}$, therefore the sequence $\{z_n\}$ also converges implying thereby the convergence of $\{z_{2n}\}$ being a subsequence of the convergent sequence $\{z_n\}$. On taking $x = v$ and $y = x_{2n+1}$ in (2.4) one gets

\[ d(Sv, Tx_{2n+1}) \leq \phi(d(Fv, Gx_{2n+1}), d(Fv, Sv), d(Gx_{2n+1}, Tx_{2n+1})). \]

Letting $n \to \infty$, we have

\[ d(Sv, u) \leq \phi(d(u, u), d(u, Sv), d(u, u)) \]

\[ = \phi(d(u, Sv)) < d(Sv, u), \text{ a contradiction.} \]

Therefore, $Sv = u = Fv$, which shows that the pair $(S, F)$ has a point of coincidence.

As $Sx \subseteq Gx$, $Sv = u$ implies that $u \in GX$. Let $w \in G^{-1}u$, then $Gw = u$.

From (3.4), we get

\[ d(z_{2n}, Tw) = d(Sx_{2n}, Tw) \leq \phi(d(Fx_{2n}, Gw), d(Fx_{2n}, Sx_{2n}), d(Gw, Tw)). \]
Letting \( n \to \infty \), we have
\[
d(u, Tw) \leq \phi(\{d(u, u), d(u, u), d(u, Tw)\})
\]
\[
= \phi(d(u, Tw)) < d(u, Tw), \text{ a contradiction.}
\]

Therefore, \( u = Tw \).

Thus we have shown that \( u = Sv = Fv = Tw = Gw \), which amounts to say that both pairs have point of coincidence. If one assumes \( GX \) to be complete, then an analogous argument establishes this claim.

The remaining two cases pertain essentially to the previous cases. Indeed if \( SX \) is complete, then \( u \in SX \subseteq GX \) and if \( TX \) is complete, then \( u \in TX \subseteq FX \). Thus \( S \) and \( F \), \( T \) and \( G \) have a point of coincidence.

Since the pairs \( (S, F) \) and \( (T, G) \) are weakly compatible at \( v \) and \( w \) respectively, then
\[
Su = S(Fv) = F(Sv) = Fu
\]
\[
Tu = T(Gw) = G(Tw) = Gu.
\]
If \( Su \neq u \), then
\[
d(Su, u) = d(Su, Tw)
\]
\[
\leq \phi(\{d(Fu, Gw), d(Fu, Su), d(Gw, Tw)\})
\]
\[
= \phi(\{d(Su, u), 0, 0\})
\]
\[
= \phi(d(Su, u)) < d(Su, u), \text{ a contradiction.}
\]

Therefore, \( Su = u \). Similarly, we can show that \( Tu = u \).

Thus \( u \) is the common fixed point of \( S, T, F \) and \( G \).

Finally, we prove that \( S, T, F \) and \( G \) have a unique common fixed point.

Let \( t \) be another common fixed point of \( S, T, F \) and \( G \).
From (3.4), we have
\[
d(t, u) = d(St, Tu) \leq \phi(\{d(Ft, Gu), d(Ft, St), d(Gu, Tu)\})
\]
\[
= \phi(\{d(t, u), d(t, t), d(u, u)\})
\]
\[
< d(t, u), \text{ a contradiction unless } d(t, u) = 0, \text{ that is, } t = u.
\]

Hence \( u \) is the unique common fixed point of \( S, T, F \) and \( G \) in \( X \).

We give the following example to illustrate Theorem 3.2.5.

**Example 3.2.6.** Let \( X = \{a_1, a_2, a_3, a_4, a_5\} \), where \( a_1, a_2, a_3, a_4, a_5 \) are positive constants. We define \( d : X \times X \to [0, \infty) \) by

1. \( d(x, x) = 0 \), for all \( x \in X \),
2. \( d(x, y) = d(y, x) \), for all \( x, y \in X \),
3. \( d(a_1, a_2) = 3k \),
(4) \(d(a_1, a_3) = d(a_2, a_3) = k\),

(5) \(d(a_1, a_4) = d(a_2, a_4) = d(a_3, a_4) = 2k\),

(6) \(d(a_1, a_5) = d(a_2, a_5) = d(a_3, a_5) = d(a_4, a_5) = 2k\),

where \(k > 0\) is a constant.

If \(\phi : \mathbb{R}^+ \to \mathbb{R}^+\), \(\phi(t) = (3/5)\max\{t^1, t^2, t^3\}\), then \(\phi\) is a \(\mathcal{S}\) function. We next define

\(S, T, F, G : X \to X\) by

- \(Sx = a_3\) if \(x \neq a_4\),
- \(Tx = a_1\) if \(x = a_4\),
- \(Gx = I(x) = \text{the identity mapping},\)
- \(Fx = a_3\) if \(x = a_3\),
- \(Gx = a_2, a_5\).

Then all the conditions of Theorem 3.2.5 are satisfied and \(a_3\) is a unique common fixed point of \(S, T, F\) and \(G\).

**E.A. property in generalized metric spaces**

In this section, we prove common fixed point theorems for \(\mathcal{U}\) function and \(\mathcal{S}\) function using the E.A. property.

**Theorem 3.3.1.** Let \((X, d)\) be Hausdorff and complete g.m.s. and let \(\phi : \mathbb{R}^+ \to \mathbb{R}^+\) be a \(\mathcal{U}\) function. Let \(S, T, F, G\) be self maps on \(X\) satisfying (3.1), (3.2), (3.5) and the following:

(3.6) pair \(\{S, F\}\) or \(\{T, G\}\) satisfy the E. A. property.

If the range of one of the maps \(S, T, F\) or \(G\) is a complete subspace of \(X\) then \(S, T, F\) and \(G\) have a unique common fixed point in \(X\).

**Proof.** If the pair \(\{T, G\}\) satisfies the E.A. property, then there exists a sequence \(\{x_n\}\) in \(X\) such that \(Tx_n \to z\) and \(Gx_n \to z\) for some \(z \in X\) as \(n \to \infty\). Since \(TX \subset FX\), there exists a sequence \(\{y_n\}\) in \(X\) such that \(Tx_n = Fy_n\). Hence \(Fy_n \to z\) as \(n \to \infty\). Also, since \(SX \subset GX\), there exists a sequence \(\{x'_n\}\) in \(X\) such that \(Sy'_n = Gy'_n\). Hence \(Sy'_n \to z\) as \(n \to \infty\). Suppose that \(FX\) is a complete subspace of \(X\). Then, \(\forall y \in X\), \(Fy \to u\) for some \(u \in X\). Subsequently, we have \(Sy_n \to Fu, Tx_n \to Fu, Fy_n \to Fu\) as \(n \to \infty\).
From (3.2), we have

\[ d(Su, Tx_n) \leq \phi(\max\{ d(Fu, Gx_n), d(Fu, Su), d(Gx_n, Tx_n)\}). \]

Letting \( n \to \infty \), we have

\[ d(Su, Fu) \leq \phi(\max\{ d(Fu, Fu), d(Fu, Su), d(Fu, Fu)\}) \]

\[ = \phi(d(Fu, Su)) < d(Fu, Su), \] implies that, \( Fu = Su. \)

The weak compatibility of \( F \) and \( S \) implies that \( FSu = SFu \) and then \( SSu = SFu = FFu. \)

On the other hand, since \( SX \subseteq GX \), there exists a \( v \in X \) such that \( Su = Gv. \)

We now show that \( Gv = Tv. \)

From (2.2), we get

\[ d(Su, Tv) \leq \phi(\max\{ d(Fu, Gv), d(Fu, Su), d(Gv, Tv)\}) \]

\[ = \phi(d(Gv, Tv)) < d(Gv, Tv), \] implies that, \( Gv = Tv. \)

This implies that \( Su = Fu = Gv = Tv. \) The weak compatibility of \( T \) and \( G \) implies that \( TGv = GTv \) and \( GGv = GTv = TGv = Tv. \)

Let us show that \( Su \) is the common fixed point of \( S, T, F \) and \( G. \)

Using (3.2), we have

\[ d(SSu, Tv) \leq \phi(\max\{ d(FSu, Gv), d(FSu, SSu), d(Gv, Tv)\}) \]

\[ = \phi(d(SSu, Tv)) < d(Gv, Tv), \] which gives that \( SSu = Tv = Su. \)

Therefore, \( Su = SSu = FSu \) and \( Su \) is the common fixed point of \( S \) and \( F. \) Similarly, we can prove that \( Tv \) is the common fixed point of \( T \) and \( G. \) Since \( Su = Tv, \) we conclude that \( Su \) is the common fixed point of \( S, T, F \) and \( G. \) The proof is similar when \( GX \) is assumed to be complete subspace of \( X. \) The cases in which \( SX \) or \( TX \) is a complete subspace of \( X \) are similar to the cases in which \( GX \) or \( FX \) respectively, is complete since \( SX \subseteq GX \) and \( TX \subseteq FX \). If \( Su = Tu = Fu = Gu = u \) and \( Sv = Tv = Fv = Gv = v, \) then (2.2) gives

\[ d(Su, Tv) \leq \phi(\max\{ d(Fu, Gv), d(Fu, Su), d(Gv, Tv)\}) \]

\[ < d(Fu, Gv), \] which implies that, \( u = v. \)

Therefore, \( u = v, \) and the common fixed point is unique.

**Theorem 3.3.2.** Let \((X, d)\) be Hausdorff and complete g.m.s. and let \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) be a \( \sigma \) function. Let \( S, T, F, G \) be self maps on \( X \) satisfying (3.1), (3.4), (3.5) and (3.6). If the
range of one of the maps S, T, F or G is a complete subspace of X then S, T, F and G have a unique common fixed point in X.

**Proof.** If the pair \{T, G\} satisfies the E.A. property, then there exists a sequence \{x_n\} in X such that \(T_{x_n} \to z\) and \(G_{x_n} \to z\) for some \(z \in X\) as \(n \to \infty\). Since \(TX \subseteq FX\), there exists a sequence \{y_n\} in X such that \(T_{x_n} = F_{y_n}\). Hence \(F_{y_n} \to z\) as \(n \to \infty\). Also, since \(SX \subseteq GX\), there exists a sequence \{y'_n\} in X such that \(S_{y_n'} = G_{y_n'}\). Hence \(S_{y_n'} \to z\) as \(n \to \infty\). Suppose that \(TX\) is a complete subspace of X. Then, \(z = Fu\) for some \(u \in X\). Subsequently, we have \(S_{y_n'} \to Fu\), \(T_{x_n} \to Fu\), \(F_{y_n} \to Fu\) as \(n \to \infty\).

From (3.4), we have

\[
d(Su, Tx_n) \leq \phi(d(Fu, G_{x_n}), d(Fu, Su), d(G_{x_n}, Tx_n)).
\]

Letting \(n \to \infty\), we have

\[
d(Su, Fu) \leq \phi(d(Fu, Fu), d(Fu, Su), d(Fu, Fu)) = \phi(0, d(Fu, Su), 0) < d(Fu, Su),
\]

which implies that \(Fu = Su\).

The weak compatibility of \(F\) and \(S\) implies that \(FSu = SFu\) and then \(SSu = SFu = FSu = FFu\).

On the other hand, since \(SX \subseteq GX\), there exists a \(v \in X\) such that \(Su = Gv\). We now show that \(Gv = Tv\).

From (3.4), we have

\[
d(Su, Tv) \leq \phi(d(Fu, Gv), d(Fu, Su), d(Gv, Tv)) = \phi(0, 0, d(Gv, Tv)) < d(Gv, Tv),
\]

which implies that \(Gv = Tv\). This implies that \(Su = Fu = Gv = Tv\).

The weak compatibility of \(T\) and \(G\) implies that \(TGv = GTv\) and \(GGv = GTv = TGv = TTv\).

Let us show that \(Su\) is the common fixed point of \(S, T, F\) and \(G\).

From (3.4), we get

\[
d(SSu, Tv) \leq \phi(d(FSu, Gv), d(FSu, SSu), d(Gv, Tv)) = \phi(d(SSu, Tv), 0, 0) < d(SSu, Tv),
\]

which implies that \(SSu = Tv = Su\).

Therefore, \(Su = SSu = FSu\) and \(Su\) is the common fixed point of \(S\) and \(F\). Similarly, we can prove that \(Tv\) is the common fixed point of \(T\) and \(G\). Since \(Su = Tv\) we conclude that \(Su\) is the common fixed point of \(S, T, F\) and \(G\). The proof is similar when \(GX\) is assumed to be complete subspace of X. The cases in which \(SX\) or \(TX\) is a complete subspace of X.
are similar to the cases in which GX or FX respectively, is complete since SX \subseteq GX and
if Su = Tu = Fu = Gu = u and Sv = Tv = Fv = Gv = v, then (3.4) gives
de(Su, Tv) \leq \phi(d(Fu, Gv), d(Fu, Su), d(Gv, Tv)) < d(Fu, Gv), which implies that u = v.

Therefore, u = v, and the common fixed point is unique.

**CLR** property in generalized metric spaces

Here, we prove common fixed point theorems for \( W \) function and \( S \) function using the
\( (CLR) \) property.

**Theorem 3.4.1.** Let \((X, d)\) be Hausdorff and complete g.m.s. and let \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be a
\( W \) function. Let \( S, T, F, G \) be self maps on \( X \) satisfying (3.2), (3.5) and the following:
(3.7) \( TX \subseteq FX \) and the pair \( \{T, G\} \) satisfies (CLR_G) property,
or
\( SX \subseteq GX \) and the pair \( \{S, F\} \) satisfies (CLR_F) property.

Then \( S, T, F \) and \( G \) have a unique common fixed point.

**Proof.** Without loss of generality, we assume that \( TX \subseteq FX \) and the pair \( \{T, G\} \) satisfies
(\( CLR_G \)) property, then there exists some sequence \( \{x_n\} \) in \( X \) such that \( Tx_n \) and \( Gx_n \)
converges to \( Gx \), for some \( x \) in \( X \) as \( n \to \infty \). Since \( TX \subseteq FX \) there exists a sequence \( \{y_n\} \)
in \( X \) such that \( Tx_n = Fy_n \) hence \( Fy_n \to Gx \) as \( n \to \infty \).

We shall show that \( \lim_{n \to \infty} Sy_n = Gx \).
Let \( \lim_{n \to \infty} Sy_n = z \).
Taking \( x = y_n \) and \( y = x_n \) in (3.2), we have
\[
d(Sy_n, Tx_n) \leq \phi(\max\{d(Fy_n, Gx_n), d(Fy_n, Sy_n), d(Gx_n, Tx_n)\}).
\]
Letting \( n \to \infty \), we have
\[
d(z, Gx) \leq \phi(\max\{d(Gx, Gx), d(Gx, z), d(Gx, Gx)\})
\]
\[
= \phi(d(Gx, z)) < d(z, Gx),
\]
which implies that \( z = Gx \).
Subsequently, we have \( Tx_n, Gx_n, Fy_n \) and \( Sy_n \) converges to \( z \).

We shall show that \( Tx = z \).
Taking \( x = y_n \) and \( y = x \) in (3.2), we have
\[
d(Sy_n, Tx) \leq \phi(\max\{d(Fy_n, Gx), d(Fy_n, Sy_n), d(Gx, Tx)\}).
\]
Letting $n \to \infty$, we have
\[ d(z, Tx) \leq \phi\left(\max\{d(Gx, Gx), d(z, z), d(z, Tx)\}\right) \]
\[ = \phi(d(z, Tx)) < d(z, Tx), \]
which implies that $z = Tx = Gx$.

Since, the pair $(T, G)$ is weakly compatible, it follows that $Tz = Gz$.
Also, since $TX \subseteq FX$, there exists some $y$ in $X$ such that $Tx = Fy(=z)$.

We next show that $Fy = Sy(=z)$.
Taking $y = x_n, x = y$ in (3.2), we have
\[ d(Sy, Tx) \leq \phi\left(\max\{d(Fy, Gx_n), d(Fy, Sy), d(Gx_n, Tx)\}\right). \]

Letting $n \to \infty$, we have
\[ d(Sy, z) \leq \phi\left(\max\{d(z, z), d(z, Sy), d(z, z)\}\right) \]
\[ = \phi(d(z, Sy)) < d(Sy, z), \]
which implies that $Sy = z = Fy$.

But the pair $\{S, F\}$ is weakly compatible, it follows that $Sz = Fz$.

Next, we claim that $Sz = Tz$.
Let, if possible, $Sz \neq Tz$.
Taking $x = z, y = z$ in (3.2), we have
\[ d(Sz, Tz) \leq \phi\left(\max\{d(Fz, Gz), d(Fz, Sz), d(Gz, Tz)\}\right) \]
\[ = \phi(d(Sz, Tz)) < d(Sz, Tz), \]
a contradiction.
Thus, we get $Sz = Tz$.

Hence, $Sz = Tz = Fz = Gz$.

We now show that $z = Sz$.
Taking $x = z, y = x$ in (3.2), we have
\[ d(Sz, Tx) \leq \phi\left(\max\{d(Fz, Gx), d(Fz, Sz), d(Gx, Tx)\}\right) \]
\[ = \phi(d(Sz, z)) < d(Sz, z), \]
which implies that $z = Sz = Tz = Fz = Gz$, that is, $z$ is the common fixed point of $S, T, F$ and $G$.

If $Sz = Tz = Fz = Gz = z$ and $St = Tt = Ft = Gt = t$.

Now, we show that $Sz = Tt$.
Let, if possible, $Sz \neq Tt$.
From (3.2), we have
\[ d(Sz, Tt) \leq \phi\left(\max\{d(Fz, Gt), d(Fz, Sz), d(Gt, Tt)\}\right) \]
\[ = \phi(d(Sz, Tt)) < d(Sz, Tt), \]
a contradiction.
Therefore $Sz = Tt$, implies that, $z = t$.

Hence the common fixed point is unique.

**Theorem 3.4.2.** Let $(X, d)$ be Hausdorff and complete g.m.s. and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a $S$ function. Let $S, T, F, G$ be self maps on $X$ satisfying (3.4), (3.5) and (3.7). Then $S, T, F$ and $G$ have a unique common fixed point.

**Proof.** Without loss of generality, we assume that $TX \subseteq FX$ and the pair $\{T, G\}$ satisfies (CLRG) property, then there exists some sequence $\{x_n\}$ in $X$ such that $Tx_n$ and $Gx_n$ converges to $Gx$, for some $x$ in $X$ as $n \to \infty$. Since $TX \subseteq FX$ there exists a sequence $\{y_n\}$ in $X$ such that $Tx_n = Fy_n$, hence $Fy_n \to Gx$ as $n \to \infty$.

We shall show that $\lim_{n\to \infty} Sy_n = Gx$.

Let $\lim_{n\to \infty} Sy_n = z$.

Let, if possible, $Gx \neq z$.

Taking $x = y_n$ and $y = x_n$ in (3.4), we have

$$d(Sy_n, Tx_n) \leq \phi(d(Fy_n, Gx_n), d(Fy_n, Sy_n), d(Gx_n, Tx_n)).$$

Letting $n \to \infty$, we have

$$d(z, Gx) \leq \phi(d(Gx, Gx), d(Gx, z), d(Gx, Gx))$$

$$= \phi(0, d(Gx, z), 0) < d(z, Gx),$$ a contradiction.

Thus, we get $z = Gx$.

Subsequently, we have $Tx_n, Gx_n, Fy_n$ and $Sy_n$ converges to $z$.

We shall show that $Tx = z$.

Taking $x = y_n$ and $y = x_n$ in (3.4), we have

$$d(Sy_n, Tx_n) \leq \phi(d(Fy_n, Gx_n), d(Fy_n, Sy_n), d(Gx_n, Tx_n)).$$

Letting $n \to \infty$, we have

$$d(z, Tx) \leq \phi(d(Gx, Gx), d(z, z), d(z, Tx))$$

$$= \phi(0, 0, d(z, Tx)) < d(z, Tx),$$

which implies that $z = Tx = Gx$.

Since, the pair $(T, G)$ is weakly compatible, it follows that $Tz = Gz$.

Also, since $TX \subseteq FX$, there exists some $y$ in $X$ such that $Tx = Fy(=z)$.

We next show that $Ty = Sy(=z)$.

Taking $y = x_n$, $x = y$ in (3.4), we have

$$d(Sy, Tx_n) \leq \phi(d(Fy_n, Gx_n), d(Fy_n, Sy_n), d(Gx_n, Tx_n)).$$
Letting \( n \to \infty \), we have
\[
d(Sy, z) \leq \phi(d(z, z), d(z, Sy), d(z, z))
\]
\[= \phi(0, d(z, Sy), 0) < d(Sy, z),
\]
which implies that \( Sy = z = Fy \).

But the pair \( \{S, F\} \) is weakly compatible, it follows that \( Sz = Fz \).

Next, we claim that \( Sz = Tz \).

Putting \( x = z, y = z \) in (3.4), we have
\[
d(Sz, Tz) \leq \phi(d(Fz, Gz), d(Fz, Sz), d(Gz, Tz))
\]
\[= \phi(d(Sz, Tz), 0, 0) < d(Sz, Tz),
\]
which implies that \( Sz = Tz \). Hence, \( Sz = Tz = Fz = Gz \).

We now show that \( z = Sz \).

Taking \( x = z, y = x \) in (3.4), we have
\[
d(Sz, Tx) \leq \phi(d(Fz, Gx), d(Fz, Sz), d(Gx, Tx))
\]
\[= \phi(d(Sz, z), 0, 0) < d(Sz, z),
\]
which implies that \( z = Sz = Tz = Fz = Gz \), that is, \( z \) is the common fixed point of \( S, T, F \) and \( G \).

If \( Sz = Tz = Fz = Gz = z \) and \( St = Tt = Ft = Gt = t \), then (3.4) gives
\[
d(Sz, Tt) \leq \phi(d(Fz, Gt), d(Fz, Sz), d(Gt, Tt))
\]
\[= \phi(d(Sz, Tt), 0, 0) < d(Sz, Tt),
\]
which implies that \( Sz = Tt \) or \( z = t \).

Therefore, \( z = t \) and the common fixed point is unique.

**Generalized \((\psi, \Phi, \Gamma)\)-weakly contractive mappings in generalized metric spaces**

For proving our main results, we need the following Lemma:

**Lemma 3.5.1.**[42] Let \( \{a_n\} \) be a sequence of non-negative real numbers. If
\[
(3.8) \quad \psi(a_{n+1}) \leq \Phi(a_n) - \Gamma(a_n)
\]
for all \( n \in \mathbb{N} \), where \( \psi \in \Psi, \Phi \in \Phi, \Gamma \in \Gamma \), and
\[
(3.9) \quad \psi(t) - \Phi(t) + \Gamma(t) > 0 \text{ for all } t > 0
\]
then the following hold:

(i) \( a_{n+1} \leq a_n \) if \( a_n > 0 \),

(ii) \( a_n \to 0 \) as \( n \to \infty \).
Now, we prove some theorems for a pair of mappings for generalized \((\psi, \Phi, \Psi)\)-weakly contractive mappings in generalized metric spaces.

**Theorem 3.5.2.** Let \(f\) and \(g\) be self mappings of a Hausdorff g.m.s. \((X, d)\) satisfying the followings:

1. \((3.10)\) \(gX \subseteq fX,\)
2. \((3.11)\) \(fx\) or \(gx\) is a complete subspace of \(X,\)
3. \((3.12)\) \(\psi(d(gx, gy)) \leq \Phi(N(fx, fx)) - \Phi(N(fx, fx)),\) for all \(x, y\) in \(X,\)

where \(\psi \in \Phi, \Psi \subseteq \Phi,\) and \(\Phi, \Psi \subseteq \Phi,\) and satisfy condition (3.9) with \(N(fx, fy) = \max\{d(fx, fy), d(fx, gx), d(fy, gy), \} + \delta,\)

Then \(f\) and \(g\) have a unique coincidence point in \(X.\)

Moreover, if \(f\) and \(g\) are weakly compatible, then \(f\) and \(g\) have a unique common fixed point.

**Proof.** Let \(x_0\) be an arbitrary point in \(X.\) Since \(gX \subseteq fX,\) we can define the sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) by

\[(3.13)\] \(y_n = fx_{n+1} = gx_n\) for all \(n \geq 0.\)

Moreover, we assume that if \(y_n = y_{n+1}\) for some \(n\) in \(\mathbb{N},\) then there is nothing to prove. Substituting \(x = x_n\) and \(y = x_{n+1}\) in (3.12), using (3.13), we have

\[\psi(d(y_n, y_{n+1})) = \psi(d(gx_n, gx_{n+1})) \]
\[\leq \Phi(N(fx_n, fx_{n+1})) - \Phi(N(fx_n, fx_{n+1})) \]
\[(3.14)\] \[= \Phi(N(y_{n-1}, y_n)) - \Phi(N(y_{n-1}, y_n)),\] where \(N(y_{n-1}, y_n) = \max\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_{n}, y_{n+1}), \} + \delta.\)

If \(d(y_{n-1}, y_n) < d(y_n, y_{n+1}),\) then from (3.14), we get

\[\psi(d(y_{n-1}, y_{n+1})) \leq \Phi(d(y_n, y_{n+1})) - \Phi(d(y_n, y_{n+1})),\]

which implies that, \(d(y_{n-1}, y_{n+1}) = 0,\) that is, \(y_n = y_{n+1},\) which is a contradiction. Therefore \(d(y_n, y_{n+1}) < d(y_{n-1}, y_n),\) then from (3.14), we obtain

\[\psi(d(y_n, y_{n+1})) \leq \Phi(d(y_{n-1}, y_n)) - \Phi(d(y_{n-1}, y_n)).\]

From (ii) of Lemma 3.5.1, we obtain that

\[(3.15)\] \[\lim_{n \to \infty} \Phi(d(y_{n-1}, y_n)) = 0.\]

Next, we prove that \(\{y_n\}\) is a g.m.s. Cauchy sequence. Suppose that \(\{y_n\}\) is not a g.m.s. Cauchy sequence. Then there exists \(\varepsilon > 0,\) for which we can find subsequences \(\{y_{m(k)}\}\) and \(\{y_{n(k)}\}\) of \(\{y_n\}\) with \(m(k) > n(k) > k\) such that
Further corresponding to \( m(k) \), we can choose \( n(k) \) in such a way that it is the smallest integer with \( n(k) > m(k) \) satisfying (3.16). Then

(3.17) \[ d(y_{n(k)-1}, y_{m(k)}) < \varepsilon. \]

Now, using (3.16), (3.17) and the rectangular inequality, we have

\[
\varepsilon \leq d(y_{n(k)}, y_{m(k)}) \\
\leq d(y_{n(k)}, y_{n(k)-2}) + d(y_{n(k)-2}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) \\
< d(y_{n(k)}, y_{n(k)-2}) + d(y_{n(k)-2}, y_{n(k)-1}) + \varepsilon.
\]

Letting \( k \to \infty \) in the above inequality and using (3.15), we obtain

(3.18) \[ \lim_{k \to \infty} d(y_{n(k)}, y_{m(k)}) = \varepsilon. \]

Again using the rectangular inequality, we have

\[
d(y_{n(k)}, y_{m(k)}) \leq d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) \\
d(y_{n(k)-1}, y_{m(k)}) \leq d(y_{n(k)-1}, y_{n(k)}) + d(y_{n(k)}, y_{m(k)}).
\]

Taking \( k \to \infty \) in the above inequalities and using (3.15) and (3.18), we get

(3.19) \[ \lim_{k \to \infty} d(y_{n(k)-1}, y_{m(k)-1}) = \varepsilon. \]

Substituting \( x = x_{n(k)} \) and \( y = x_{m(k)} \) in (3.13), we have

\[
\psi(d(gx_{n(k)}, gx_{m(k)})) \leq \overline{d}(N(fx_{n(k)}, fx_{m(k)}) - \overline{d}(N(fx_{n(k)}, fx_{m(k)})), that is,
\]

(3.20) \[ \psi(d(y_{n(k)}, y_{m(k)})) \leq \overline{d}(N(y_{n(k)-1}, y_{m(k)-1})) - \overline{d}(N(y_{n(k)-1}, y_{m(k)-1})) \\
= \max\{ d(y_{n(k)-1}, y_{n(k)-1}), d(y_{m(k)-1}, y_{m(k)-1}), \frac{d(y_{m(k)-1}, y_{n(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})}{\overline{d}(N(y_{n(k)-1}, y_{m(k)-1}))} \}. \]

From (3.18), (3.19) and (3.21), we conclude that

\[
\lim_{k \to \infty} d(y_{n(k)-1}, y_{m(k)-1}) = \varepsilon.
\]

Letting \( k \to \infty \) in (3.20) and using the lower semi-continuity of \( \overline{d} \) and the continuities of \( \psi \) and \( \overline{d} \), we obtain

\[
\psi(\varepsilon) \leq \overline{d}(\varepsilon) - \overline{d}(\varepsilon),
\]

which implies that \( \varepsilon = 0 \), by (3.9), a contradiction with \( \varepsilon > 0 \).

It follows that \( \{y_n\} \) is a g.m.s. Cauchy sequence.

Since \( fX \) is complete, so there exists a point \( u \) in \( fX \) such that

\[
\lim_{k \to \infty} y_{n(k)} = \lim_{k \to \infty} y_{m(k)} = u.
\]

We claim that \( fp = u \).
From (3.12), we have
\[ \psi(d(fx_{n+1}, gp)) = \psi(d(gx_n, gp)) \]
\[ \leq \|N(gx_n, gp)\| - \|N(gx_n, gp)\|, \]
where
\[ N(gx_n, gp) = \max \{d(fx_n, fp), d(fx_n, gx_n), d(fp, gp), \} \]
Making limit as \( n \to \infty \), we have
\[ \lim_{n \to \infty} N(gx_n, gp) = \max \{d(fp, gp), d(fp, fp), d(fp, gp), \} \]
(3.24) \[ \lim_{n \to \infty} N(gx_n, gp) = d(fp, gp). \]
So, from (3.23) and (3.24), we have
\[ \psi(d(fp, gp)) \leq \|d(fp, gp)\| - \|d(fp, gp)\|, \]
which implies that, \( d(fp, gp) = 0 \), that is,
(3.25) \[ fp = gp = u. \]
Therefore, \( p \) is a point of coincidence of \( f \) and \( g \).
The uniqueness of the point of coincidence is a consequence of condition (3.12).
Now, we show that there exists a common fixed point of \( f \) and \( g \). Since \( f \) and \( g \) are weakly compatible, by (3.25), we have \( gfp = fgp \), and
(3.26) \[ gu = gfp = fgp = fu. \]
If \( p = u \), then \( p \) is a common fixed point of \( f \) and \( g \).
If \( p \neq u \), then by (3.12), we have
\[ \psi(d(gp, gu)) \leq \|N(gp, gu)\| - \|N(gp, gu)\|, \]
where
\[ N(gp, gu) = \max \{d(fp, fu), d(fp, gp), d(fu, gu), \} \]
\[ = \max \{d(u, gu), d(u, u), 0, \} \]
(3.9) \[ \psi(d(u, gu)) \leq \|d(u, gu)\| - \|d(u, gu)\|. \]
which by (3.9) implies that, \( d(u, gu) = 0 \), that is, \( u = gu = fu \).
Consequently, \( u \) is the unique common fixed point of \( f \) and \( g \).

Denote by \( \Lambda \) the set of functions \( \mathbb{R} : [0, \infty) \to [0, \infty) \) satisfying the following hypotheses:
(1) \( \psi \) is a Lebesgue-integrable mapping on each compact subset of \([0,
(h2) for every $\varepsilon > 0$, we have

$$\lim_{t \to 0^+} \beta(t) \geq 0.$$  

Now, we prove some results for mappings of integral type.

**Theorem 3.5.3.** Let $(X, d)$ be a Hausdorff g.m.s. and $f, g : X \to X$ be self mappings satisfying (3.10), (3.11) and the following:

$$\begin{align*}
\beta_1 (x, y) & \leq 0, \\
\beta_2 (x, y) & \leq 0, \\
\beta_3 (x, y) & \leq 0,
\end{align*}$$

for all $x, y$ in $X$, where $\beta_1, \beta_2, \beta_3 \in \Lambda$ and satisfy condition (3.9). If $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.

**Proof.** On taking $\psi(t) = 0$, $\beta_1 (x, y) = 0$, $\beta_2 (x, y) = 0$ and $\beta_3 (x, y) = 0$ in Theorem 3.5.2, we get Theorem 3.5.3.

Taking $\beta_3 (x, y) = (1-k) \beta_2 (x, y)$ for $k \in [0, 1)$ in Theorem 3.5.3, we obtain the following result:

**Corollary 3.5.4.** Let $(X, d)$ be a Hausdorff g.m.s. and $f, g : X \to X$ be self mappings satisfying (3.10), (3.11) and the following:

$$\begin{align*}
\beta_1 (x, y) & \leq k \beta_2 (x, y), \\
\beta_3 (x, y) & \leq k \beta_2 (x, y),
\end{align*}$$

for all $x, y$ in $X$, where $\beta_1, \beta_2 \in \Lambda$ and satisfy condition (3.9). If $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.

**Remark 3.5.5.** If $N(fx, fy) = d(fx, fy)$, then (3.12) reduces to

$$\psi(d(gx, gy)) \leq \psi(d(fx, fy)) - \psi(d(fx, fy)),$$

which is condition (2.3) of Theorem 1 [42].

**Remark 3.5.6.** If $f$ is the identity mapping, then (3.27) reduces to

$$\psi(d(gx, gy)) \leq \psi(d(x, y)) - \psi(d(x, y)).$$

**Example 3.5.7.** Let $X = [0, 10] \cup \{11, 12, 13, \ldots\}$ and

$$\begin{align*}
d(x, y) &= \max \{ \frac{x+y}{2}, 0, 10, \frac{y-x}{2} \}, \\
&\in [0, 10], \frac{y-x}{2} \notin [0, 10] \frac{y-x}{2} \neq \frac{y-x}{2}.
\end{align*}$$

Then $(X, d)$ is a Hausdorff and g.m.s.

Let $\psi, \beta, \gamma : [0, \infty) \to [0, \infty)$ be defined as

$$\begin{align*}
\beta(t) &= \begin{cases} \\
0 & \text{for } 0 \leq t \leq 10, \\
\frac{t^2}{2} & \text{for } t > 10, \\
\gamma(t) &= \begin{cases} \\
1 & \text{for } 0 \leq t \leq 10, \\
\frac{t^2}{2} & \text{for } t > 10.
\end{cases}
\end{cases}
$$
Let \( g : X \to X \) be defined as
\[
g_x = \begin{cases} 
\frac{\bar{x}}{2}, & 0 \leq \bar{x} \leq 10, \\
-10, & \bar{x} \in \{11, 12, 13, \ldots\}
\end{cases}
\]
Without loss of generality, assume that \( x > y \) and discuss the following cases:

Case 1. \((x \in [0, 10])\). Then
\[
\psi(d(gx, gy)) = \left\{x - \frac{1}{5} \bar{x}^2 \right\} - \left\{y - \frac{1}{5} \bar{y}^2 \right\}
\]
\[
= (x - y) - \frac{1}{5} (x - y) (x + y) \leq (x - y) - \frac{1}{5} (\bar{x} - \bar{y})^2
\]
\[
= d(x, y) - \frac{1}{5} (\bar{x} - \bar{y})^2
\]
\[
= d(x, y) - \frac{1}{5} (d(gx, gy)).
\]

Case 2. \((x \in \{12, 13, \ldots\})\). Then
\[
d(gx, gy) = d(x-10, y-\frac{1}{5} \bar{y}^2), \text{ if } y \in [0, 10],
\]
or, \(d(gx, gy) = x - 10 + y - \frac{1}{5} \bar{y}^2 \leq x + y - 10\).

and
\[
d(gx, gy) = d(x-10, y-10), \text{ if } y \in \{11, 12, 13, \ldots\},
\]
or, \(d(gx, gy) = x - 10 + y - 10 < x + y - 10\).

Consequently, we have
\[
\psi(d(gx, gy)) = (d(gx, gy))^2 \leq (x + y - 10)^2 < (x + y - 10) (x + y + 10)
\]
\[
= (x + y)^2 - 100 < (x + y)^2 - \frac{1}{5}
\]
\[
= d(x, y) - \frac{1}{5} (d(gx, gy)).
\]

Case 3. \((x = 11)\). Then \(y \in [0, 10], gx = 1\) and \(d(gx, gy) = 1 - (y - \frac{1}{5} \bar{y}^2) \leq 1\).

So, we have \(\psi(d(gx, gy)) \leq \psi(1) = 1\).
Again \(d(x, y) = 11 + y\).
So, \(d(gx, gy) = (11 + y)^2 = \frac{1}{5}
\]
\[
= 121 + y^2 + 22 y - \frac{1}{5}
\]
\[
= 604 + 22 y + y^2 > 1 = \psi(d(gx, gy)).
\]

Considering all the above cases, we conclude that the inequality (3.28) remains valid for
\[\psi, \bar{\psi}, \text{ and } \overline{\psi} \text{ constructed as above and consequently, } g \text{ has a unique fixed point.}
\]
Clearly, \(0\) is the unique fixed point of \(g\).
Weakly compatible maps with E.A. and (CLR) properties in generalized metric spaces

In this section, some common fixed point theorems using E.A. and (CLR) properties in generalized metric spaces for weakly compatible maps are proved.

**Theorem 3.6.1.** Let $f$ and $g$ be self mappings of a Hausdorff g.m.s $(X, d)$ satisfying (3.10), (3.12) and the following:

1. $f$ and $g$ are weakly compatible,
2. $f$ and $g$ satisfy the E.A. property.

If the range of $f$ or $g$ is a complete subspace of $X$, then $f$ and $g$ have a unique common fixed point in $X$.

**Proof.** Since $f$ and $g$ satisfy the E.A. property, there exists a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} f(x_n) = z,$$

for some $z \in X$.

Since $gx_n \in fX$, there exists a sequence $\{y_n\}$ in $X$ such that $gx_n = fy_n$.

Let us suppose that $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

From (3.12), we have

$$\psi(d(gx_n, gy_n)) \leq \|N(fx_n, fy_n)\| - \|N(fx_n, fy_n)\|,$$

where $N(fx_n, fy_n) = \max\{d(fx_n, fy_n), d(fx_n, gx_n), d(fy_n, gy_n), 2\}$. Letting $n \to \infty$, we have

$$\lim_{n \to \infty} N(fx_n, fy_n) = \max\{d(z, z), d(z, t), d(z, t), \frac{1}{2}\|z - t\| \} = d(z, t).$$

Thus, from (3.33) and (3.34), we get

$$\psi(d(z, t)) \leq \|N(fx_n, fy_n)\| - \|N(fx_n, fy_n)\|,$$

which implies that $d(z, t) = 0$, that is, $z = t$.

Hence, $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

Now, suppose that $fX$ is complete subspace of $X$. Then, there exists $u$ in $X$ such that $z = fu$.

Subsequently, we have...
\[ \lim_{n \to \infty} \psi(d(x_n, u_n)) = \lim_{n \to \infty} \psi(d(x_n, u_n)) = \lim_{n \to \infty} \psi(d(x_n, u_n)) = \lim_{n \to \infty} \psi(d(x_n, u_n)) = z = u. \]

Now, we show that \( u = v. \)

From (3.12), we have

\[ \psi(d(gx_n, gu)) \leq \psi(N(fx_n, fu)) - \psi(N(fx_n, fu)). \]

Letting \( n \to \infty \), we have

(3.35) \[ \psi(d(z, gu)) \leq \psi(\lim N(fx_n, fu)) - \psi(\lim N(fx_n, fu)), \]

where

\[ N(fx_n, fu) = \max\{d(fx_n, fu), d(fx_n, gx_n), d(fu, gu), \} = d(z, gu). \]

Letting \( n \to \infty \), we have

(3.36) \[ \lim N(fx_n, fu) = \max\{d(z, z), d(z, z), d(z, gu), \} = d(z, gu). \]

Thus, from (3.35) and (3.36), we get

\[ \psi(d(z, gu)) \leq \psi(d(z, gu)) - \psi(d(z, gu)), \]

which implies that \( d(z, gu) = 0 \), that is, \( z = gu = u. \)

Since \( f \) and \( g \) are weakly compatible, therefore, \( gfu = fgu \), implies that, \( ffu = fg = gfu = ggu. \)

Now, we claim that \( gu \) is the common fixed point of \( f \) and \( g \).

From (3.12), we have

\[ \psi(d(gu, ggu)) \leq \psi(N(fu, ffu)) - \psi(N(fu, ffu)) \]

\[ = \psi(d(fu, ffu)) - \psi(d(fu, ffu)) \]

\[ = \psi(d(gu, ggu)) - \psi(d(gu, ggu)), \]

which implies that, \( gu = ggu = ffu. \)

Therefore, \( gu \) is the common fixed point of \( f \) and \( g \).

For the uniqueness, let \( z \) and \( w \) be two common fixed points of \( f \) and \( g \).

From (3.12), we have

\[ \psi(d(z, w)) = \psi(d(gz, gw)) \]

(3.37) \[ \leq \psi(N(fz, fw)) - \psi(N(fz, fw)), \]

where

\[ N(fz, fw) = \max\{d(fz, fw), d(fz, gz), d(fw, gw), \} = d(z, w). \]

(3.38) \[ \leq \max\{d(z, w), 0, \} = d(z, w). \]

From (3.37) and (3.38), we get

\[ \psi(d(z, w)) \leq \psi(d(z, w)) - \psi(d(z, w)), \]

which implies that, \( d(z, w) = 0 \), that is, \( z = w. \)

Therefore, \( f \) and \( g \) have a unique common fixed point in \( X. \)
Theorem 3.6.2. Let f and g be self mappings of a Hausdorff g.m.s (X, d) satisfying (3.10), (3.12), (3.30) and the following:
(3.39) f and g satisfy (CLRf) property.
Then f and g have a unique common fixed point in X.

Proof. Since f and g satisfy the (CLRf) property, there exists a sequence \{x_n\} in X such that

\[
\lim_{n \to \infty} \psi(d(gx_n, gx)) \leq \lim_{n \to \infty} \psi(d(fx, gx)) = d(fx, gx),
\]
for some x in X.

From (3.12), we have

\[
\psi(d(fx, gx)) \leq \psi(N(fx, fx)) - \psi(N(fx, fx)),
\]
Letting \(n \to \infty\), we have

(3.40) \[
\lim_{n \to \infty} N(fx_n, fx) = \max \{d(fx_n, fx), d(fx_n, gx_n), d(fx, gx), d(fx, gx)\}.
\]

Letting \(n \to \infty\), we have

(3.41) \[
\lim_{n \to \infty} N(fx_n, fx) = d(fx, gx).
\]
Thus, from (3.40) and (3.41), we get

\[
\psi(d(fx, gx)) \leq \psi(d(fx, gx)) - \psi(d(fx, gx)),
\]
which implies that \(d(fx, gx) = 0\), that is, \(gx = fx\).

Now, let \(z = fx = gx\).

Since f and g are weakly compatible, therefore, \(fgx = gfx\), implies that,
\(fz = fgx = gfx = gz\).

Now, we claim that \(gz = z\).

From (3.12), we have

\[
\psi(d(gz, z)) = \psi(d(gz, gx)) \leq \psi(N(fz, fx)) - \psi(N(fz, fx)),
\]
where

(3.42) \[
N(fz, fx) = \max \{d(fz, fx), d(fz, gz), d(fx, gx), d(fx, gx)\} = d(gz, z).
\]

(3.43) \[
= \max \{d(gz, z), 0, 0, d(gz, z)\} = d(gz, z) \cdot 2.
\]

From (3.42) and (3.43), we get

\[
\psi(d(gz, z)) \leq \psi(d(gz, z)) - \psi(d(gz, z)),
\]
which implies that \(d(gz, z) = 0\), that is, \(gz = z\).
Hence, \( gz = z = fz \). So, \( z \) is the common fixed point of \( f \) and \( g \).

For the uniqueness, let \( w \) be another common fixed point of \( f \) and \( g \).

From (3.12), we have

\[
\psi(d(z, w)) = \psi(d(gz, gw)) 
\leq \psi(N(fz, fw)) - \varphi(N(fz, fw)),
\]

where

\[
N(fz, fw) = \max \{d(fz, fw), d(fz, gz), d(fw, gw)\}.
\]

(3.44)

\[
= \max \{d(z, w), 0, \varphi(d(z, w))\} = d(z, w).
\]

(3.45)

From (3.44) and (3.45), we get

\[
\psi(d(z, w)) \leq \psi(d(z, w)) - \varphi(d(z, w)),
\]

which implies that, \( d(z, w) = 0 \), that is, \( z = w \).

Therefore, \( f \) and \( g \) have a unique common fixed point in \( X \).

**Fixed points for \( \varphi \)-weakly expansive mappings in G-metric spaces**

Here, we introduce the notion of \( \varphi \)-weakly expansive mappings in G-metric spaces as follows:

**Definition 3.7.1.** Let \((X, G)\) be a G-metric space. A mapping \( f : X \rightarrow X \) is said to be a \( \varphi \)-weakly expansive, if there exists a map \( \varphi : [0, \infty) \rightarrow (-\infty, 0] \) with \( \varphi(0) = 0 \) and \( \varphi(t) < 0 \) for all \( t > 0 \) such that

\[
G(fx, fy, fz) \geq G(x, y, z) - \varphi(G(x, y, z)), \text{ for all } x, y, z \text{ in } X.
\]

**Definition 3.7.2.** Let \((X, G)\) be a G-metric space. A mapping \( f : X \rightarrow X \) is said to be a \( \varphi \)-weakly expansive with respect to \( g : X \rightarrow X \), if there exists a map \( \varphi : [0, \infty) \rightarrow (-\infty, 0] \) with \( \varphi(0) = 0 \) and \( \varphi(t) < 0 \) for all \( t > 0 \) such that

\[
G(fx, fy, fz) \geq d(gx, gy, gz) - \varphi(d(gx, gy, gz)), \text{ for all } x, y, z \text{ in } X.
\]

Now, we prove the existence of fixed point for a \( \varphi \)-weakly expansive self map in G-complete metric space.

**Theorem 3.7.3.** Let \((X, G)\) be a G-complete metric space. Let \( f \) be a self mapping on \( X \) satisfying the following:

(3.46) \( f \) be \( \varphi \)-weakly expansive, that is,

\[
G(fx, fy, fz) \geq G(x, y, z) - \varphi(G(x, y, z)),
\]
for all \( x, y, z \in X \), where \( \varphi : [0, \infty) \to (-\infty, 0] \) is a continuous and non-decreasing map \((3.47)\). It's onto.

Then \( f \) has a unique fixed point.

**Proof.** Since \( f \) is an onto mapping, therefore for each \( x_0 \) in \( X \), there exists \( x_1 \) in \( X \) such that \( f(x_1) = x_0 \).

Continuing this process, we can define \( \{x_n\} \) by \( x_n = f x_{n+1} \), \( n = 0, 1, 2, \ldots \).

Without loss of generality, we suppose that \( x_{n+1} \neq x_n \) for all \( n \geq 1 \).

From (3.46), we have
\[
G(x_{n+1}, x_n, x_n) \geq G(x_{n+1}, x_n, x_n) - \varphi(G(x_{n+1}, x_n, x_n)),
\]
implies that, \( G(x_{n+1}, x_n, x_n) > G(x_{n+1}, x_n, x_n) \), for all \( n \geq 1 \).

Hence the sequence \( \{G(x_{n+1}, x_n, x_n)\} \) is strictly decreasing and bounded below. Thus, there exists \( r \geq 0 \), such that, \( G(x_{n+1}, x_n, x_n) \to r \) as \( n \to \infty \).

Letting \( n \to \infty \) in (3.48), we have
\[
r \geq r - \varphi(r), \quad \text{a contradiction, unless } r = 0.
\]

Hence
\[
G(x_{n+1}, x_n, x_n) \to 0 \quad \text{as } n \to \infty.
\]

Now, we show that \( \{x_n\} \) is a G-Cauchy sequence. If possible, let \( \{x_n\} \) is not a G-Cauchy sequence. Then there exists \( \varepsilon > 0 \) for which we can find subsequences \( \{x_{m(k)}\} \) and \( \{x_{n(k)}\} \) of \( \{x_n\} \) with \( n(k) > m(k) > k \) such that
\[
G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geq \varepsilon.
\]

Further, corresponding to \( m(k) \), we can choose \( n(k) \) in such a way that it is the smallest integer with \( n(k) > m(k) \) and satisfying (3.50).

Then
\[
G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) < \varepsilon.
\]

Then, we have
\[
\varepsilon \leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}),
\]
\[
< \varepsilon + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}).
\]

Letting \( k \to \infty \) and using (3.49), we have
\[
\lim_{k \to \infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon.
\]

Again using triangular inequality,
\[ G(x_n(k), x_m(k), x_m(k)) \leq G(x_n(k), x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_m(k)-1, x_{m(k)-1}) + G(x_{m(k)-1}, x_m(k), x_m(k)), \]
\[ G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) \leq G(x_{n(k)-1}, x_n(k), x_n(k)) + G(x_{n(k)}, x_m(k), x_m(k)) + G(x_m(k), x_{m(k)-1}, x_{m(k)-1}). \]

Letting \( k \to \infty \) in the above two inequalities and using (3.49), we get

\[ \lim_{n \to \infty} G(x_{m(k)-1}, x_{n(k)}, x_{n(k)}) = \varepsilon. \]

Putting \( x = x_{m(k)}, y = x_{n(k)} \) and \( z = x_{n(k)} \) in (3.46), we get
\[ G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1}) = G(fx_{m(k)}, fx_{n(k)}, fx_{n(k)}), \]
\[ \geq G(x_{m(k)}, x_n(k), x_n(k)) - \varphi(G(x_{m(k)}, x_{n(k)}, x_{n(k)})). \]

Letting \( k \to \infty \) and using (3.53), (3.54), we get
\[ \varepsilon \geq \varepsilon - \varphi(\varepsilon), \]
implies that, \( \varphi(\varepsilon) \geq 0 \), a contradiction.

Hence \( \{x_n\} \) is a \( G \)-Cauchy sequence in \( X \). Since \( X \) is \( G \)-complete, therefore there exists \( q \) in \( X \) such that \( fx_{n+1} = x_n \to q \), as \( n \to \infty \). Consequently, we can find a \( p \) in \( X \) such that \( fp = q \).

Now, we show that \( p = q \). Let, if possible, \( p \neq q \).

Putting \( x = p, y = x_{n+1} \) and \( z = x_{n+1} \) in (3.46), we get
\[ G(q, x, x_n) = G(fp, fx_{n+1}, fx_{n+1}) \]
\[ \geq G(p, x_{n+1}, x_{n+1}) - \varphi(G(p, x_{n+1}, x_{n+1})). \]

Letting \( n \to \infty \), we get
\[ G(q, q, q) \geq G(p, q, q) - \varphi(G(p, q, q)), \]
that is,
\[ \varphi(G(p, q, q)) \geq G(p, q, q) > 0, \]
a contradiction.

Hence \( p = q \), that is, \( fp = q \).

Hence \( q \) is the fixed point of \( f \).

For the uniqueness, let \( z \) be another fixed point of \( f \) such that \( q \neq z \).

From (3.46), we have
\[ G(q, z, z) = G(fq, fz, fz) \]
\[ \geq G(q, z, z) - \varphi(G(q, z, z)), \]
that is,
\[ \varphi(G(q, z, z)) \geq 0, \]
a contradiction.

Hence \( q = z \).

So, \( f \) has a unique fixed point.

**Example 3.7.4.** Let \( X = \mathbb{R} \) and let \( G(x, y, z) = \max\{ |x| - |y|, |y| - |z|, |z| - |x| \} \), then \( (X, G) \) is a \( G \)-complete metric space.
Let \( f_x = 2x, x \in X \) and define \( \varphi : [0, \infty) \rightarrow (-\infty, 0] \) by \( \varphi(t) = \frac{-t}{2} \).

Without loss of generality, assume that \( x > y > z \).

Then \( G(f_x, f_y, f_z) = 2 \cdot -\cdot \).

\( \varphi(G(x, y, z)) = \varphi(\cdot - \cdot) = \frac{-\cdot - \cdot}{2} \).

Therefore,

\[
G(x, y, z) - \varphi(G(x, y, z)) = \frac{-\cdot + \cdot - \cdot - \cdot}{2}.
\]

Clearly,

\[
G(f_x, f_y, f_z) \geq G(x, y, z) - \varphi(G(x, y, z)).
\]

Also \( f \) is onto and 0 is the unique fixed point of \( f \).

Next, the result for a pair of mappings is established.

Manro et al. [64] proved the following fixed point theorem on G-metric spaces:

**Theorem 3.7.5.** Let \( f \) and \( g \) be two weakly reciprocally continuous self mappings of a complete G-metric space \((X, G)\) satisfying the following:

1. \( gX \subseteq fX \)
2. For any \( x, y \) in \( X \) and \( q > 1 \), we have that \( G(fx, fy, fz) \geq q G(gx, gy, gz) \).

If \( f \) and \( g \) are either compatible or R-weakly commuting of type (A) or R-weakly commuting of type (A) or R-weakly commuting of type (P), then \( f \) and \( g \) have a unique common fixed point.

Now, Theorem 3.7.5 is extended using the notion of \( \varphi \)-weakly expansive mappings as follows:

**Theorem 3.7.6.** Let \( f \) and \( g \) be two weakly reciprocally continuous self mappings of a complete G-metric space \((X, G)\) satisfying the following conditions:

(3.55) \( gX \subseteq fX \),

(3.56) \( G(fx, fy, fz) \geq G(gx, gy, gz) - \varphi(G(gx, gy, gz)) \),

for all \( x, y, z \) in \( X \) (i.e., it is \( \varphi \)-weakly expansive), where \( \varphi : [0, \infty) \rightarrow (-\infty, 0] \) is a non-increasing map with \( \varphi(0) = 0 \) and \( \varphi(t) < 0 \) for all \( t > 0 \). If \( f \) and \( g \) are compatible, then \( f \) and \( g \) have a unique common fixed point.

**Proof.** Let \( x_0 \in X \). Since \( gX \subseteq fX \), there exists a sequence of points \( \{x_n\} \) such that \( gx_n = fx_{n+1} \).

Define a sequence \( \{y_n\} \) in \( X \) by

(3.57) \( y_n = gx_n = fx_{n+1} \).
Moreover, we assume that, if \( y_n = y_{n+1} \) for some \( n \in \mathbb{N} \), then there is nothing to prove. Now, we assume that \( y_n \neq y_{n+1} \) for all \( n \in \mathbb{N} \).

From (3.56), we have
\[
G(y_{n-1}, y_n, y_n) = G(fx_n, fx_{n+1}, fx_{n+1}) \geq G(gx_n, gx_{n+1}, gx_{n+1}) - \varphi(G(gx_n, gx_{n+1}, gx_{n+1}))
\]
(3.58)

\[
= G(y_n, y_{n+1}, y_{n+1}) - \varphi(G(y_n, y_{n+1}, y_{n+1})), \text{ that is,}
G(y_{n-1}, y_n, y_n) > G(y_n, y_{n+1}, y_{n+1}).
\]

Let \( u_n = G(y_n, y_{n+1}, y_{n+1}) \), then \( 0 \leq u_n \leq u_{n-1} \) for all \( n > 0 \).

It follows that the sequence \( \{u_n\} \) is monotonically decreasing and bounded below.

So, there exists some \( r \geq 0 \) such that
\[
\lim_{n \to \infty} G(y_n, y_{n+1}, y_{n+1}) = r.
\]

Using this in (3.58), then as \( n \to \infty \), we have
\[
0 \geq \varphi(r) \geq r - r = 0, \text{ that is,}
\]
\[\
\varphi(r) = 0, \text{ implies that, } r = 0.
\]

Therefore,
(3.59)
\[
\lim_{n \to \infty} G(y_n, y_{n+1}, y_{n+1}) = 0.
\]

Now, we prove that \( \{y_n\} \) is a \( G \)-Cauchy sequence. Let, if possible, \( \{y_n\} \) is not a \( G \)-Cauchy sequence. Then, there exists \( \varepsilon > 0 \) for which we can find subsequences \( \{y_{m(k)}\} \) and \( \{y_{n(k)}\} \) of \( \{y_n\} \) with \( n(k) > m(k) > k \) such that
(3.60)

\[
G(y_{m(k)}, y_{m(k)}, y_{m(k)}) \geq \varepsilon.
\]

Let \( m(k) \) be the least positive integer exceeding \( n(k) \) satisfying (3.60) and such that
(3.61)
\[
G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}) < \varepsilon, \text{ for every integer } k.
\]

Then, we have
\[
\varepsilon \leq G(y_{m(k)}, y_{m(k)}, y_{m(k)})
\leq G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}) + G(y_{m(k)-1}, y_{m(k)}, y_{m(k)})
\leq \varepsilon + G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}).
\]
(3.62)

In other words,
\[
0 \leq G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}) = G(y_{m(k)-1}, y_{m(k)-1}, y_{m(k)}),
\]
but
\[
G(y_{m(k)-1}, y_{m(k)-1}, y_{m(k)}) \leq G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}).
\]

Letting \( k \to \infty \) and using (3.59), we find
\[
G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}) \to 0.
\]

Using this in (3.62), we get
(3.63)
\[
\lim_{k \to \infty} G(y_{m(k)}, y_{m(k)}, y_{m(k)}) = \varepsilon.
\]
By the triangular inequality,
\[ G(\text{yn}(k), \text{ym}(k), \text{ym}(k)) \leq G(\text{yn}(k), \text{ym}(k)-1, \text{ym}(k)-1) + G(\text{ym}(k)-1, \text{ym}(k), \text{ym}(k)) \]
\[ + G(\text{yn}(k)-1, \text{ym}(k), \text{ym}(k)) \]
\[ G(\text{yn}(k)-1, \text{ym}(k)-1, \text{ym}(k)-1) \leq G(\text{yn}(k), \text{ym}(k), \text{ym}(k)) + G(\text{ym}(k), \text{ym}(k), \text{ym}(k)) \]
\[ + G(\text{yn}(k), \text{ym}(k)-1, \text{ym}(k)-1). \]

Letting \( k \to \infty \) in the above two inequalities and using (3.59)-(3.63), we get
(3.64) \[ \lim_{n\to\infty} G(\text{yn}(k)-1, \text{ym}(k)-1, \text{ym}(k)-1) = \varepsilon. \]

From (3.56), we have
\[ G(\text{yn}(k)-1, \text{ym}(k)-1, \text{ym}(k)-1) = G(\text{fxm}(k), \text{fxm}(k), \text{fxm}(k)) \]
\[ \geq G(\text{gxn}(k), \text{gxn}(k), \text{gxn}(k)) - \varphi(G(\text{gxn}(k), \text{gxn}(k), \text{gxn}(k))) \]
\[ = G(\text{yn}(k), \text{ym}(k), \text{ym}(k)) - \varphi(G(\text{yn}(k), \text{ym}(k), \text{ym}(k))). \]

Letting \( k \to \infty \) and using (3.64), we get
\[ \varepsilon \geq \varepsilon - \varphi(\varepsilon), \] a contradiction.

Hence \( \{y_n\} \) is a G-Cauchy sequence in \( X \). Since \( (X, G) \) is a G-complete metric space, there exists a point \( z \) in \( X \) such that \( \lim_{n\to\infty} y_n = z. \)

Therefore, from (3.57), we have
\[ \lim_{n\to\infty} y_n = \lim_{n\to\infty} g\text{xn} = \lim_{n\to\infty} f\text{xn+1} = z. \]

Suppose that \( f \) and \( g \) are compatible mappings. Now, by the reciprocal continuity of \( f \) and \( g \), we obtain \( \lim_{n\to\infty} f\text{gx}_n = fz \) or \( \lim_{n\to\infty} g\text{fx}_n = gz. \)

Let \( \lim_{n\to\infty} f\text{gx}_n = fz \), then the compatibility of \( f \) and \( g \) gives \( \lim_{n\to\infty} G(f\text{gx}_n, f\text{gx}_n, f\text{gx}_n) = 0. \)

Hence \( \lim_{n\to\infty} g\text{fx}_n = gz. \)

Now, we claim that, \( fz = gz. \)

Let, if possible, \( fz \neq gz. \)

From (3.57), we get
\[ \lim_{n\to\infty} g\text{fx}_{n+1} = \lim_{n\to\infty} g\text{gx}_n = fz. \]

Therefore, from (3.56), we get
\[ G(fz, f\text{gx}_n, f\text{gx}_n) \geq G(gz, g\text{gx}_n, g\text{gx}_n) - \varphi(G(gz, g\text{gx}_n, g\text{gx}_n)). \]

Letting \( n \to \infty \), we get
\[ G(fz, fz, fz) \geq G(gz, fz, fz) - \varphi(G(gz, fz, fz)), \] that is,
\[ \varphi(G(gz, fz, fz)) \geq G(gz, fz, fz) > 0, \] a contradiction.

Hence \( fz = gz. \)
Again, the compatibility of \( f \) and \( g \) implies the commutativity at a coincidence point. Hence \( gfz = fgz = ffz = ggz \).

Now, we claim that, \( gz = ggz \).

Using (3.56), we obtain

\[
G(fz, fgz, fgz) \geq G(gz, ggz, ggz) - \varphi(G(gz, ggz, ggz)),
\]

that is,

\[
G(gz, ggz, ggz) \geq G(gz, ggz, ggz) - \varphi(G(gz, ggz, ggz)).
\]

Thus, we get

\[
0 \geq \varphi(G(gz, ggz, ggz)) \geq 0,
\]

that is,

\[
\varphi(G(gz, ggz, ggz)) = 0,
\]

implies that, \( gz = ggz \).

Also, we get \( gz = ggz = fgz \), and so, \( gz \) is the common fixed point of \( f \) and \( g \).

Next, suppose that \( \lim_{n \to \infty} gfx_n = gz \).

The assumption \( gX \subseteq fX \), implies that, \( gz = fu \), for some \( u \in X \), and therefore,

\[
\lim_{n \to \infty} gfx_n = fu.
\]

The compatibility of \( f \) and \( g \) implies that \( \lim_{n \to \infty} fgnx_n = fu \).

By virtue of (3.57), this gives

\[
\lim_{n \to \infty} fgnx_n = \lim_{n \to \infty} ggx_n = fu.
\]

Now, we claim that \( fu \neq gu \).

Let, if possible, \( fu \neq gu \).

From (3.56), we have

\[
G(fu, fgu, fgu) \geq G(gu, ggu, ggu) - \varphi(G(gu, ggu, ggu)).
\]

Letting \( n \to \infty \), we get

\[
G(fu, fu, fu) \geq G(gu, fu, fu) - \varphi(G(gu, fu, fu)),
\]

that is,

\[
\varphi(G(gu, fu, fu)) \geq G(gu, fu, fu) > 0,
\]

a contradiction.

Hence \( fu = gu \).

Again, the compatibility of \( f \) and \( g \) implies the commutativity at a coincidence point. Hence \( gfu = fgu = ffu = ggu \).

Finally, we show that, \( gu = ggu \).

From (3.56), we obtain

\[
G(fu, fgu, fgu) \geq G(gu, ggu, ggu) - \varphi(G(gu, ggu, ggu)),
\]

that is,

\[
G(gu, ggu, ggu) \geq G(gu, ggu, ggu) - \varphi(G(gu, ggu, ggu)).
\]

Thus, we get
\[ 0 \geq \varphi(G(gu, ggu, ggu)) \geq 0, \text{ that is,} \]
\[ \varphi(G(\text{gu, ggu, ggu})) = 0, \] implies that, \( \text{gu} = \text{ggu}. \)

Also, we get \( \text{gu} = \text{ggu} = f\text{gu}, \) and so, \( \text{gu} \) is the common fixed point of \( f \) and \( g. \)

For the uniqueness, let \( v \) and \( w \) be two common fixed points of \( f \) and \( g. \)

From (3.56), we have

\[ G(fv, fw, fw) \geq G(gv, gw, gw) - \varphi(G(gv, gw, gw)), \] that is,

\[ G(v, w, w) \geq G(v, w, w) - \varphi(G(v, w, w)). \]

From here, we get

\[ 0 \geq \varphi(G(v, w, w)) \geq 0, \] implies that, \( \varphi(G(v, w, w)) = 0, \) and hence, \( v = w. \)

Hence \( f \) and \( g \) have a unique common fixed point.

Next, we prove a common fixed point theorem for variants of \( R \)-weak commutative mappings (\( R \)-weakly commuting of type \( (A_g), (A_f) \) and \( (P) \)) as follows:

**Theorem 3.7.7.** Let \( f \) and \( g \) be two weakly reciprocally continuous self mappings of a complete \( G \)-metric space \((X, G)\) satisfying (3.55), (3.56) and the following:

(3.65) \( f \) and \( g \) are \( R \)-weakly commuting of type \( (A_g) \) or \( R \)-weakly commuting of type \( (A_f) \) or \( R \)-weakly commuting of type \( P. \)

Then \( f \) and \( g \) have a unique common fixed point.

**Proof.** From Theorem (3.7.6), \( \{y_n\} \) is a \( G \)-Cauchy sequence in \( X. \) Since \((X, G)\) is complete, there exists a point \( z \) in \( X \) such that

\[ \lim_{n \to \infty} y_n = z. \]

Therefore, by (3.57), we have

\[ \lim_{n \to \infty} y_n = \lim_{n \to \infty} g_{x_n} = \lim_{n \to \infty} f_{x_{n+1}} = z. \]

Now, suppose that \( f \) and \( g \) are \( R \)-weakly commuting of type \( (A_f) \). The weak reciprocal continuity of \( f \) and \( g \) implies that \( \lim_{n \to \infty} f_{g_{x_n}} = fz \) or \( \lim_{n \to \infty} g_{f_{x_n}} = gz. \)

Let us first assume that \( \lim_{n \to \infty} f_{g_{x_n}} = fz. \)

Then the \( R \)-weak commutativity of type \( (A_f) \) of \( f \) and \( g \) yields

\[ G(g_{g_{x_n}}, f_{g_{x_n}}, f_{g_{x_n}}) \leq R G(g_{x_n}, f_{x_n}, f_{x_n}), \] that is,

\[ \lim_{n \to \infty} G(g_{g_{x_n}}, f_{x_n}, f_{x_n}) \leq R G(z, z, z) = 0, \] that is, \( \lim_{n \to \infty} g_{g_{x_n}} = fz. \)

Now, we claim that \( fz = gz. \)

Let, if possible, \( fz \neq gz. \)

Again, using (3.56), we get

\[ G(fz, fg_{x_n}, fg_{x_n}) \geq G(gz, gg_{x_n}, gg_{x_n}) - \varphi(G(gz, gg_{x_n}, gg_{x_n})). \]

Letting \( n \to \infty, \) we get
\[ G(f_z, f_z, f_z) \geq G(g_z, f_z, f_z) - \varphi(G(g_z, f_z, f_z)), \] that is,
Thus, we get \( g_z = f_z \).

Again, by using the R-weak commutativity of type (A_c), we have

\[
G(\text{gg}_z, \text{fg}_z, \text{fg}_z) \leq R \ G(\text{g}_z, \text{f}_z, \text{f}_z) = 0,
\]

that is, \( \text{gg}_z = \text{fg}_z \).

Therefore, \( \text{ff}_z = \text{fg}_z = \text{gf}_z = \text{gg}_z \).

From (3.56), we have

\[
G(\text{f}_z, \text{fg}_z, \text{fg}_z) \geq G(\text{g}_z, \text{gg}_z, \text{gg}_z) - \varphi(G(\text{g}_z, \text{gg}_z, \text{gg}_z)),
\]

that is,

\[
G(\text{g}_z, \text{gg}_z, \text{gg}_z) \geq G(\text{g}_z, \text{gg}_z, \text{gg}_z) - \varphi(G(\text{g}_z, \text{gg}_z, \text{gg}_z)).
\]

Thus, we get

\[
0 \geq \varphi(G(\text{g}_z, \text{gg}_z, \text{gg}_z)) \geq 0,
\]

that is,

\[
\varphi(G(\text{g}_z, \text{gg}_z, \text{gg}_z)) = 0,
\]

implies that, \( \text{g}_z = \text{gg}_z \).

Then, we also get, \( \text{g}_z = \text{gg}_z = \text{fg}_z \), and so, \( \text{g}_z \) is the common fixed point of \( f \) and \( g \).

Similar proof works in the case where \( \lim_{n \to \infty} \text{g}_x_n = \text{g}_z \).

Suppose that \( f \) and \( g \) are R-weakly commuting of type (A_g). Again, as done above, we can easily prove that \( f_z \) is a common fixed point of \( f \) and \( g \).

Finally, suppose that \( f \) and \( g \) are R-weakly commuting of type (P). The weak reciprocal continuity of \( f \) and \( g \) implies that \( \lim \ f\text{g}_x_n = f_z \) or \( \lim \ \text{g}_x_n = g_z \).

Let us first assume that \( \lim \ f\text{g}_x_n = f_z \). Then the R-weak commutativity of type (P) of \( f \) and \( g \) yields

\[
G(\text{ff}_x_n, \text{gg}_x_n, \text{gg}_x_n) \leq R \ G(\text{f}_x_n, \text{g}_x_n, \text{g}_x_n),
\]

and therefore

\[
\lim_{n \to \infty} G(\text{ff}_x_n, \text{gg}_x_n, \text{gg}_x_n) \leq R \ G(z, z, z) = 0,
\]

that is,

\[
\lim_{n \to \infty} G(\text{ff}_x_n, \text{gg}_x_n, \text{gg}_x_n) = 0.
\]

Using (3.55) and (3.57), we have, \( f\text{g}_x_{n+1} = \text{ff}_x_n \to f_z \) as \( n \to \infty \), which gives, \( \text{gg}_x_n \to f_z \) as \( n \to \infty \).

Now, we claim that \( f_z = g_z \).

Let, if possible, \( f_z \neq g_z \).

From (3.56), we have

\[
G(f_z, f\text{g}_x_n, f\text{g}_x_n) \geq G(g_z, g\text{g}_x_n, g\text{g}_x_n) - \varphi(G(g_z, g\text{g}_x_n, g\text{g}_x_n)).
\]

Letting \( n \to \infty \), we get

\[
G(f_z, f_z, f_z) \geq G(g_z, f_z, f_z) - \varphi(G(g_z, f_z, f_z)),
\]

that is,
\[ \varphi(G(gz, fz, fz)) \geq G(gz, fz, fz) > 0, \text{ a contradiction.} \]

Thus, we get \( gz = fz \).

Again, by using the R-weak commutativity of type (P), we have

\[ G(ffz, ggz, ggz) \leq R G(fz, gz, gz) = 0, \text{ that is, } ffz = ggz. \]

Therefore, \( ffz = fgz = gfz = ggz \).

Finally, we show that \( gz = ggz \).

From (3.56), we have

\[ G(fz, fgz, fgz) \geq G(gz, ggz, ggz) - \varphi(G(gz, ggz, ggz)), \text{ that is,} \]

\[ G(gz, ggz, ggz) \geq G(gz, ggz, ggz) - \varphi(G(gz, ggz, ggz)). \]

Thus, we get

\[ 0 \geq \varphi(G(gz, ggz, ggz)) \geq 0, \text{ that is,} \]

\[ \varphi(G(gz, ggz, ggz)) = 0, \text{ implies that, } gz = ggz. \]

Then, we also get, \( gz = ggz = fgz \), and so, \( gz \) is the common fixed point of \( f \) and \( g \).

Similar proof works in the case where \( \lim_{n \to \infty} gfx_n = gz \).

For the uniqueness, let \( v \) and \( w \) be two common fixed points of \( f \) and \( g \).

From (3.56), we have

\[ G(fv, fw, fw) \geq G(gv, gw, gw) - \varphi(G(gv, gw, gw)), \text{ that is,} \]

\[ G(v, w, w) \geq G(v, w, w) - \varphi(G(v, w, w)). \]

From here, we get

\[ 0 \geq \varphi(G(v, w, w)) \geq 0, \text{ implies that, } \varphi(G(v, w, w)) = 0, \text{ and hence, } v = w. \]

Hence \( f \) and \( g \) have a unique common fixed point.

**Example 3.7.8.** Let \( X = [0, 1] \) and \( G(x, y, z) = \max\{ x - y, y - z, z - x \} \), for all \( x, y, z \) in \( X \).

Clearly \( (X, G) \) is a G-metric space.

Define \( f, g : X \to X \) by \( fx = \frac{1}{8}, \quad gx = \frac{4}{8} \).

So, \( gX = [0, \frac{1}{8}] \subseteq [0, \frac{1}{4}] = fX \).

Let \( \{x_n\} \) be a sequence in \( X \) such that \( x_n = \frac{1}{8} \) for each \( n \).

Also, let \( \varphi : [0, \infty) \to (-\infty, 0] \) be defined by \( \varphi(t) = -t^2 \) for all \( t \in [0, \infty) \).

Clearly, for \( x > y > z \), all the conditions of Theorems 3.7.6 and 3.7.7 are satisfied.