Common fixed point theorems for minimal commutative mappings in metric spaces

In this chapter, some common fixed theorems for $\varphi$-weakly expansive maps, weak contraction, generalized weak contraction along with compatible/weakly compatible maps, variants of R-weak commutative mappings and E.A./$(\text{CLR}_g)$ properties are proved. It consists of five sections. In first section, the concept of $\varphi$-weakly expansive maps in metric spaces is introduced. In Section 2.2, the result of Manro et al. [64] for $\varphi$-weakly expansive pair of mappings in metric spaces is generalized. In third section, some common fixed point theorems for $(\psi - \varphi )$-weakly contractive and generalized $(\psi - \varphi )$-weakly contractive pair of maps are proved, which generalizes the results of Moradi et al. [66]. In Section 2.4, some common fixed point theorems for $(\psi - \varphi )$-weakly contractive and generalized $(\psi - \varphi )$-weakly contractive pair of maps using E.A. property and $(\text{CLR}_g)$ property are proved. In last section, a theorem using variants of R-weak commutative mappings along with weakly compatible maps is proved.

Introduction

The notion of $\varphi$-weakly expansive mappings in metric spaces is introduced as follows:

**Definition 2.1.1.** Let $(X, d)$ be a metric space. A self map $f$ on $X$ is said to be expansive, if there exists a real number $a > 1$ such that

$$d(fx, fy) \geq a d(x, y).$$

Alber and Guerre-Delabriere [8] introduced the notion of weak contraction as follows:

**Definition 2.1.2.** Let $(X, d)$ be a metric space. A mapping $f : X \to X$ is said to be $\varphi$-weakly contraction, if there exists a map $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)), \text{ for all } x, y \in X.$$
Using these concepts, we introduce the notion of \( \varphi \)-weakly expansive mappings in metric spaces as follows:

**Definition 2.1.3.** Let \((X, d)\) be a metric space. A mapping \(f : X \to X\) is said to be a \(\varphi\)-weakly expansive, if there exists a map \(\varphi : [0, \infty) \to (-\infty, 0]\) with \(\varphi(0) = 0\) and \(\varphi(t) < 0\) for all \(t > 0\) such that

\[
d(fx, fy) \geq d(x, y) - \varphi(d(x, y)), \quad \text{for all } x, y \in X.
\]

**Definition 2.1.4.** Let \((X, d)\) be a metric space. A mapping \(f : X \to X\) is said to be a \(\varphi\)-weakly expansive with respect to \(g : X \to X\), if there exists a map \(\varphi : [0, \infty) \to (-\infty, 0]\) with \(\varphi(0) = 0\) and \(\varphi(t) < 0\) for all \(t > 0\) such that

\[
d(fx, fy) \geq d(gx, gy) - \varphi(d(gx, gy)), \quad \text{for all } x, y \in X.
\]

Now, we show the existence of a fixed point for \(\varphi\)-weakly expansive mappings in complete metric spaces in the form of following theorem:

**Theorem 2.1.5.** Let \((X, d)\) be a complete metric space. Let \(f\) be a self mapping on \(X\) satisfying the following:

- \(f\) be \(\varphi\)-weakly expansive, that is,
  \[
  d(fx, fy) \geq d(x, y) - \varphi(d(x, y)),
  \quad \text{for all } x, y \in X,
  \]
  where \(\varphi : [0, \infty) \to (-\infty, 0]\) is a continuous and non-decreasing map with \(\varphi(0) = 0\) and \(\varphi(t) < 0\) for all \(t > 0\),
- \(f\) is onto.

Then \(f\) has a unique fixed point.

**Proof.** Since \(f\) is an onto mapping, therefore for each \(x_0\) in \(X\), there exists \(x_1\) in \(X\) such that \(fx_1 = x_0\).

Continuing this process, define \(\{x_n\}\) by \(x_n = fx_{n+1}, n = 0, 1, 2, \ldots\)

Without loss of generality, suppose that \(x_{n-1} \neq x_n\) for all \(n \geq 1\).

From (2.1), we have

\[
d(x_n, x_{n-1}) = d(fx_{n+1}, fx_n)
\]

that, \(d(x_n, x_{n-1}) \geq d(x_{n+1}, x_n) - \varphi(d(x_{n+1}, x_n))\), implies

Hence the sequence \(\{d(x_{n+1}, x_n)\}\) is strictly decreasing and bounded below. Thus, there exists \(r \geq 0\) such that, \(d(x_{n+1}, x_n) \to r\) as \(n \to \infty\).

Letting \(n \to \infty\) in (2.3), we have

\[
r \geq r - \varphi(r), \quad \text{a contradiction, unless } r = 0.
\]
Hence

(2.4) \quad d(x_{n+1}, x_n) \to 0 \text{ as } n \to \infty.

Now, we show that \( \{x_n\} \) is a Cauchy sequence. If possible, let \( \{x_n\} \) is not a Cauchy sequence. Then there exists \( \varepsilon > 0 \) for which we can find subsequences \( \{x_{n(k)}\} \) and \( \{x_{m(k)}\} \) of \( \{x_n\} \) with \( n(k) > m(k) > k \) such that

(2.5) \quad d(x_{m(k)}, x_{n(k)}) \geq \varepsilon.

Further, corresponding to \( m(k) \), we can choose \( n(k) \) in such a way that it is the smallest integer with \( n(k) > m(k) \) and satisfying (2.5).

Then

(2.6) \quad d(x_{m(k)}, x_{n(k)-1}) < \varepsilon.

Then, we have

(2.7) \quad \varepsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})

\leq \varepsilon + d(x_{n(k)-1}, x_{n(k)}).

Letting \( k \to \infty \) and using (2.4), we have

(2.8) \quad \lim_{k \to \infty} d(x_{n(k)}, x_{n(k)}) = \varepsilon.

Again using triangular inequality,

\[ d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \]

\[ d(x_{n(k)-1}, x_{m(k)-1}) \leq d(x_{n(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)}, x_{m(k)-1}). \]

Letting \( k \to \infty \) in the above two inequalities and using (2.4), we get

(2.9) \quad \lim_{k \to \infty} d(x_{n(k)-1}, x_{n(k)-1}) = \varepsilon.

Putting \( x = x_{m(k)} \) and \( y = x_{n(k)} \) in (2.1), we get

\[ d(x_{m(k)-1}, x_{n(k)-1}) = d(fx_{m(k)}, fx_{n(k)}) \]

\[ \geq d(x_{m(k)}, x_{n(k)}) - \varphi(d(x_{m(k)}, x_{n(k)})). \]

Letting \( k \to \infty \) and using (2.8), (2.9), we get

\[ \varepsilon \geq \varepsilon - \varphi(\varepsilon), \text{ implies that, } \varphi(\varepsilon) \geq 0, \text{ a contradiction}. \]

Hence \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, therefore there exists \( q \) in \( X \) such that \( f_{x_{n+1}} = x_n \to q \), as \( n \to \infty \). Consequently, we can find a \( p \) in \( X \) such that \( fp = q \).

Now, we show that \( p = q \). Let, if possible, \( p \neq q \).

Putting \( x = p \) and \( y = x_{n+1} \) in (2.1), we get

\[ d(q, x_n) = d(fp, fx_{n+1}) \]

\[ \geq d(p, x_{n+1}) - \varphi(d(p, x_{n+1})). \]
Letting \( n \to \infty \), we get
\[
d(q, q) \geq d(p, q) - \varphi(d(p, q)),
\]
that is,
\[
\varphi(d(p, q)) \geq d(p, q) - \varphi(d(p, q)),
\]
Hence \( p = q \), that is, \( f(q) = q \).
Hence \( q \) is the fixed point of \( f \).
For the uniqueness, let \( z \) be another fixed point of \( f \) such that \( q \neq z \).
From (2.1), we have
\[
d(q, z) = d(f(q), f(z))
\]
\[
\geq d(q, z) - \varphi(d(q, z)),
\]
that is,
\[
\varphi(d(q, z)) \geq d(q, z) - \varphi(d(q, z)),
\]
Hence \( q = z \).
So, \( f \) has a unique fixed point.

**Example 2.1.6.** Let \( X = \mathbb{R} \) and let \( d(x, y) = \| x - y \| \), then \( (X, d) \) is a complete metric space.
Let \( f(x) = 2x, x \in X \) and define \( \varphi : [0, \infty) \to (-\infty, 0] \) by \( \varphi(t) = \frac{-t}{2} \).
\[
d(f(x), f(y)) = 2 \| x - y \|.
\]
\[
\varphi(d(x, y)) = \varphi(\| x - y \|) = \frac{-\| x - y \|}{2}.
\]
Therefore,
\[
d(x, y) - \varphi(d(x, y)) = \| x - y \| + \frac{-\| x - y \|}{2} = \frac{3\| x - y \|}{2}.
\]
Clearly,
\[
d(f(x), f(y)) \geq d(x, y) - \varphi(d(x, y)).
\]
Also \( f \) is onto and \( 0 \) is the unique fixed point of \( f \).

Jungck [46] introduced the notion of compatible maps as follows:

**Definition 2.1.7.** A pair of self-mappings \((f, g)\) of a metric space \((X, d)\) is said to be compatible if
\[
\lim_{n \to \infty} d(fg(x_n), gf(x_n)) = 0,
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such that
\[
\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = z
\]
for some \( z \) in \( X \).

Pant [77] introduced the notion of pointwise R-weak commutativity in metric spaces to extend the scope of the study of common fixed point theorems from the class of compatible to wider class of pointwise R-weakly commuting mappings as follows:

**Definition 2.1.8.** A pair of self-mappings \((f, g)\) of a metric space \((X, d)\) is said to be R-weakly commuting at a point \( x \) in \( X \) if
\[
\varphi(d(f(x_n), g(x_n))) \leq \varphi(d(x_n)) + \varphi(d(f(x_n), g(x_n)))
\]
for every \( \{x_n\} \) in \( X \) and some \( \varphi \) satisfying certain conditions.
\[ d(fgx, gfx) \leq R \ d(fx, gx) \] for some \( R > 0 \).

**Definition 2.1.9.** Two self-maps \( f \) and \( g \) of a metric space \( (X, d) \) are called pointwise R-weakly commuting on \( X \), if given \( x \) in \( X \), there exists \( R > 0 \) such that

\[ d(fgx, gfx) \leq R \ d(fx, gx). \]

Further, Pathak et al. [81] generalized the notion of R-weakly commuting mappings to R-weakly commuting mappings of type \( (A_f) \) and \( (A_g) \) as follows:

**Definition 2.1.10.** Two self-maps \( f \) and \( g \) of a metric space \( (X, d) \) are called R-weakly commuting of type \( (A_f) \) if there exists some \( R > 0 \) such that

\[ d(ffx, gfx) \leq R \ d(fx, gx) \quad \text{for all } x \text{ in } X. \]

Similarly, two self-mappings \( f \) and \( g \) of a metric space \( (X, d) \) are called R-weakly commuting of type \( (A_g) \) if there exists some \( R > 0 \) such that

\[ d(fgx, ggx) \leq R \ d(fx, gx) \quad \text{for all } x \text{ in } X. \]

Kumar et al. [60] introduced the notion of R-weakly commuting mappings of type \( (P) \) as follows:

**Definition 2.1.11.** Two self-maps \( f \) and \( g \) of a metric space \( (X, d) \) are called R-weakly commuting of type \( (P) \) if there exists some \( R > 0 \) such that

\[ d(ffx, ggx) \leq R \ d(fx, gx) \quad \text{for all } x \text{ in } X. \]

Pant [78] introduced a new notion of continuity, known as reciprocal continuity, as follows:

**Definition 2.1.12.** Two self-mappings \( f \) and \( g \) are called reciprocally continuous if

\[
\lim_{n \to \infty} fg(x_n) = fz \quad \text{and} \quad \lim_{n \to \infty} gf(x_n) = gz,
\]

whenever \( \{x_n\} \) is a sequence such that

\[
\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = z \quad \text{for some } z \text{ in } X.
\]

If \( f \) and \( g \) are both continuous, then they are obviously reciprocally continuous, but the converse need not be true.

Pant et al. [80] generalized the notion of reciprocal continuity to weak reciprocal continuity as follows:

**Definition 2.1.13.** Two self-mappings \( f \) and \( g \) are called weakly reciprocally continuous if

\[
\lim_{n \to \infty} fg(x_n) = fz \quad \text{or} \quad \lim_{n \to \infty} gf(x_n) = gz,
\]

whenever \( \{x_n\} \) is a sequence such that

\[
\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = z \quad \text{for some } z \text{ in } X.
\]
Fixed point theorems for \( \varphi \)-weakly expansive mappings in metric spaces

In this section, we prove some common fixed point theorems for \( \varphi \)-weakly expansive mappings in metric spaces.

Manro et al. [64] proved the following fixed point theorem on complete metric spaces:

**Theorem 2.2.1.** Let \( f \) and \( g \) be two weakly reciprocally continuous self mappings of a complete metric space \((X, d)\) satisfying the following:

\[
gX \subseteq fX,
\]
for any \( x, y \) in \( X \) and \( q > 1 \), we have that

\[
d(fx, fy) \geq q \ d(gx, gy).
\]

If \( f \) and \( g \) are either compatible or \( R \)-weakly commuting of type \( (A_g) \) or \( R \)-weakly commuting of type \( (A_f) \) or \( R \)-weakly commuting of type \( P \), then \( f \) and \( g \) have a unique common fixed point.

Now, we extend this result using the notion of \( \varphi \)-weakly expansive mapping.

**Theorem 2.2.2.** Let \( f \) and \( g \) be two weakly reciprocally continuous self mappings of a complete metric space \((X, d)\) satisfying the following:

\[
(2.10) \quad gX \subseteq fX,
\]

\[
(2.11) \quad d(fx, fy) \geq d(gx, gy) - \varphi(d(gx, gy)),
\]

for all \( x, y \) in \( X \) (i.e., it is \( \varphi \)-weakly expansive), where \( \varphi : [0, \infty) \to (-\infty, 0] \) is a non-increasing and continuous map with \( \varphi(0) = 0 \) and \( \varphi(t) < 0 \) for all \( t > 0 \). If \( f \) and \( g \) are compatible, then \( f \) and \( g \) have a unique common fixed point.

**Proof.** Let \( x_0 \in X \). Since \( gX \subseteq fX \), there exists a sequence of points \( \{x_n\} \) such that

\[
x_{n+1} = g_{n+1} = f_{n+1}.
\]

Define a sequence \( \{y_n\} \) in \( X \) by

\[
y_n = y_{n+1} = f_{n+1}.
\]

Moreover, we assume that, if \( y_n = y_{n+1} \) for some \( n \in \mathbb{N} \), then there is nothing to prove.

From (2.11), we have

\[
d(y_n, y_{n-1}) = d(f_{n+1}, f_n)
\]

\[
\geq d(g_{n+1}, g_n) - \varphi(d(g_{n+1}, g_n))
\]

\[
= d(y_{n+1}, y_n) - \varphi(d(y_{n+1}, y_n)), \text{ that is,}
\]

\[
d(y_n, y_{n-1}) > d(y_{n+1}, y_n).
\]
Hence the sequence \( \{d(y_{n+1}, y_n)\} \) is strictly decreasing and bounded below. Thus, there exists \( r \geq 0 \), such that \( \lim_{n \to \infty} r(\frac{1}{n+1}, \frac{1}{n}) = r \).

From (2.13), we deduce

\[
0 \geq \varphi(d(y_{n+1}, y_n)) \geq d(y_{n+1}, y_n) - d(y_n, y_{n-1}).
\]

Letting \( n \to \infty \), we get

\[
0 \geq \varphi(r) \geq r - r = 0,
\]

that is,

\[
\varphi(r) = 0, \text{ implies that, } r = 0.
\]

Therefore,

\[
(2.15) \quad \lim_{n \to \infty} r(\frac{1}{n+1}, \frac{1}{n}) = 0.
\]

Now, we will show that \( \{y_n\} \) is a Cauchy sequence. Let, if possible, \( \{y_n\} \) is not a Cauchy sequence. So, there exists an \( \varepsilon > 0 \) and the subsequences \( \{y_{m(k)}\} \) and \( \{y_{n(k)}\} \) of \( \{y_n\} \) such that \( n(k) \) is minimal in the sense that \( n(k) > m(k) > k \) and \( d(y_{m(k)}, y_{n(k)}) > \varepsilon \).

Therefore, \( d(y_{m(k)}, y_{n(k)-1}) \leq \varepsilon \).

By the triangular inequality, we have

\[
(2.16) \quad 0 \leq d(y_{m(k)}, y_{n(k)}) = d(y_{m(k)-1}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)}) \leq 2d(y_{m(k)}, y_{n(k)-1}) + \varepsilon + d(y_{n(k)-1}, y_{n(k)}),
\]

Letting \( k \to \infty \) in the above inequality and using (2.15), we get

\[
(2.17) \quad \lim_{k \to \infty} d(y_{m(k)}, y_{n(k)}) = \lim_{k \to \infty} d(y_{m(k)-1}, y_{n(k)-1}) = \varepsilon.
\]

From (2.11), we have

\[
d(y_{m(k)-1}, y_{n(k)-1}) = d(fy_{m(k)}, fy_{n(k)}) \\
\geq d(gy_{m(k)}, gy_{n(k)}) - \varphi(d(gy_{m(k)}, gy_{n(k)}))
\]

\[
= d(y_{m(k)}, y_{n(k)}) - \varphi(d(y_{m(k)}, y_{n(k)})).
\]

Letting \( k \to \infty \) and using (2.17), we get

\[
\varepsilon \geq \varepsilon - \varphi(\varepsilon), \text{ a contradiction.}
\]

Hence \( \{y_n\} \) is a Cauchy sequence in \( X \).

Since \( X \) is complete, there exists a point \( z \) in \( X \) such that \( \lim_{n \to \infty} y_n = z \).

Therefore, by (2.12), we have

\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} gx_n = \lim_{n \to \infty} fx_{n+1} = z.
\]

Suppose that \( f \) and \( g \) are compatible mappings. Now, by the reciprocal continuity of \( f \) and \( g \), we obtain \( \lim_{n \to \infty} fgy_n = fz \) or \( \lim_{n \to \infty} fgy_n = fz \).
\[ \lim_{n \to \infty} \mathbf{f}x_n = \mathbf{g}z. \]
Let $\lim_{n \to \infty} f_{g x_n} = f z$, then the compatibility of $f$ and $g$ gives $\lim_{n \to \infty} d(f_{g x_n}, g_{g x_n}) = 0$. Hence $\lim_{n \to \infty} g_{g x_n} = f z$.

Now, we claim that, $f z = g z$.

Let, if possible, $f z \neq g z$.

From (2.12), we get $\lim_{n \to \infty} g_{g x_n+1} = \lim_{n \to \infty} g_{g x_n} = f z$.

Therefore, from (2.11), we get

$$d(f z, f_{g x_n}) \geq d(g z, g_{g x_n}) - \phi(d(g z, g_{g x_n})).$$

Letting $n \to \infty$, we get

$$d(f z, f z) \geq d(g z, f z) - \phi(d(g z, f z)), \text{ that is,}$$

$$\phi(d(g z, f z)) \geq d(g z, f z) > 0,$$

a contradiction.

Hence $f z = g z$.

Again, the compatibility of $f$ and $g$ implies the commutativity at a coincidence point.

Hence $g f z = f g z = f f z = g g z$.

Using (2.11), we obtain

$$d(f z, f g z) \geq d(g z, g g z) - \phi(d(g z, g g z)), \text{ that is,}$$

$$d(g z, g g z) \geq d(g z, g g z) - \phi(d(g z, g g z)).$$

Thus, we get

$$0 \geq \phi(d(g z, g g z)) \geq 0,$$

that is,

$$\phi(d(g z, g g z)) = 0,$$

implies that, $g z = g g z$.

Also, we get $g z = g g z = f g z$, and so, $g z$ is the common fixed point of $f$ and $g$.

Next, suppose that $\lim_{n \to \infty} g_{f x_n} = g z$.

The assumption $g X \subseteq f X$, implies that, $g z = f u$, for some $u \in X$, and therefore, $\lim_{n \to \infty} g_{f x_n} = f u$.

The compatibility of $f$ and $g$ implies that $\lim_{n \to \infty} f_{g x_n} = f u$.

Now, we claim that $f u = g u$.

Let, if possible, $f u \neq g u$.

By virtue of (2.12), we have $\lim_{n \to \infty} g_{f x_n+1} = \lim_{n \to \infty} g_{g x_n} = f u$.

From (2.11), we have $\lim_{n \to \infty} g_{f x_n} = g u$. 

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\[ d(fu, fgx_n) \geq d(gu, ggx_n) - \varphi(d(gu, ggx_n)). \]
Letting \( n \to \infty \), we get

\[
d(fu, fu) \geq d(gu, fu) - \varphi(d(gu, fu)),
\]
that is,

\[
\varphi(d(gu, fu)) \geq d(gu, fu) > 0,
\]
a contradiction.

Hence \( fu = gu \).

Again, the compatibility of \( f \) and \( g \) implies the commutativity at a coincidence point.

Hence \( gfu = fgu = ffu = ggu \).

Finally, using (2.11), we obtain

\[
d(fu, fgu) \geq d(gu, ggu) - \varphi(d(gu, ggu)),
\]
that is,

\[
d(gu, ggu) \geq d(gu, ggu) - \varphi(d(gu, ggu)).
\]

Thus, we get

\[
0 \geq \varphi(d(gu, ggu)) \geq 0,
\]
that is,

\[
\varphi(d(gu, ggu)) = 0,
\]
implies that, \( gu = ggu \).

Also, we get \( gu = ggu = fgu \), and so, \( gu \) is the common fixed point of \( f \) and \( g \).

For the uniqueness, let \( v \) and \( w \) be two common fixed points of \( f \) and \( g \).

From (2.11), we have

\[
d(fv, fw) \geq d(gv, gw) - \varphi(d(gv, gw)),
\]
that is,

\[
d(v, w) \geq d(v, w) - \varphi(d(v, w)).
\]

From here, we get

\[
0 \geq \varphi(d(v, w)) \geq 0,
\]
implies that, \( \varphi(d(v, w)) = 0 \), and hence, \( v = w \).

Hence \( f \) and \( g \) have a unique common fixed point.

Next, we prove a common fixed point Theorem for variants of \( R \)-weak commutative mappings (\( R \)-weakly commuting of type \( (Ag) \), \( (Af) \) and \( (P) \)) as follows:

**Theorem 2.2.3.** Let \( f \) and \( g \) be two weakly reciprocally continuous self mappings of a complete metric space \( (X, d) \) satisfying (2.10), (2.11) and the following:

\[
(2.19) \quad f \text{ and } g \text{ are } R \text{-weakly commuting of type } (Ag) \text{ or } R \text{-weakly commuting of type } (Af) \text{ or } R \text{-weakly commuting of type } (P).
\]

Then \( f \) and \( g \) have a unique common fixed point.

**Proof.** From Theorem (2.2.1), \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there exists a point \( z \) in \( X \) such that \( \lim_{n \to \infty} y_n = z \).

Therefore, by (2.12), we have

\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} gx_n = \lim_{n \to \infty} fx_{n+1} = z.
\]
Now, suppose that \( f \) and \( g \) are \( R \)-weakly commuting of type \((A_f)\). The weak reciprocal continuity of \( f \) and \( g \) implies that \( \lim_{n \to \infty} fgx_n = fz \) or \( \lim_{n \to \infty} gfx_n = gz \).

Let us first assume that \( \lim_{n \to \infty} fgx_n = fz \).

Then the \( R \)-weak commutativity of type \((A_f)\) of \( f \) and \( g \) yields,
\[
d(ggx_n, fgx_n) \leq R \ d(fx_n, gx_n),
\]
and therefore
\[
\lim_{n \to \infty} d(ggx_n, fz) \leq R \ d(z, z) = 0, \text{ that is, } \lim_{n \to \infty} ggx_n = fz.
\]

Now, we claim that \( fz = gz \).

Let, if possible, \( fz \neq gz \).

Again, using (2.11), we get
\[
d(fz, fgx_n) \geq d(gz, ggx_n) - \varphi(d(gz, ggx_n)).
\]

Letting \( n \to \infty \), we get
\[
d(fz, fz) \geq d(gz, fz) - \varphi(d(gz, fz)), \text{ that is,}
\]
\[
\varphi(d(gz, fz)) \geq d(gz, fz) > 0, \text{ a contradiction, implies that, } fz = gz.
\]

Again, by using the \( R \)-weak commutativity of type \((A_f)\), we have
\[
d(ggz, fgz) \leq R \ d(gfz, fz) = R \ d(z, z) = 0, \text{ that is, } ggz = fgz.
\]

Therefore, \( ffz = fgz = gfz = ggz \).

Using (2.11), we have
\[
d(fz, fgz) \geq d(gz, ggz) - \varphi(d(gz, ggz)), \text{ that is,}
\]
\[
d(gz, ggz) \geq d(gz, ggz) - \varphi(d(gz, ggz)).
\]

Thus, we get
\[
0 \geq \varphi(d(gz, ggz)) \geq 0, \text{ that is,}
\]
\[
\varphi(d(gz, ggz)) = 0, \text{ implies that, } gz = ggz.
\]

Then, we also get, \( gz = ggz = fgz \), and so, \( gz \) is the common fixed point of \( f \) and \( g \).

Similar proof works in the case where \( \lim_{n \to \infty} gfx_n = gz \).

Suppose that \( f \) and \( g \) are \( R \)-weakly commuting of type \((A_g)\). Again, as done above, we can easily prove that \( fz \) is a common fixed point of \( f \) and \( g \).

Finally, suppose that \( f \) and \( g \) are \( R \)-weakly commuting of type \((P)\). The weak reciprocal continuity of \( f \) and \( g \) implies that \( \lim_{n \to \infty} fgx_n = fz \) or \( \lim_{n \to \infty} gfx_n = gz \).

Let us first assume that \( \lim_{n \to \infty} fgx_n = fz \). Then the \( R \)-weak commutativity of type \((P)\) of \( f \) and \( g \) yields
and therefore
\[ d(\text{ff}x_n, \text{gg}x_n) \leq R \ d(\text{fx}_n, \text{gx}_n), \]
and therefore
\[ \lim_{n \to \infty} d(\text{ff}x_n, \text{gg}x_n) \leq R \ d(z, z) = 0, \]
that is, \[ \lim_{n \to \infty} d(\text{ff}x_n, \text{gg}x_n) = 0. \]
Using (2.10) and (2.12), we have, \( f\text{gx}_{n+1} = \text{ff}x_n \to fz \) as \( n \to \infty \), which gives \( \text{gg}x_n \to fz \) as \( n \to \infty \).

Now, we claim that \( fz = gz \).

Let, if possible, \( fz \neq gz \).

From (2.11), we get
\[ d(fz, \text{fg}x_n) \geq d(gz, \text{gg}x_n) - \varphi(d(gz, \text{gg}x_n)). \]

Letting \( n \to \infty \), we get
\[ d(fz, fz) \geq d(gz, fz) - \varphi(d(gz, fz)), \]
that is,
\[ \varphi(d(gz, fz)) \geq d(gz, fz) > 0, \]
a contradiction.

Thus, we get \( gz = fz \).

Again, by using the R-weak commutativity of type (P), we have
\[ d(\text{ff}z, \text{gg}z) \leq R \ d(fz, gz) = 0, \]
that is, \( \text{ff}z = \text{gg}z \).

Using (2.11), we have
\[ d(fz, \text{fg}z) \geq d(gz, \text{gg}z) - \varphi(d(gz, \text{gg}z)), \]
that is,
\[ d(gz, \text{gg}z) \geq d(gz, \text{gg}z) - \varphi(d(gz, \text{gg}z)). \]

Thus, we get
\[ 0 \geq \varphi(d(gz, \text{gg}z)) \geq 0, \]
that is,
\[ \varphi(d(gz, \text{gg}z)) = 0, \]
implies that, \( gz = \text{gg}z \).

Then, we also get, \( gz = \text{gg}z = \text{fg}z \), and so, \( gz \) is the common fixed point of \( f \) and \( g \).

Similar proof works in the case where \( \lim_{n \to \infty} \text{gf}x_n = gz \).

For the uniqueness, let \( v \) and \( w \) be two common fixed points of \( f \) and \( g \).

From (2.11), we have
\[ d(fv, fw) \geq d(gv, gw) - \varphi(d(gv, gw)), \]
that is,
\[ d(v, w) \geq d(v, w) - \varphi(d(v, w)). \]

From here, we get
\[ 0 \geq \varphi(d(v, w)) \geq 0, \]
implies that, \( \varphi(d(v, w)) = 0, \) and hence, \( v = w. \)

Hence \( f \) and \( g \) have a unique common fixed point.
Example 2.2.4. Let $X = [0, 1]$ be equipped with the Euclidean metric $d(x, y) = |x - y|$, for all $x, y$ in $X$.

Define $f, g : X \to X$ by

$$f(x) = \begin{cases} \frac{3}{4} & \text{if } x = 0 \\ \frac{1}{4} & \text{if } x = 1 \\ x & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{8} & \text{if } x = 0 \\ \frac{7}{8} & \text{if } x = 1 \\ x & \text{otherwise} \end{cases}$$

So, $gX = [0, \frac{1}{4}] \subseteq [0, \frac{1}{4}] = fX$.

Let $\{x_n\}$ be a sequence in $X$ such that $x_n = \frac{1}{2^n}$ for each $n$.

Also, let $\varphi : [0, \infty) \to (-\infty, 0]$ be defined by $\varphi(t) = -\frac{t}{2}$ for all $t \in [0, \infty)$.

Here, $f(x_n) = f(\frac{1}{2^n}) = \frac{1}{4^n}$.

So, $\lim_{n \to \infty} f(x_n) = 0$.

Also $\lim_{n \to \infty} f(g(x_n)) = \lim_{n \to \infty} f(\frac{1}{8^n}) = \lim_{n \to \infty} \frac{1}{4^n} = 0 = f(0)$.

So, we can say that $f$ and $g$ are weakly reciprocally continuous.

Also, $d(fx, fy) = |\frac{1}{4^n} - \frac{1}{8^n} - |, d(gx, gy) = |\frac{1}{8^n} - \frac{1}{8^n} - |, \varphi(d(gx, gy)) = -|\frac{1}{4^n} - \frac{1}{8^n} - |$.

Clearly,

Again $d(fx, fy) \geq d(gx, gy) - \varphi(d(gx, gy))$.

$\varphi(d(gx, gy)) = d(g(\frac{1}{4^n}), f(\frac{1}{8^n})) = d(\frac{1}{4^n}, \frac{1}{8^n}) = \frac{1}{4^n}$.

and $d(fx_n, gx_n) = d(\frac{1}{4^n}, \frac{1}{8^n}) = \frac{1}{4^n}$.

Clearly $d(ggx_n, fgx_n) < R d(fx_n, gx_n)$, where $R > \frac{1}{8}$.

Hence $f$ and $g$ are $R$-weakly commuting maps of type $(A_I)$. So, all the conditions of Theorems 2.2.2 and 2.2.3 are satisfied.

Weakly compatible maps for weak and generalized weak contractions

In this section, we prove some common fixed point theorems for weak and generalized weak contractions.

Khan et al. [56] addressed a new category of fixed point problems with the help of a control function and called it altering distance function.
Definition 2.3.1. A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(i) $\psi(0) = 0$,
(ii) $\psi$ is continuous and monotonically non-decreasing.

Khan et al. [56] proved the following fixed point theorem using altering distance function as follows:

**Theorem 2.3.2.** Let $(X, d)$ be a complete metric space. Let $\psi$ be an altering distance function and $f : X \rightarrow X$ be a self-mapping which satisfies the following inequality:

$$\psi(d(fx, fy)) \leq c \psi(d(x, y))$$

for all $x, y \in X$ and for some $0 < c < 1$. Then $f$ has a unique fixed point.

Altering distance has been used in metric fixed point theory in a number of papers. Some of the works utilizing the concept of altering distance function are noted in [12, 73, 95, 96].

Chaudhary et al. ([26] and [27]) extend the notion of altering distance to two variables and three variables.

An interesting generalization of the contraction principle was suggested by Alber and Guerre-Delabriere [8] in complete metric spaces as follows:

**Definition 2.3.3.** A mapping $T : X \rightarrow X$, where $(X, d)$ is a metric space, is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)),$$

where $x, y \in X$ and $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

If one takes $\varphi(t) = kt$ where $0 < k < 1$, then (2.21) reduces to (2.1.1).

Weakly contractive mappings have been dealt with in a number of papers. Some of these works are noted in [12, 24, 90, 107].

Rhoades [90] proved the following Theorem:

**Theorem 2.3.4.** Let $T : X \rightarrow X$ be a weakly contractive mapping on a complete metric space $(X, d)$, then $T$ has a unique fixed point.

In fact, Alber and Guerre-Delabriere [8] assumed an additional condition on $\varphi$ which is $\lim_{t \to \infty} \varphi(t) = \infty$. But Rhoades [90] obtained the result noted in Theorem 2.3.4 without using this particular assumption.
It may be observed that though the function \( \varphi \) has been defined in the same way as the altering distance function, the way it has been used in Theorem 2.3.4 is completely different from the use of altering distance function.

Dutta et al. \cite{35} proved the following Theorem:

**Theorem 2.3.5.** Let \( (X, d) \) be a complete metric space and let \( T : X \to X \) be a self-mapping satisfying the inequality

\[
\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)),
\]

where \( \varphi, \psi : [0, \infty) \to [0, \infty) \) are both continuous and monotone non-decreasing functions with \( \varphi(0) = \psi(0) = 0 \) if and only if \( t = 0 \).

Then \( T \) has a unique fixed point.

Beg et al. \cite{16} generalized Theorem 2.3.5 in the following form:

**Theorem 2.3.6.** Let \( (X, d) \) be a metric space and let \( f \) be a weakly contractive mapping with respect to \( g \), that is,

\[
\psi(d(fx, fy)) \leq \psi(d(gx, gy)) - \varphi(d(gx, gy)),
\]

for all \( x, y \in X \), where \( \varphi, \psi : [0, \infty) \to [0, \infty) \) are two mappings with \( \varphi(0) = \psi(0) = 0 \), \( \psi \) is continuous non-decreasing and \( \varphi \) is lower semi-continuous.

If \( fX \subseteq gX \) and \( gX \) is a complete subspace of \( X \), then \( f \) and \( g \) have coincidence point in \( X \).

Moradi et al. \cite{66} proved the following Theorem:

**Theorem 2.3.7.** Let \( T \) be self mapping on a complete metric space \( (X, d) \) satisfying the following:

\[
\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)),
\]

for all \( x, y \in X \) (known as \( (\psi - \varphi) \)-weakly contractive), where \( \varphi, \psi : [0, \infty) \to [0, \infty) \) are two mappings with \( \varphi(0) = \psi(0) = 0 \), \( \psi \) is continuous non-decreasing and \( \varphi \) is lower semi-continuous.

Also suppose that either

(i) \( \psi \) is continuous and \( \lim_{t \to \infty} \varphi(t) = 0 \), if \( \lim_{t \to \infty} \varphi(t) = 0 \).

(ii) \( \psi \) is monotone non-decreasing and \( \lim_{t \to \infty} \varphi(t) = 0 \), if \( \{t_n\} \) is bounded and \( \lim_{t \to \infty} \varphi(t) = 0 \).

Then \( T \) has a unique fixed point.

Now, we prove our results relaxing the condition of completeness on metric space for a pair of weakly compatible mappings.

**Theorem 2.3.8.** Let \( f \) and \( g \) be self mappings on a metric space \( (X, d) \) satisfying the followings:

\[
\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)),
\]

for all \( x, y \in X \) (known as \( (\psi - \varphi) \)-weakly contractive), where \( \varphi, \psi : [0, \infty) \to [0, \infty) \) are two mappings with \( \varphi(0) = \psi(0) = 0 \), \( \psi \) is continuous non-decreasing and \( \varphi \) is lower semi-continuous.

## Footnotes

\footnote{\cite{35} Dutta et al.}

\footnote{\cite{16} Beg et al.}

\footnote{\cite{66} Moradi et al.}
(2.24) \( gX \subseteq fX \),
(2.25) \( gX \) or \( fX \) is complete,
(2.26) \( \psi(d(gx, gy)) \leq \psi(d(fx, fy)) - \varphi(d(fx, fy)) \),
for all \( x, y \in X \), \( (\psi \circ \varphi) \) weakly contractive, where \( \varphi, \psi : [0, \infty) \to [0, \infty) \) are two mappings with \( \varphi(0) = \psi(0) = 0 \), \( \varphi(t) > 0 \) and \( \psi(t) > 0 \) for all \( t > 0 \).

Suppose also that either
(a) \( \psi \) is continuous and \( \lim_{n \to \infty} \varphi(\frac{t}{n}) = 0 \), if \( \lim_{n \to \infty} \psi (\frac{t}{n}) = 0 \).

or
(b) \( \psi \) is monotone non-decreasing and \( \lim_{n \to \infty} \varphi(\frac{t}{n}) = 0 \), if \( \{t_n\} \) is bounded and
\[ \lim_{n \to \infty} \varphi(\frac{t}{n}) = 0. \]

Then \( f \) and \( g \) have a unique point of coincidence in \( X \).

Moreover, if \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point.

**Proof.** Let \( x_0 \in X \). From (2.24), one can construct sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) by
\[ y_n = fx_{n+1} = gx_n, \quad n = 0, 1, 2, \ldots \]

Moreover, we assume that if \( y_n = y_{n+1} \) for some \( n \in \mathbb{N} \), then there is nothing to prove. Now, we assume that \( y_n \neq y_{n+1} \) for all \( n \in \mathbb{N} \).

From (2.26), we have
\[ \psi(d(y_{n+1}, y_n)) = \psi(d(gx_{n+1}, gx_n)) \]
\[ \leq \psi(d(fx_{n+1}, fx_n)) - \varphi(d(fx_{n+1}, fx_n)) \]
\[ = \psi(d(y_n, y_{n-1})) - \varphi(d(y_n, y_{n-1})), \]

for all \( n \in \mathbb{N} \).

Hence, the sequence \( \{\psi(d(y_{n+1}, y_n))\} \) is monotone decreasing and bounded below.

From (2.27), we deduce that
\[ \lim_{n \to \infty} \psi(d(y_{n+1}, y_n)) = 0. \]

If (a) holds, then by hypothesis \( \lim_{n \to \infty} \varphi(d(y_n, y_{n-1})) = 0 \).

If (b) holds, then from (2.28), we have
\[ d(y_{n+1}, y_n) < d(y_n, y_{n-1}), \quad \text{for all } n \in \mathbb{N}. \]

Hence \( \{d(y_{n+1}, y_n)\} \) is monotonically decreasing and bounded below.
By hypothesis, \( \lim_{n \to \infty} d(y_n, y_{n-1}) = 0 \).

Therefore, in every case, we conclude that
\[
\lim_{n \to \infty} d(y_n, y_{n-1}) = 0.
\]

Now, we claim that \( \{y_n\} \) is a Cauchy sequence. Indeed, if it is false, then there exists \( \varepsilon > 0 \) and the subsequences \( \{y_{m(k)}\} \) and \( \{y_{n(k)}\} \) of \( \{y_n\} \) such that \( n(k) > m(k) > k \) and \( d(y_{m(k)}, y_{n(k)}) \leq \varepsilon \) and by using the triangular inequality, obtain

\[
\varepsilon < d(y_{m(k)}, y_{n(k)}) \leq d(y_{m(k)}, y_{n(k)-1}) + d(y_{m(k)-1}, y_{n(k)})
\]

\[
\leq d(y_{m(k)}, y_{n(k)}) + d(y_{m(k)-1}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)})
\]

(2.30)

Letting \( k \to \infty \) in the above inequality and using (2.29), we get

\[
\lim_{k \to \infty} d(y_{m(k)-1}, y_{n(k)-1}) = \varepsilon.
\]

For all \( k \in \mathbb{N} \), from (2.26), we have

\[
\phi(d(y_{m(k)}, y_{n(k)})) \leq \phi(d(y_{m(k)-1}, y_{n(k)-1})) - \phi(d(y_{m(k)-1}, y_{n(k)-1})).
\]

If (a) holds, then

\[
\lim_{k \to \infty} \phi(d(y_{m(k)-1}, y_{n(k)-1})) = \lim_{k \to \infty} \phi(d(y_{m(k)}, y_{n(k)})) = \phi(\varepsilon),
\]

(2.31)

Now, from (2.32), we conclude that

\[
\lim_{k \to \infty} \phi(d(y_{m(k)-1}, y_{n(k)-1})) = 0.
\]

By hypothesis \( \lim_{k \to \infty} d(y_{m(k)-1}, y_{n(k)-1}) = 0 \), a contradiction.

If (b) holds, then from (2.32), we have

\[
\varepsilon < d(y_{m(k)}, y_{n(k)}) < d(y_{m(k)-1}, y_{n(k)-1}), \text{ and so, } d(y_{m(k)}, y_{n(k)}) \to \varepsilon^- \text{ and } d(y_{m(k)-1}, y_{n(k)-1}) \to \varepsilon^+ \text{ as } k \to \infty.
\]

Hence

\[
\lim_{k \to \infty} \phi(d(y_{m(k)-1}, y_{n(k)-1})) = \lim_{k \to \infty} \phi(d(y_{m(k)}, y_{n(k)})) = \phi(\varepsilon^-),
\]

where \( \phi(\varepsilon^-) \) is the right limit of \( \phi \) at \( \varepsilon \).

Therefore, from (2.32), we get

\[
\lim_{k \to \infty} \phi(d(y_{m(k)-1}, y_{n(k)-1})) = 0.
\]

By hypothesis \( \lim_{k \to \infty} d(y_{m(k)-1}, y_{n(k)-1}) = 0 \), a contradiction.

Thus \( \{y_n\} \) is a Cauchy sequence.

Since \( fX \) is complete, so there exists a point \( z \in fX \) such that \( \lim_{n \to \infty} y_n = \lim_{n \to \infty} f_{x_{n+1}} = z \).

Now, we show that \( z \) is the common fixed point of \( f \) and \( g \).

Since \( z \in fX \), so there exists a point \( p \in X \) such that \( fp = z \).
If (a) holds, then from (2.26), for all \( n \in \mathbb{N} \), we have
\[
\psi(d(fp, gp)) = \lim_{n \to \infty} \psi(d(gp, gxn)) \\
\leq \lim_{n \to \infty} \psi(d(fp, fxn)) - \lim_{n \to \infty} \varphi(d(fp, fxn)) \\
(2.33)
\]
\[
\leq \lim_{n \to \infty} \psi(d(fp, fxn)) - \lim_{n \to \infty} \varnothing(d(fp, fxn)) \\
\leq \lim_{n \to \infty} \psi(d(fp, fzn)).
\]
Using condition (a) and \( \lim_{n \to \infty} y_n = z \), we get
\[
\psi(d(fp, gp)) \leq \psi(d(z, z)) = \psi(0) = 0,
\]
and so \( d(gp, fp) = 0 \) (note that \( \varphi \) and \( \psi \) are non-negative with \( \varphi(0) = \psi(0) = 0 \)), which implies that \( gp = fp = z \).

If (b) holds, then from (2.26), we have
\[
\psi(d(fp, gp)) = \lim_{n \to \infty} \psi(d(gp, gxn)) \\
\leq \lim_{n \to \infty} \psi(d(fp, fxn)) - \lim_{n \to \infty} \varphi(d(fp, fxn)) \\
(2.34)
\]
\[
\leq \lim_{n \to \infty} \psi(d(fp, fzn)).
\]
Using condition (b) and \( \lim_{n \to \infty} y_n = z \), we get
\[
\psi(d(fp, gp)) \leq \psi(d(z, z)) = \psi(0) = 0,
\]
and so \( d(gp, fp) = 0 \) (note that \( \varphi \) and \( \psi \) are non-negative with \( \varphi(0) = \psi(0) = 0 \)), which implies that \( gp = fp = z \).

Now, we show that \( z = fp = gp \) is a common fixed point of \( f \) and \( g \).

Since \( fp = gp \) and \( f, g \) are weakly compatible maps, we have \( fz = fgp = gfp = gz \).

We claim that \( fz = gz = z \).

Let, if possible, \( gz \neq z \).

If (a) holds, then from (2.26), we have
\[
\psi(d(gz, z)) = \psi(d(gz, gp)) \leq \psi(d(fz, fp)) - \varphi(d(fz, fp)) \\
= \psi(d(gz, z)) - \varphi(d(gz, z)) \\
< \psi(d(gz, z)), \text{a contradiction.}
\]

If (b) holds, then we have
\[
d(gz, z) < d(gz, z), \text{a contradiction.}
\]
Hence \( gz = z = fz \), so \( z \) is the common fixed point of \( f \) and \( g \).

For the uniqueness, let \( u \) be another common fixed point of \( f \) and \( g \), so that \( fu = gu = u \).

We claim that \( z = u \).
Let, if possible, \( z \neq u \).

If (a) holds, then from (2.26), we have

\[
\psi(d(z, u)) = \psi(d(gz, gu))
\]

\[
\leq \psi(d(fz, fu)) - \varphi(d(fz, fu))
\]
\[ = \psi(d(z, u)) - \varphi(d(z, u)) \]

If (b) holds, then we have \( \psi(d(z, u)) < \psi(d(z, u)) \), a contradiction.

Thus, we get \( z = u \). Hence \( z \) is the unique common fixed point of \( f \) and \( g \).

**Example 2.3.9.** Let \( X = [0, 1] \) be endowed with the Euclidean metric \( d(x, y) = \frac{1}{2} |x - y| \) for all \( x, y \in X \) and let \( g_x = \frac{1}{5} x \) and \( f_x = \frac{3}{5} x \) for each \( x \in X \). Then
\[
d(gx, gy) = \frac{1}{5} \frac{1}{2} |x - y| = \frac{1}{10} |x - y| \quad \text{and} \quad d(fx, fy) = \frac{3}{5} \frac{1}{2} |x - y| = \frac{3}{10} |x - y| .
\]

Let \( \psi(t) = 5t \) and \( \varphi(t) = t \). Then
\[
\psi(d(gx, gy)) = \psi(\frac{1}{5} \frac{1}{2} |x - y|) = 5 \frac{1}{10} |x - y| = \frac{1}{2} |x - y| .
\]
\[
\psi(d(fx, fy)) = \psi(\frac{3}{5} \frac{1}{2} |x - y|) = 5 \frac{3}{10} |x - y| = \frac{3}{2} |x - y| .
\]
\[
\varphi(d(fx, fy)) = \varphi(\frac{3}{5} \frac{1}{2} |x - y|) = \frac{3}{5} \frac{1}{2} |x - y| = \frac{3}{10} |x - y| .
\]

Now
\[
\psi(d(fx, fy)) - \varphi(d(fx, fy)) = (\frac{3}{2} - \frac{3}{10}) |x - y| = \frac{12}{10} |x - y| .
\]

So \( \psi(d(gx, gy)) < \psi(d(fx, fy)) - \varphi(d(fx, fy)) \).

From here, we conclude that \( f, g \) satisfies the relation (2.26).

Also \( gX = [0, \frac{1}{5}] \subseteq [0, \frac{3}{5}] = fX \), \( gX \) is complete and \( f, g \) are weakly compatible. Hence all the conditions of Theorem 2.3.8 are satisfied.

Here \( 0 \) is the unique common fixed point of \( f \) and \( g \).

**Moradi et al. [66]** proved the following Theorems:

**Theorem 2.3.10.** Let \( T \) be self mapping on a complete metric space \( (X, d) \) satisfying the following:
\[
\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)),
\]
for all \( x, y \in X \) (known as \( (\psi - \varphi) \) weakly contractive), where \( \psi, \varphi : [0, \infty) \to [0, \infty) \).

Also suppose that either

(i) \( \psi \) is continuous and \( \lim_{t \to \infty} \psi(t) = 0 \), \( \varphi(t) > 0 \) and \( \varphi(t) \to 0 \) for all \( t > 0 \).

or

(ii) \( \psi \) is monotone non-decreasing and \( \lim_{t \to \infty} \varphi(t) = 0 \), if \( \lim_{t \to \infty} \varphi(t) = 0 \).

Then
\[
\lim_{t \to \infty} \varphi(t) = 0 .
\]
Then T has a unique fixed point.

**Theorem 2.3.11.** Let T be a self mapping on a complete metric space \((X, d)\) satisfying the following:

\[
\psi(d(Tx, Ty)) \leq \psi(N(x, y)) - \varphi(N(x, y)),
\]

where

\[
N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx) + d(y, Ty)}{2}\},
\]

for all \(x, y \in X\) (generalized \((\psi - \varphi)\) weakly contractive), where \(\varphi : [0, \infty) \to [0, \infty)\) is a mapping \(\lim_{t \to \infty} \varphi(t) = 0\) and \(\lim_{t \to 0} \varphi(t) = 0\) if \(\{t_n\}\) is bounded and \(\varphi(0) = 0\) and \(\varphi(t) > 0\) for all \(t > 0\).

Also suppose that either

(iii) \(\psi\) is continuous
or

(iv) \(\psi\) is monotone non-decreasing and for all \(k > 0\), \(\varphi(k) \geq \psi(k) - \psi(k^-)\), where \(\psi(k^-)\) is the left limit of \(\psi\) at \(k\).

Then T has a unique common fixed point.

Now, we prove our results relaxing the condition of completeness on metric space for pair of weakly compatible mappings.

**Theorem 2.3.12.** Let \(f\) and \(g\) be self mappings of a metric space \((X, d)\) satisfying (2.24), (2.25) and the following:

\[
(2.35) \quad \psi(d(gx, gy)) \leq \psi(N(fx, fy)) - \varphi(N(fx, fy)),
\]

where

\[
N(fx, fy) = \max\{d(fx, fy), d(fx, gx), d(fy, gy), \frac{d(fx, gx) + d(fy, gy)}{2}\},
\]

for all \(x, y \in X\) (generalized \((\psi - \varphi)\) weakly contractive), where \(\varphi : [0, \infty) \to [0, \infty)\) is a mapping \(\lim_{t \to \infty} \varphi(t) = 0\) and \(\lim_{t \to 0} \varphi(t) = 0\) if \(\{t_n\}\) is bounded and \(\varphi(0) = 0\) and \(\varphi(t) > 0\) for all \(t > 0\).

Suppose also that either

(c) \(\psi\) is continuous
or

(d) \(\psi\) is monotone non-decreasing and for all \(k > 0\), \(\varphi(k) \geq \psi(k) - \psi(k^-)\), where \(\psi(k^-)\) is the left limit of \(\psi\) at \(k\).

Then \(f\) and \(g\) have a unique point of coincidence in \(X\).
Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

**Proof.** Let $x_0 \in X$. From (2.24), one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X by

$y_n = fx_{n+1} = gx_n$, $n = 0, 1, 2, \ldots$

Moreover, we assume that if $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then there is nothing to prove. From (2.35), we have $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$.

From (2.36),

$$\psi(d(y_{n+1}, y_n)) \leq \psi(N(y_n, y_{n-1})) - \varphi(N(y_n, y_{n-1})), \quad \text{where} \quad N(y_n, y_{n-1}) = \max \{ d(y_n, y_{n-1}), d(y_n, y_{n+1}), d(y_{n-1}, y_n), \frac{1}{2} + 1, \frac{1}{2} + 1 \}.$$  

(2.37)

If $d(y_n, y_{n-1}) < d(y_n, y_{n+1})$, then from (2.36) and $y_n \neq y_{n+1}$, we conclude that

$$\psi(d(y_{n+1}, y_n)) \leq \psi(d(y_n, y_{n+1})) - \varphi(d(y_n, y_{n+1})) < \psi(d(y_n, y_{n+1})),$$

a contradiction.

Therefore, $d(y_n, y_{n+1}) \leq d(y_n, y_{n-1})$.

Hence the sequence $\{(d(y_n, y_{n+1}))\}$ is monotonically decreasing and bounded below. From (2.36) and (2.37), we have

$$\lim_{n \to \infty} \psi(d(y_{n+1}, y_n)) = \psi(d(y_n, y_{n+1})) - \varphi(d(y_n, y_{n+1})).$$

Therefore, the sequence $\{(d(y_n, y_{n+1}))\}$ is monotonically decreasing and bounded below. Thus, there exists $r \geq 0$ such that $\lim d(y_{n+1}, y_n) = r$. From (2.39), we have

$$\lim_{n \to \infty} \varphi(d(y_{n+1}, y_n)) = 0,$$

implies that, $\lim d(y_{n+1}, y_n) = 0$.

Now, we claim that $\{y_n\}$ is a Cauchy sequence. Indeed, if it is false, then there exists $\varepsilon > 0$ and subsequences $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ such that $\sigma(k)$ is minimal in the sense that

$$\varepsilon < d(y_{m(k)}, y_{n(k)}) \leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)}),$$

(2.41)

Letting $k \to \infty$ in the above inequality and using (2.38), we get

$$\lim_{k \to \infty} (d(y_{m(k)}, y_{n(k)})) = \lim_{k \to \infty} (d(y_{m(k)-1}, y_{n(k)})) = \varphi(N(y_{m(k)-1}, y_{n(k)-1})).$$

From (2.35), for all $k \in \mathbb{N}$, we have

$$\psi(d(y_{m(k)}, y_{n(k)})) \leq \psi(N(y_{m(k)-1}, y_{n(k)-1})) - \varphi(N(y_{m(k)-1}, y_{n(k)-1})).$$

(2.43)
where

\[
N(y_{m(k)-1}, y_{n(k)-1}) = \max \{ d(y_{m(k)-1}, y_{n(k)-1}), d(y_{m(k)-1}, y_{n(k)}), d(y_{n(k)-1}, y_{n(k)}), d(y_{m(k)-1}, y_{n(k)}), d(y_{n(k)-1}, y_{n(k)-1}) + d(y_{m(k)-1}, y_{n(k)}), d(y_{n(k)-1}, y_{n(k)-1}) + d(y_{m(k)-1}, y_{n(k)}) \}.
\]

If (2.42) and (2.44) holds, then we conclude that

\[
\lim_{k \to \infty} \psi(N(y_{m(k)-1}, y_{n(k)-1})) = \varepsilon.
\]

If \( \psi \) is continuous, then

\[
\lim_{k \to \infty} \psi(d(y_{m(k)}, y_{n(k)})) = \lim_{k \to \infty} \psi(N(y_{m(k)-1}, y_{n(k)-1})) = \psi(\varepsilon).
\]

From (2.43), we conclude that

\[
\lim_{k \to \infty} \varphi(N(y_{m(k)-1}, y_{n(k)-1})) = 0.
\]

Since \( N(y_{m(k)-1}, y_{n(k)-1}) \) is bounded, we conclude that \( \lim_{k \to \infty} N(y_{m(k)-1}, y_{n(k)-1}) = 0 \), a contradiction.

If \( \psi \) is monotone non-decreasing, then from (2.43), we have

\[
\varepsilon \leq d(y_{m(k)}, y_{n(k)}) \leq N(y_{m(k)-1}, y_{n(k)-1}) \quad \text{for all } k \in \mathbb{N}, \quad \text{and so } d(y_{m(k)}, y_{n(k)}) \to \varepsilon^+.
\]

Hence \( \lim_{k \to \infty} \psi(d(y_{m(k)}, y_{n(k)})) = \lim_{k \to \infty} \psi(N(y_{m(k)-1}, y_{n(k)-1})) = \psi(\varepsilon^+) \), where \( \psi(\varepsilon^+) \) is the right limit of \( \psi \) at \( \varepsilon \).

Therefore, from (2.43), we have

\[
\lim_{k \to \infty} \varphi(N(y_{m(k)-1}, y_{n(k)-1})) = 0.
\]

Since \( \{N(y_{m(k)-1}, y_{n(k)-1})\} \) is bounded, we conclude that \( \lim_{k \to \infty} N(y_{m(k)-1}, y_{n(k)-1}) = 0 \), a contradiction.

Thus \( \{y_n\} \) is a Cauchy sequence.

Since \( fX \) is complete, so there exists a point \( z \in fX \) such that \( \lim y_n = \lim f_{n+1} = z \).

Now, we show that \( z \) is the common fixed point of \( f \) and \( g \). Since \( z \in fX \), so there exists a point \( p \in X \) such that \( fp = z \).

We claim that \( fp = gp \). Let, if possible, \( fp \neq gp \).

From (2.35), we have

\[
\psi(d(gp, fp)) = \lim_{n \to \infty} \psi(d(gp, g_{n+1}))
\]

\[
\leq \lim_{n \to \infty} \psi(N(fp, f_n)) - \lim_{n \to \infty} \varphi(N(fp, f_n)),
\]

\[
= \psi(d(fp, gp)) - \varphi(d(fp, gp)), \quad \text{since}
\]

\[
41
\]

\[
\]
N(fp, fxn) = \max\{d(fp, fxn), d(fp, gp), d(fxn, gxn), \frac{\frac{d(fp, gp)}{2}}{2}\}.

Letting limit as \( n \to \infty \), we have
\[
\lim_{n \to \infty} N(fp, fxn) = \max\{d(fp, z), d(fp, gp), d(z, z), \frac{\frac{d(fp, gp)}{2}}{2}\} = \max\{0, d(fp, gp), 0, \frac{d(fp, gp)}{2}\} = d(fp, gp).
\]

If (c) holds, then we have
\[
\psi(d(gp, fp)) < \psi(d(gp, gp)) , \text{ a contradiction.}
\]
If (d) holds, then we have
\[
d(gp, fp) < d(gp, fp) , \text{ a contradiction.}
\]
Hence \( fp = gp = z \).

Now we show that \( z = fp = gp \) is a common fixed point of \( f \) and \( g \).

Since \( fp = gp \) and \( f, g \) are weakly compatible maps, we have \( fz = fgp = gfp = gz \).

We claim that \( fz = gz = z \).
Let, if possible \( gz \neq z \).
From (2.35), we have
\[
\psi(d(gz, z)) = \psi(d(gz, gp))
\]
\[
\leq \psi(N(fz, fp)) - \varphi(N(fp, gp))
\]
\[
= \psi(d(gz, z)) - \varphi(d(gz, z)), \text{ since}
N(fz, fp) = \max\{d(fz, fp), d(fz, gz), d(fp, gp), \frac{\frac{d(fz, gz)}{2}}{2}\}
= \max\{d(gz, z), d(gz, gz), d(gp, gp), \frac{\frac{d(gz, gz)}{2}}{2}\} = d(gz, z).
\]
If (c) holds, then we have
\[
\psi(d(gz, z)) < \psi(d(gz, z)) , \text{ a contradiction.}
\]
If (d) holds, then we have
\[
d(gz, z) < d(gz, z) , \text{ a contradiction.}
\]
Hence \( gz = z = fz \), so \( z \) is the common fixed point of \( f \) and \( g \).

For the uniqueness, let \( u \) be another common fixed point of \( f \) and \( g \), so that \( fu = gu = u \).
We claim that \( z = u \).
Let, if possible, \( z \neq u \).
From (2.35), we have
\[
\psi(d(z, u)) = \psi(d(gz, gu))
\]
\[ \leq \psi(N(fz, fu)) - \phi(N(fz, fu)) \]
= \psi(d(fz, fu)) - \varphi(d(fz, fu))

If (c) holds, then we have $\psi(d(z, u)) \leq \varphi(d(z, u))$. If (d) holds, then we have $d(z, u) < d(z, u)$, a contradiction.

Thus, we get $z = u$. Hence $z$ is the unique common fixed point of $f$ and $g$.

**Corollary 2.3.13.** Let $f$ and $g$ be self mappings on a metric space $(X, d)$ satisfying (2.24), (2.25) and the following:

\begin{equation}
\psi(d(gx, gy)) \leq \psi(d(fx, fy)) - \varphi(d(fx, fy)),
\end{equation}

for all $x, y \in X$, $(\psi \leq \varphi)$ weakly contractive, where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are two mappings with $\psi(0) = \varphi(0) = 0$, $\varphi(t) > 0$ and $\psi(t) > 0$ for all $t > 0$.

1. $\psi$ is continuous and $\lim_{t \to \infty} \psi(t) = 0$, if $\lim_{t \to \infty} \varphi(t) = 0$.
2. $\psi$ is monotone non-decreasing and $\lim_{t \to \infty} \varphi(t) = 0$, if $\{t_n\}$ is bounded and $\lim_{t \to \infty} \varphi(t) = 0$.

Then $f$ and $g$ have a unique point of coincidence in $X$.

Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.

**Proof.** By putting $N(fx, fy) = d(fx, fy)$ in Theorem 2.3.8, we get the result.

**Example 2.3.14.** Let $X = [0, 1]$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$ and let $gx = \frac{1}{5}x$ and $fx = \frac{3}{5}x$ for each $x \in X$. Then

$d(gx, gy) = \frac{1}{5}|x - y|$ and $d(fx, fy) = \frac{3}{5}|x - y|$.

Let $\psi(t) = 5t$ and $\varphi(t) = t$. Then

$\psi(d(gx, gy)) = \psi(\frac{1}{5}|x - y|) = 5^1 \frac{1}{5}|x - y| = \frac{5}{5}|x - y|.$

$\psi(d(fx, fy)) = \psi(\frac{3}{5}|x - y|) = 5^3 \frac{3}{5}|x - y| = 3 \frac{3}{5}|x - y|.$

$\varphi(d(fx, fy)) = \varphi(\frac{5}{5}|x - y|) = \frac{5}{5}|x - y|.$

Now

$\psi(d(fx, fy)) - \varphi(d(fx, fy)) = (3 - \frac{3}{5}) \frac{3}{5}|x - y| = 12 \frac{3}{5}|x - y|.$

So $\psi(d(gx, gy)) < \psi(d(fx, fy)) - \varphi(d(fx, fy)).$
Now, we conclude that \( f, g \) satisfy (2.45).

Also \( gX = [0, \frac{1}{3}] \subseteq [0, \frac{3}{5}] = fX \). \( gX \) is complete and \( f, g \) are weakly compatible. Hence all the conditions of Corollary 2.3.13 are satisfied. Here 0 is the unique common fixed point of \( f \) and \( g \).

**E.A. and \((\text{CLR}_f)\) properties for weakly contractive maps**

In this section, we prove some common fixed point theorems for weakly contractive maps using E.A. and \((\text{CLR}_f)\) properties:

**Theorem 2.4.1.** Let \((X, d)\) be a metric space and let \( f \) and \( g \) be weakly compatible self-maps of \( X \) satisfying (2.26), (a), (b) and the followings:

1. \((2.46)\) \( f \) and \( g \) satisfy the E.A. property,
2. \((2.47)\) \( fX \) is closed subset of \( X \).

Then \( f \) and \( g \) have a unique common fixed point.

**Proof.** Since \( f \) and \( g \) satisfy the E.A. property, there exists a sequence \( \{x_n\} \) in \( X \) such that 

\[
\lim_{n \to \infty} x_n = x_0 \text{ for some } x_0 \in X.
\]

Then \( fX \) is closed subset of \( X \), therefore, for \( z \in X \), we have

\[
\lim_{n \to \infty} f^{n}z = f^{n}z = x_0 \text{ for some } x_0 \in X.
\]

Now, \( fX \) is closed subset of \( X \), therefore, for \( z \in X \), we have

\[
\lim_{n \to \infty} f^{n}z = f^{n}z = x_0.
\]

We claim that \( fz = gz \).

From (2.26), we have

\[
\psi(d(gx_n, gz)) \leq \psi(d(fx_n, fz)) - \varphi(d(fx_n, fz)).
\]

Letting \( n \to \infty \), we have

\[
\psi(d(fz, gz)) \leq \lim_{n \to \infty} \psi(d(fx_n, fz)) - \lim_{n \to \infty} \varphi(d(fx_n, fz))
\]

\[
= \psi(d(fz, fz)) - \varphi(d(fz, fz))
\]

If (a) holds, then

\[
= \psi(0) - \varphi(0).
\]

If (b) holds, then

\[
\psi(d(fz, gz)) \leq 0, \text{ implies that, } d(fz, gz) = 0, \text{ that is, } fz = gz.
\]

Therefore, \( fz = gz \).

Now, we show that \( gz \) is the common fixed point of \( f \) and \( g \).

Suppose that \( gz \neq ggz \).

Since \( f \) and \( g \) are weakly compatible \( gfz = fgz \) and therefore \( ffz = ggz \).
From (2.26), we have
\[
\psi(d(gz, ggz)) \leq \psi(d(fz, fgz)) - \varphi(d(fz, fgz))
\]
\[
= \psi(d(gz, gfz)) - \varphi(d(gz, gfz))
\]
\[
= \psi(d(gz, ggz)) - \varphi(d(gz, ggz)).
\]
If (a) holds, then
\[
\psi(d(gz, ggz)) < \psi(d(gz, ggz)),
\]
a contradiction.

If (b) holds, then
\[
d(gz, ggz) < d(gz, ggz),
\]
a contradiction.

Hence \(ggz = gz\). Hence \(gz\) is the common fixed point of \(f\) and \(g\).

Finally, we show that the fixed point is unique.

Let \(u\) and \(v\) be two common fixed points of \(f\) and \(g\) such that \(u \neq v\).

From (2.26), we have
\[
\psi(d(u, v)) = \psi(d(fu, fv))
\]
\[
\leq \psi(d(fu, fv)) - \varphi(d(fu, fv))
\]
\[
= \psi(d(u, v)) - \varphi(d(u, v)).
\]
If (a) holds, then we have
\[
\psi(d(u, v)) < \psi(d(u, v)),
\]
a contradiction.

If (b) holds, then we have
\[
d(u, v) < d(u, v),
\]
a contradiction.

Therefore, \(u = v\), which proves the uniqueness.

**Theorem 2.4.2.** Let \((X, d)\) be a metric space and let \(f\) and \(g\) be weakly compatible self-mappings of \(X\) satisfying (2.26), (a), (b) and the following:

(2.48) \(f\) and \(g\) satisfy (CLRf) property.

Then \(f\) and \(g\) have a unique common fixed point.

**Proof.** Since \(f\) and \(g\) satisfy the (CLRf) property, there exists a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} \frac{x_n + x_n}{2} = \lim_{n \to \infty} \frac{x_n + x_n}{2} = f(x)\) for some \(x \in X\).

Now, we claim that \(fx = gx\).

From (2.26), we have
\[
\psi(d(gx_n, gx)) \leq \psi(d(fx_n, fx)) - \varphi(d(fx_n, fx)) \quad \text{for all} \ n \in \mathbb{N}.
\]
Letting \(n \to \infty\), we have
\[
\psi(d(fx, gx)) \leq \lim_{n \to \infty} \psi(d(fx_n, fx)) - \lim_{n \to \infty} \varphi(d(fx_n, fx))
\]
\[
= \psi(d(fx, fx)) - \varphi(d(fx, fx))
\]
If (a) holds, then we have \( = \psi(0) - \varphi(0) \).

\[ \psi(d(fx, gx)) \leq 0, \] implies that, \( d(fx, gx) = 0 \), that is, \( gx = fx \).

If (b) holds, then we have \( d(fx, gx) \leq 0 \), that is, \( gx = fx \).

Let \( w = fx = gx \).

Since \( f \) and \( g \) are weakly compatible \( gfx = fgx \), implies that, \( fw = fx = gw \).

Now, we claim that \( gw = w \).

Let, if possible, \( gw \neq w \).

If (a) holds, then from (2.26), we have

\[ \psi(d(gw, w)) = \psi(d(gw, gx)) \leq \psi(d(fw, fx)) - \varphi(d(fw, fx)) \]

\[ < \psi(d(fw, fx)) = \psi(d(gw, w)), \] a contradiction.

If (b) holds, then we have

\[ d(gw, w) < d(gw, w), \] a contradiction.

Thus, we get \( gw = w = fw \).

Hence \( w \) is the common fixed point of \( f \) and \( g \).

For the uniqueness, let \( u \) be another common fixed point of \( f \) and \( g \) such that \( fu = u = gu \).

Now, we claim that \( w = u \).

Let, if possible, \( w \neq u \).

If (a) holds, then from (2.26), we have

\[ \psi(d(w, u)) = \psi(d(gw, gu)) \leq \psi(d(fw, fu)) - \varphi(d(fw, fu)) \]

\[ = \psi(d(w, u)) - \varphi(d(u, u)) < \psi(d(w, u)), \] a contradiction.

If (b) holds, then we have

\[ d(w, u) < d(w, u), \] a contradiction.

Thus, we get, \( w = u \). Hence \( w \) is the unique common fixed point of \( f \) and \( g \).

**Example 2.4.3.** Let \( X = [0, 1] \) be endowed with the Euclidean metric \( d(x, y) = |x - y| \) and let \( gx = \frac{1}{x} \) and \( fx = \frac{3}{x} \) for each \( x \in X \). Then

\[ d(gx, gy) = \frac{1}{5} \left| \frac{1}{x} - \frac{1}{y} \right| = \frac{5}{5} \left| x - y \right| \]

\[ d(fx, fy) = \frac{3}{5} \left| x - y \right| \].

Let \( \psi(t) = 5t \) and \( \varphi(t) = t \). Then

\[ \psi(d(gx, gy)) = \psi\left(\frac{1}{5} \left| \frac{1}{x} - \frac{1}{y} \right| \right) = 5 \left| \frac{1}{x} - \frac{1}{y} \right| = \frac{5}{5} \left| x - y \right|. \]
\( \psi(d(fx, fy)) = \psi(3 - \frac{3}{5}) = 3^2 - \frac{3}{5} = 3 - \frac{3}{5} \).

\( \varphi(d(fx, fy)) = \varphi(3 - \frac{3}{5}) = 3^2 - \frac{3}{5} \).

Now

\[ \psi(d(fx, fy)) - \varphi(d(fx, fy)) = \left(3 - \frac{3}{5}\right) \frac{3}{5} - \frac{3}{5} = \frac{12}{5} \frac{3}{5} - \frac{3}{5}. \]

So \( \psi(d(gx, gy)) < \psi(d(fx, fy)) - \varphi(d(fx, fy)) \).

Now, we conclude that \( f, g \) satisfy (2.26).

Consider the sequence \( \{x_n\} = \{1\} \) so that \( \lim_{n \to \infty} \frac{d}{\infty} = \lim_{n \to \infty} \frac{d}{\infty} = 0 = f(0) \), hence the pair \( f, g \) satisfy the (CLRf) property. Also \( f \) and \( g \) are weakly compatible. From here, we also deduce that \( \lim_{n \to \infty} \frac{d}{\infty} = \lim_{n \to \infty} \frac{d}{\infty} = 0 \), where \( 0 \in X \), implies that \( f \) and \( g \) satisfy E.A. property. Hence all the conditions of Theorem 2.4.1 and 2.4.2 are satisfied. Here \( 0 \) is the unique common fixed point of \( f \) and \( g \).

Now, we prove common fixed point theorems using E.A. and (CLRf) properties for generalized weakly contractive maps.

**Theorem 2.4.4.** Let \((X, d)\) be a metric space and let \( f \) and \( g \) be weakly compatible self-maps of \( X \) satisfying (2.35), (2.46), (2.47), (c) and (d).

Then \( f \) and \( g \) have a unique common fixed point.

**Proof.** Since \( f \) and \( g \) satisfy the E.A. property, there exists a sequence \( \{x_n\} \) in \( X \) such that

\[ \lim_{n \to \infty} \frac{d}{\infty} = \lim_{n \to \infty} \frac{d}{\infty} = x_0 \text{ for some } x_0 \in X. \]

Now, \( fX \) is closed subset of \( X \), therefore, for \( z \in X \), we have \( \lim_{n \to \infty} \frac{d}{\infty} = \frac{d}{\infty} = fz \).

We claim that \( fz = gz \). Suppose that \( fz \neq gz \).

From (2.35), we have

\[ \psi(d(gx_n, gz)) \leq \psi(N(fx_n, fz)) - \varphi(N(fx_n, fz)). \]

Letting \( n \to \infty \), we have

\[ \psi(d(fz, gz)) \leq \lim_{n \to \infty} \psi(N(fx_n, fz)) - \lim_{n \to \infty} \varphi(N(fx_n, fz)) \]

\[ = \psi(d(fz, gz)) - \varphi(d(fz, gz)), \text{ since} \]

\[ \lim_{n \to \infty} N(fx_n, fz) = d(fz, gz). \]

If (c) holds, then we have

\[ \psi(d(fz, gz)) < \psi(d(fz, gz)), \text{ a contradiction.} \]

If (d) holds, then we have

\[ d(fz, gz) < d(fz, gz), \text{ a contradiction.} \]

Therefore, \( fz = gz \).
Now, we show that $gz$ is the common fixed point of $f$ and $g$.

Suppose that $gz \neq ggz$.

Since $f$ and $g$ are weakly compatible $gfz = fgz$ and therefore $ffz = ggz$.

From (2.35), we have

$$\psi(d(gz, ggz)) \leq \psi(N(fz, fgz)) - \varphi(N(fz, fgz)) = \psi(N(gz, gfz)) - \varphi(N(gz, gfz)) = \psi(N(gz, ggz)) - \varphi(N(gz, ggz))$$

$$= \psi(d(gz, ggz)) - \varphi(d(gz, ggz)),$$

since $N(gz, ggz) = d(gz, ggz)$.

If (c) holds, then we have

$$\psi(d(gz, ggz)) < \psi(d(gz, ggz)),$$ a contradiction.

If (d) holds, then we have

$$d(gz, ggz) < d(gz, ggz),$$ a contradiction.

Hence $ggz = gz$. Hence $gz$ is the common fixed point of $f$ and $g$.

Finally, we show that the fixed point is unique.
Let $u$ and $v$ be two common fixed points of $f$ and $g$ such that $u \neq v$.

Now

$$\psi(d(u, v)) = \psi(d(gu, gv))$$

$$\leq \psi(N(fu, fv)) - \varphi(N(fu, fv))$$

$$= \psi(N(u, v)) - \varphi(N(u, v))$$

$$= \psi(d(u, v)) - \varphi(d(u, v)),$$ since $N(u, v) = d(u, v)$.

If (c) holds, then we have

$$\psi(d(u, v)) < \psi(d(u, v)),$$ a contradiction.

If (d) holds, then we have

$$d(u, v) < d(u, v),$$ a contradiction.

Therefore, $u = v$, which proves the uniqueness.

**Corollary 2.4.5.** Let $(X, d)$ be a metric space and let $f$ and $g$ be weakly compatible self-maps of $X$ satisfying (2.45), (2.46), (2.47), (c) and (d).

Then $f$ and $g$ have a unique common fixed point.

**Proof.** By putting $N(fx, fy) = d(fx, fy)$ in Theorem 2.4.4, we get the result.

**Theorem 2.4.6.** Let $(X, d)$ be a metric space and let $f$ and $g$ be weakly compatible self-mappings of $X$ satisfying (2.35), (2.48), (c) and (d).

Then $f$ and $g$ have a unique common fixed point.
Proof. Since $f$ and $g$ satisfy the $(CLR_f)$ property, there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} \psi(d(fx_n, gx_n)) = \lim_{n \to \infty} \psi(N(fx_n, fx)) = fx$ for some $x \in X$.

Now, we claim that $fx = gx$.

From (2.35), we have

$$\psi(d(gx_n, gx)) \leq \psi(N(fx_n, fx)) - \phi(N(fx_n, fx))$$

Letting $n \to \infty$, we have

$$\psi(d(fx, gx)) \leq \lim_{n \to \infty} \psi(N(fx_n, fx)) - \lim_{n \to \infty} \phi(N(fx_n, fx))$$

If (c) holds, then we have

$$\psi(0) - \phi(0), \text{ since } \lim N(fx_n, fx) = d(fx, fx) = 0.$$

If (d) holds, then we have

$$\psi(d(fx, gx)) \leq 0, \text{ implies that } d(fx, gx) = 0, \text{ that is, } fx = gx.$$

If (d) holds, then we have

$$d(fx, gx) \leq 0, \text{ that is, } fx = gx.$$

Thus, we get, $gx = fx$. Let $w = fx = gx$.

Since $f$ and $g$ are weakly compatible $gfx = fgx$, implies that, $fw = fgx = gfx = gw$.

Now, we claim that $gw = w$.

Let, if possible, $gw \neq w$.

From (2.35), we have

$$\psi(d(gw, w)) = \psi(d(gw, gx))$$

$$\leq \psi(N(fw, fx)) - \phi(N(fw, fx))$$

$$= \psi(d(fw, fx)) - \phi(d(fw, fx))$$

If (c) holds, then we have

$$\psi(d(gw, w)) < \psi(d(gw, w)), \text{ a contradiction.}$$

If (d) holds, then we have

$$d(gw, w) < d(gw, w), \text{ a contradiction.}$$

Thus, we get $gw = w = fw$.

Hence $w$ is the common fixed point of $f$ and $g$.

For the uniqueness, let $u$ be another common fixed point of $f$ and $g$ such that $fu = u = gu$.

We claim that $w = u$. Let, if possible, $w \neq u$.

From (2.35), we have

$$\psi(d(w, u)) = \psi(d(gw, gu))$$
\[ \leq \psi(N(fw, fu)) - \varphi(N(fw, fu)) = \psi(d(fw, fu)) - \varphi(d(fw, fu)) \]

If (c) holds, then we have \( \psi(d(w, u)) - \varphi(d(w, u)) \), since \( N(fw, fu) = d(fw, fu) \).

\[ \psi(d(w, u)) < \psi(d(w, u)), \text{ a contradiction.} \]

If (d) holds, then we have \( d(w, u) < d(w, u) \), a contradiction.

Thus, we get \( w = u \).

Hence \( w \) is the unique common fixed point of \( f \) and \( g \).

**Corollary 2.4.7.** Let \( (X, d) \) be a metric space and let \( f \) and \( g \) be weakly compatible self-maps of \( X \) satisfying (2.45), (2.48), (c) and (d).

Then \( f \) and \( g \) have a unique common fixed point.

**Proof.** By putting \( N(fx, fy) = d(fx, fy) \) in Theorem 2.4.6, we get the result.

**Example 2.4.8.** Let \( X = [0, 1] \) be endowed with the Euclidean metric \( d(x, y) = |x - y| \) and let \( g_x = \frac{1}{5} x \) and \( f_x = \frac{3}{5} x \) for each \( x \in X \). Then

\[ d(gx, gy) = \frac{1}{5} |x - y| \quad \text{and} \quad d(fx, fy) = \frac{3}{5} |x - y|. \]

Let \( \psi(t) = 5t \) and \( \varphi(t) = t \). Then

\[ \psi(d(gx, gy)) = \psi(\frac{1}{5} |x - y|) = \frac{5}{1} |x - y| = |x - y|. \]

\[ \psi(d(fx, fy)) = \psi(\frac{3}{5} |x - y|) = \frac{5}{3} |x - y| = \frac{3}{5} |x - y|. \]

\[ \varphi(d(fx, fy)) = \varphi(\frac{3}{5} |x - y|) = \frac{3}{5} |x - y|. \]

Now

\[ \psi(d(fx, fy)) - \varphi(d(fx, fy)) = (3 - \frac{3}{5}) |x - y| = \frac{12}{5} |x - y|. \]

So \( \psi(d(gx, gy)) < \psi(d(fx, fy)) - \varphi(d(fx, fy)) \).

From here, we conclude that \( f, g \) satisfy the relation (2.45).

Consider the sequence \( \{x_n\} = \{1, \ldots, n\} \) so that \( \lim_{n \to \infty} f^n x_n = \lim_{n \to \infty} g^n x_n = 0 = f(0) \), hence the pair \( (f, g) \) satisfy the (CLRf) property. Also \( f \) and \( g \) are weakly compatible. From here, we also deduce that \( \lim_{n \to \infty} f^n x_n = \lim_{n \to \infty} g^n x_n = 0 \), where \( 0 \in X \), implies that \( f \) and \( g \) satisfy property. Hence all the conditions of Corollaries 2.4.5 and 2.4.7 are satisfied. Here \( 0 \) is the unique common fixed point of \( f \) and \( g \).
Remark 2.4.9. We conclude the following:

(i) E.A. property allows to replace the completeness requirement of the space with a more natural condition of closeness of range.

(ii) (CLR₀) property relaxes the condition of closeness of range for a pair of mappings.

Weakly compatible maps along with different variants of R-weakly commuting mappings

In this section, we show the existence of common fixed points for different variants of R-weakly commuting mappings instead of weakly compatible maps.

Theorem 2.5.1. The Theorems 2.3.8, 2.3.12, 2.4.1, 2.4.2, 2.4.4, 2.4.6 and Corollaries 2.3.13, 2.4.5, 2.4.7 remains true if a weakly compatible property is replaced by any one (retaining the rest of hypothesis) of the following:

(i) R-weakly commuting property,
(ii) R-weakly commuting property of type (A₀),
(iii) R-weakly commuting property (A₀),
(iv) R-weakly commuting property (P),
(v) weakly commuting property.

Proof. Since all the conditions of all above theorems and corollaries are satisfied, then the existence of coincidence points for both the pairs is insured. Let x be an arbitrary point of coincidence for the pairs (f, g), then using R-weak commutativity one gets

\[d(fgx, gfx) \leq R d(fx, gx) = 0,\]

which amounts to say that \(fgx = gfx\). Thus the pair (f, g) is weakly compatible. Now applying above theorems one concludes that f and g have a unique common fixed point.

In case (f, g) is an R-weakly commuting pair of type (A₀), then

\[d(fgx, g^2x) \leq d(fx, gx) = 0, \text{ which amounts to say that } fgx = g^2x.\]

Now \(d(fgx, gfx) \leq d(fgx, g^2x) + d(g^2x, gfx) \leq 0 + 0 = 0, \text{ yielding thereby } fgx = gfx.\)

In case (f, g) is an R-weakly commuting pair of type (A₀), then

\[d(fgx, f^2x) \leq d(fx, gx) = 0, \text{ which amounts to say that } gfx = f^2x.\]

Now, we have

\[d(fgx, gfx) \leq d(fgx, f^2x) + d(f^2x, gfx) \leq 0 + 0 = 0, \text{ yielding thereby, }\]

\[fgx = gfx.\]
Similarly, if pair is R-weakly commuting mapping of type (P) or weakly commuting, then (f, g) also commutes at their points of coincidence. Now, in view of above theorems, in all four cases f and g have a unique common fixed point.

**Remark 2.5.2.** Weakly compatible maps can be replaced by the properties R-weakly commuting property, R-weakly commuting property of type (A_1), R-weakly commuting property of type (A_3), R-weakly commuting property of type (P), weakly commuting property, even though they posses the fixed point.