1.1 INTRODUCTION

Fluid mechanics is a branch of science which deals with the behavior of fluids under the action of various forces. The fluid mechanics provides the knowledge for the analysis and design of a system in which fluid is a working medium. Since fluids have the ability to transport matter and its property and so fluid mechanics becomes a subject of interest for research in all branches of science and engineering like mechanical, civil, chemical, metallurgical, ecological and biological etc.

A great contribution in the development of fluid dynamics is given by Leonardo da Vinci (1452-1519) who derived the equation of conservation of mass for one dimensional steady flow, and also described the waves, jets, hydraulic jumps and eddy formation accurately. Isaac Newton (1642-1727) postulated the law of motion and the law of viscosity of linear fluid in the year 1687, known as Newtonian fluid. In 18th century many mathematicians including Daniel Bernoulli, Leonhard Euler, Jean d’ Alembert, Joseph-Louis Lagrange and Pierre-Simon Laplace contributed in the solution of ideal fluid flow problems. Leonhard Euler formulated the equations of motion in both differential and integral form (Bernoulli equations).

D’ Alembert has used the Euler and Bernoulli equations for the ideal flow to show his famous paradox known as D’ Alembert paradox which can be described as “when a sphere moves with uniform velocity in an ideal fluid at rest at infinity or fluid past through a sphere at rest, there was no drag observed on the sphere”. This result was not simply a consequence of the symmetrical shape of the sphere, but a body of the arbitrary shape at rest in a uniform stream or moving uniformly through a fluid at rest experiences no drag force. The inability of an ideal fluid to produce a force in such circumstances is known as D’Alembert’s Paradox. But in real situations this is not possible. Therefore, ideal fluid theory does not explain adequately drag on the rigid body in the fluid.

At the end of 19th century, William Froude (1810-1879) and his son Robert (1846-1924) developed laws of model testing. Lord Rayleigh (1842-1919) proposed the technique of dimensional analysis. Continuing, Osborne Reynolds (1842-1912) introduced the dimensionless Reynolds number in 1883. Navier (1785-1836) was considered as a founder of modern structural analysis on the contribution in general
theory of elasticity in the year 1826 with great accuracy and introduced the idea of elastic modulus as a property of material. Navier and Stokes (1819-1903) formulated the governing equation of motion with viscous terms. Also, another contribution by Stokes brought the fluid dynamics on the new height. He developed the theory of steady motion of incompressible fluid, friction of fluids in motion, Effects of internal friction of fluids on the motion of pendulums, equilibrium and motion of elastic solids, and the theory of sound giving the explanation of the effect of wind on the intensity of sound. Ludwig Prandtl (1875-1953) has introduced boundary layer theory in the year 1904 which was turning point in the modern fluid dynamics and known as one of the most important theory of 20th century. Boundary layer theory has proven to be the single most important advancement in modern flow analysis. He pointed that fluid flow with small viscosity forms a boundary layer near the solid surface. The development and advancement in the fluid dynamics are continued in the years ahead and in parallels fluid dynamics will be directed towards the application in the different areas for human well being; like-Environmental studies, Oceanography and Bio-fluid mechanics, etc. Application of fluid mechanics in Biology in particular- the Bio-fluid mechanics is concerned with the study of the motion of biological fluids such as blood flow in artery, oxygen transport in lungs and flow of aqueous humor in the eyes, etc. It does not involve any new development of the principles and formula, but reflects on new application in bioengineering and medical sciences. It includes the blood flow and cardiovascular diseases which are most venerable area of study for improving health of human being and to reduce mortality rate due to cardiac dysfunction. For the better understanding of cardiovascular diseases like atherosclerosis, various fluid dynamics factors namely; flow rate, pressure gradients, shear stress on the wall and on the blood cells are to be studied. Before going in details of bio-fluid mechanics, some basic property of fluid are discussed here.

1.2 FLUID AS A CONTINUUM

All fluids are made up of molecules which are separated from each other by spaces. At the microscopic level, the properties of fluids cannot be defined in these spaces due to non-existence of mass. To overcome these difficulties, a fluid is regarded as a continuum i.e. a hypothetical continuous substance. The study model on continuum hypothesis breaks down whenever the mean free path of the molecules approaches the
smallest characteristic dimension of the problem under consideration, while at macroscopic level continuum hypothesis of fluid adequately explains the behavior of fluid flow.

1.3 FLUID PROPERTIES

Manifestations which are primarily characteristic of a particular fluid and not the manner of the flow are called fluid properties. Viscosity and surface tension are examples of fluid properties and as the pressure and density of gases are primarily flow-dependent and hence are not considered fluid properties.

1.4 TYPES OF FLUIDS

(i) Ideal Fluid

An ideal fluid is one that has no viscosity i.e.

\[
\text{Ideal Fluid Flow : } \begin{cases} 
\text{Inviscid Fluid } & \vec{\nabla} \cdot \vec{q} = 0 \\
\text{Incompressible Flow } & \frac{D}{Dt} \cdot \vec{q} = 0 \quad \text{or} \quad \nabla \cdot \vec{q} = 0
\end{cases}
\]  

(1.1)

It is termed as inviscid fluid. No real fluid is inviscid. But for practical purposes and their studies fluids can be considered as inviscid e.g., in flat-plate, the flow at a large distance from the plate, fluid will behave as a non-viscous flow system. The reason for this behaviour is that the velocity gradient normal to the flow direction is very small and the viscous-shear forces are small.

(ii) Viscous Fluid

Fluid in which friction (viscosity) has significant effect on the solution. The viscosity of the fluid is defined as the ratio of the shear stress and shear rate. Thus the slope of the relation between shear-stress and shear-rate gives the viscosity (\(\mu\)). If this relation is linear then the fluid is Newtonian fluid otherwise it is non-Newtonian.

(iii) Incompressible Fluid

If density of the fluid under static condition undergo very little change despite the existence of large pressures. These fluids are invariably in the liquid state for such behaviour. For incompressible fluid it is assumed that during computation of such type of fluid, density is constant.
(iv) Compressible Fluid

A fluid is called compressible, if the pressure variations in the flow field are large enough to effect substantial change in the density of fluid.

(v) Newtonian Fluid

If the plot between shear-stress and shear-rate is a straight line that is shear stress is independent of shear rate as shown in Figure 1.1, then fluid is termed as Newtonian. This slope is called the viscosity of the fluid.

The simplest constitutive equation is Newtonian’s law of viscosity which states that shear force per unit area is proportional to the negative of the local velocity gradient.

\[ \tau_{yx} = \mu \frac{dv_x}{dy} \]  

(1.2)

where,

\[ \tau_{yx} \rightarrow \text{shear stress exerted in the x-direction \ (unit = N/m^2, [\tau] = ML^{-1}T^{-2})} \]

\[ \mu \rightarrow \text{Newtonian viscosity \ (unit = Pa.s or Poise = 0.1 Pa.s, [\mu] = ML^{-1}T^{-1})} \]

\[ v_x \rightarrow \text{velocity of the fluid along x-axis. \ (unit = m/s, [v_x] = LT^{-1})} \]

The Newtonian fluid is the basis for classical fluid mechanics. Gases and liquids like water and mineral oils exhibit characteristics of Newtonian viscosity. Also, blood behaves as a Newtonian fluid at shear rates above 100 s\(^{-1}\), Pedley (1980), Berger and Jou (2000).

(vi) Non-Newtonian Fluid

Fluids, those don’t obey Newton’s law of viscosity are known as non-Newtonian fluid.

The power law is one way to describe the behaviour of Non-Newtonian fluids

\[ \tau = k \left( \frac{dv}{dy} \right)^n \]  

(1.3)

for \( n < 1 \), fluid is called ‘pseudoplastic’ and for \( n > 1 \), the fluid is called ‘dilatant’.
1.5 TYPES OF FLOW

(i) Laminar Flow
A flow in which each fluid particle traces out a definite curve and the curves traced out by any two different fluid particles do not intersect, is said to be laminar.

(ii) Turbulent Flow
A flow in which each fluid particle does not trace out a definite curve and the curves traced out by fluid particles intersect, is called to be turbulent.

(iii) Steady Flow
A flow in which the physical properties and conditions associated with the motion of the fluid are independent of the time so that the flow pattern remains unchanged with the time, is said to be steady, mathematically, we may write as

$$\frac{\partial (\cdot)}{\partial t} = 0. \quad (1.4)$$

Here (\cdot ) is the physical property may be velocity, density, pressure, temperature etc.

(iv) Unsteady Flow
A flow in which the properties and conditions associated with the motion of the fluid depends on the time so that the flow pattern varies with time, is said to be unsteady flow.

(v) Uniform Flow
A flow in which the fluid particles possess equal velocities at each section of the flow region is called uniform flow.
(vi) Non-uniform Flow

A flow in which the fluid particles possess different velocities at each section of the flow region is called non-uniform flow.

Under certain mathematical circumstances, blood flow can be modeled by the equation of motion and energy equation, which are described as follows

1.6 GOVERNING EQUATIONS FOR VISCOUS FLUID FLOW

(i) Equation of Continuity (Conservation of Mass)

It amounts to the basic physical law, that is, the matter is conserved, it is neither being created nor destroyed.

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0
\]  

(1.5)

which is the required equation of continuity in vector notations.

For the incompressible fluid equation (1.5) reduces to \( \nabla \cdot (\vec{q}) = 0 \)

(ii) Equations of Motion (Navier-Stokes Equations)

Conservation of momentum - the equation of motion are derived from Newton’s second law and are given by

\[
\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} = -\frac{1}{\rho} \nabla p + \vec{F} + \nabla \cdot \vec{\tau}
\]  

(1.6)

These are known as Navier-Stokes equation in vector form for the motion of a viscous, incompressible fluid.

(iii) Energy Equation (Conservation of Energy)

It is based upon the conservation of energy and given by

\[
\left(\frac{\partial T}{\partial t} + \vec{q} \cdot \nabla T\right) - \nabla \cdot (\kappa \nabla T) + \rho \nabla \cdot \vec{q} = 0
\]  

(1.7)

where,

\[ \rho \rightarrow \text{density} \]

\[ \vec{q} \rightarrow \text{velocity vector field} \]
F → external force per unit mass
μ → coefficient of viscosity
K_H → heat conduction coefficient
p → Pressure
T → Temperature

1.7 FLOW THROUGH POROUS MEDIA

1.7.1 Darcy Law

Darcy’s law is the constitutive equation which describes the flow of a fluid through a porous medium. It is a simple proportional relationship between the instantaneous discharge rate through porous medium and the local hydraulic gradient. Mathematically, it is describe as

\[ Q = -KA \frac{h_a - h_b}{L} \]  

(1.8)

i.e. the total discharge, \( Q \) (unit=volume/time=\( \text{cm}^3/\text{s} \)) is equal to the product of the hydraulic conductivity \( K \), the cross-sectional area to flow \( A \) and the hydraulic gradient (change in hydraulic head \( h_a \) and \( h_b \) between two points \( a \) and \( b \), respectively divided by length of the column between \( a \) and \( b \)). As a fact of the liquid flows from high head to low head, therefore negative sign is associative with the hydraulic gradient.

The simplified form of the Darcy’s law (1.8) is

\[ Q = -K \frac{\Delta h}{L} \]  

(1.9)

where, \( Q \) is the flux, discharge per unit area (unit=length/time=\( \text{cm/s} \)) and \( \frac{\Delta h}{L} \) is the dimensionless hydraulic gradient vector.

1.8 MAGNETOHYDRODYNAMICS (MHD) AND ITS APPLICATIONS IN BIO-FLUID FLOW

Magnetohydrodynamics is the science which deals with the motion of conducting fluids in the presence of a magnetic field. Complete description of the production and interrelation of electric and magnetic field are formulated by the James Clerk Maxwell in the form of the following equations known as Maxwell’s equations.
(i) \[ \nabla \cdot \frac{\mathbf{r}}{E} = \frac{\rho}{\varepsilon_0} \quad (1.10) \]

(ii) \[ \nabla \cdot \mathbf{B} = 0 \quad (1.11) \]

(iii) \[ \nabla \times \frac{\mathbf{r}}{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.12) \]

(iv) \[ \nabla \times \frac{\mathbf{r}}{\mathbf{H}} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (1.13) \]

where,
\[ \mathbf{E} \rightarrow \text{electric field} \]
\[ \mathbf{B} \rightarrow \text{magnetic flux density} \]
\[ \rho \rightarrow \text{free electric charge density} \]
\[ \varepsilon_0 \rightarrow \text{electric permeability in free space} \]
\[ \mathbf{J} \rightarrow \text{free current density} \]
\[ \mathbf{D} \rightarrow \text{electric displacement field} \]
\[ \mathbf{H} \rightarrow \text{magnetic field} \]

It is known that when a stationary, transverse magnetic field is applied externally to a moving electrically conducting fluid like blood, electrical currents are induced which interact with the applied magnetic field produces a body force known as Lorentz force which tends to retard the movement of the flowing blood. Now a days the work on magneto-hydrodynamic has received much attention. This is due the fact that such concepts are very useful for getting a proper understanding of the functioning of different machines used by physicians for pumping of blood. Human body experience magnetic fields of moderate to high intensity in many situations. Many medical diagnostic devices concerned with the cardiovascular disease, make use of magnetic fields. One of popular device is MRI (Magnetic Resonance Imaging) which is used for diagnosis of biological problem like hemorrhages and hypertension.
1.9 NON-DIMENSIONALIZATION

In practice, in order to study the behavior of certain system, we make experiments at lab scale in which the size of models differ from the actual ones. In such cases, it becomes imperative to know the relationship between the condition of the prototype and actual object when the models and actual objects are geometrically similar. The concept of geometrical similarities was coined by Osborne Reynolds (1883). Suppose we study a model and get equations, boundary condition etc. for this system. We should then be able to get description of the actual system by making suitable changes in units with the help of former. So, it is useful to write these equations in terms of dimensionless variables so that the effect of changing the values of quantities involved in equation is dissociated from the effect of mere changes of units. The following steps are adopted for the non-dimensionlization of a system

(1) Identify all the independent and dependent variables.

(2) Replace each of the variables with a scaled quantity so that the unit of measure is to be determined.

(3) Divide the coefficient of the highest order polynomial or derivative term.

(4) Choose the characteristic unit for each variable so that the coefficients of many terms as possible become unity.

(5) Finally, rewrite the system in terms of their new dimensionless quantities.

1.10 NON-DIMENSIONAL PARAMETERS

1.10.1 Reynolds Number

An estimate of the relative importance of the inertial and viscous forces acting on unit volume of the fluid. It is named after Osborne Reynolds (1842 – 1912) who proposed it in the year 1883. Mathematically, it is given as

\[ \text{Re} = \frac{\text{Intertial force}}{\text{Viscous force}} = \frac{\nu^2 / L}{\nu L} = \frac{\nu L}{V L} \]

where,

\[ V \rightarrow \text{mean fluid velocity.} \]

\[ L \rightarrow \text{characteristic length.} \]
\( \mu \rightarrow \) dynamic viscosity of fluid.

\[ \nu = \left( \frac{\mu}{\rho} \right) \text{ -- kinematic viscosity of fluid.} \]

\( \rho \rightarrow \) density of the fluid.

The Reynolds number act as parameter of viscosity. If Re is small, the viscous forces will be predominant and the effect of viscosity will be felt in the whole flow field. On the other hand, if Re is large, then the inertial force will be predominant and in such a case, the effect of viscosity can be considered to be confined in a thin layer adjacent to the solid boundary. This thin layer in which all viscous effects are confined is called boundary layer. Outside the boundary layer flow is potential flow. However, If Re is very large; the flow ceases to be laminar and becomes turbulent. The Reynolds number at which the transition, from laminar to turbulent occurs is known as critical Reynolds number \((Re_{crit})\). The value of critical Reynolds number for a particular fluid depends on the geometry of the conduit or the nature of the surface over which fluid is flowing. For example, for the flow in pipes, flow is

(a) laminar when Reynolds number in low \((Re < 2100)\)

(b) Turbulent when Reynolds number is high \((Re > 3000)\).

For Bio-fluid flow, in particular for blood the typical Reynolds number range in the body varies from 1 in small arteriols to approximately 4000 in the largest artery (aorta). Thus the blood network spans a range in which viscous forces are dominant on one end and inertial forces are more important on the other. For a medium-sized artery, the Reynolds number is of the order 100 to 1000. The value of Reynolds number is 0.01 or less for the capillaries. For blood flow in brain, the value of Re is of order \(10^2\).

1.10.2 **Womersley Number (Witzig parameter)**

It is a dimensionless number in bio fluid mechanics, gives the relationship between pulsatile flow frequency \((\text{Unsteady})\) and viscous forces named after John R. Womersley (1907-1958). It is defined as

\[ \frac{\omega R}{\sqrt{Re}} \]  

\((1.15)\)
where, \( R \) is the tube radius, \( \omega \) is the angular frequency of oscillation and \( \nu \) is the kinematic viscosity of fluid. It appears in the solution of linearized Navier Stokes equation and incompressible flow in a tube. When the Womersley parameter is low (1 or less), *viscous forces dominate*, velocity profiles are parabolic in shape and the flow will be nearly in phase with pressure gradient (Womersley 1955, Mc Donald 1974). For Womersley parameter 10 or above, unsteady *inertial forces dominate*; it means that frequency of pulsations is sufficiently large that the velocity profile is flat.

On a blood vessel network, the blood flows from a large tube to many small tubes. Through the vessel work the frequency, density and dynamics viscosity remains same, only tube radius changes. Therefore, the Womersley number is large in large vessels and small in small vessels. For a medium sized artery, the values of \( \alpha \) ranges from 1 to 10 (Caro *et al.*1978) and for abdominal aorta is about 16. For specific flow situation, one can predict the hemodynamics situation by recognizing the appropriate Womersley parameter range. Some typical values for the Womersley number in the cardiovascular system for a canine at a heart rate of 2Hz are:

<table>
<thead>
<tr>
<th>Vessel</th>
<th>Womersley Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ascending Aorta</td>
<td>13.2</td>
</tr>
<tr>
<td>Descending Aorta</td>
<td>11.5</td>
</tr>
<tr>
<td>Abdominal Aorta</td>
<td>08.0</td>
</tr>
<tr>
<td>Femoral Artery</td>
<td>03.5</td>
</tr>
<tr>
<td>Carotid Artery</td>
<td>04.4</td>
</tr>
<tr>
<td>Arterioles</td>
<td>00.04</td>
</tr>
<tr>
<td>Capillaries</td>
<td>0.005</td>
</tr>
<tr>
<td>Venules</td>
<td>0.035</td>
</tr>
<tr>
<td>Inferior Vena Cava</td>
<td>8.80</td>
</tr>
<tr>
<td>Main Pulmonary Artery</td>
<td>15.0</td>
</tr>
</tbody>
</table>

1.10.3 **Darcy Number**

It is the ratio of medium permeability \( K \) to the square of characteristic radius \( R \) i.e.

\[
Da^2 = \frac{K}{R^2}
\]  

(1.16)
1.10.4 Nusselt Number

The non-dimensional coefficient of heat transfer is the ratio of convective heat flux to conductive heat flux i.e.

\[ \text{Nu} = \frac{h \Delta T}{\Delta T} = \frac{\text{Convection Heat flux}}{\text{Conduction Heat flux}} \]  

(1.17)

where,

\[ \delta \rightarrow \text{fluid layer thickness}, \]
\[ h \rightarrow \text{convection heat transfer coefficient}, \]
\[ \rho \rightarrow \text{thermal conductivity of the fluid}, \]

It represents the enhancement of heat transfer through a fluid layer as a result of convection relative to conduction across the same fluid layer. The Nusselt number characterizes convection of heat, that is, the larger the Nusselt number, the more effective the convection. If the Nusselt number \( \text{Nu} = 1 \) for a fluid layer, then heat transfer is purely by conduction.

1.10.5 Prandtl Number

It is ratio of the kinematic viscosity to the thermal diffusivity named after the Ludving Prandtl, represented as

\[ \text{Pr} = \frac{\nu}{\alpha} = \frac{\nu}{c_p} = \frac{d_p}{\rho} \]  

(1.18)

where,

\[ \nu \rightarrow \text{kinematic viscosity}, \]
\[ \alpha \rightarrow \text{thermal diffusivity}, \]
\[ c_p \rightarrow \text{specific heat of the fluid at constant pressure}. \]

The Prandtl number is a parameter which relates the relative thickness of the hydrodynamic and thermal boundary layers. The kinematic viscosity (\( \nu \)) of a fluid conveys information about the rate at which momentum may diffuse through the fluid.
because of molecular motion. The thermal diffusivity ($\alpha$) tells us the same thing in regard to the diffusion of heat in the fluid. Thus the ratio of these two quantities expressed the relative magnitude of diffusion of momentum and heat in the flowing fluid. But these diffusion rates are precisely the quantities that determine how thick the boundary layers will be for a given external flow field. Large diffusivities mean that the viscous or temperature influence is felt further out in the flow field. The Prandtl number is thus the connecting link between the velocity field and the temperature field. Small values of $Pr$ tell that the heat diffuses very quickly compared to the velocity (momentum). Which implies that for liquid metals the thickness of the thermal boundary layer is much bigger than the velocity boundary layer. Prandtl number, like the viscosity and thermal conductivity is a material property and it thus varies from fluid to fluid.

For air $Pr = 0.7$ (approx.), water (at 60°F) $Pr = 7.0$ (approx.)

The value of Prandtl number is very small, e.g. for mercury $Pr = 0.044$, while for highly viscous fluids it may be very large, e.g. for glycerin $Pr = 7250$.

### 1.10.6 Brinkman Number

It is the ratio of the heat produced by viscous dissipation to the heat transport by conduction, mathematically expressed as

$$Br = \frac{\mu V^2}{\kappa \Delta T}$$

(1.19)

where,

$\mu \rightarrow$ viscosity

$V \rightarrow$ characteristic velocity

$\kappa \rightarrow$ conductivity

$\Delta T \rightarrow$ Temperatures difference

It is named after Professor H.C. Brinkman, who solved the problem of flow in a circular tube with viscous heat effects. It is a measure of the extent to which viscous heating is important relative to the heat flow resulting from the impressed temperature difference $\Delta T$.

### 1.10.7 Peclet Number
In heat transfer theory, the Peclet number is defined as the ratio of heat convection and heat conduction which implies the Peclet number as the product of Reynolds number and Prandtl number.

\[ Pe = \frac{V L}{\nu} = \frac{V L}{\nu} \cdot \Re \Pr \tag{1.20} \]

For mass diffusion, it is defined as the product of Reynolds number and Schmidt number

\[ Pe = \frac{V L}{D} = \Re \Sc \tag{1.21} \]

where, \( \Sc \) is the Schmidt number which is the ratio of momentum diffusion to molecular diffusion.

\[ \Sc = \frac{\nu}{D} \tag{1.22} \]

where, \( D \rightarrow \) momentum diffusion.

It takes a very large value in various applications, so we use simple computational models. A flow has different Peclet numbers for heat and mass. This leads to double-diffusive convection.

**1.10.8 Hartmann Number**

It gives a measure of the relative importance of drag forces resulting from magnetic induction and viscous forces in Hartmann flow and determine the velocity profile for such flow. Mathematically, it is expressed as

\[ H^2 = \frac{L^2 B_0^2}{\sigma \nu} \tag{1.23} \]

where,

- \( \sigma \rightarrow \) electrical conductivity of the fluid.
- \( B_0 \rightarrow \) electromagnetic induction
- \( \rho \rightarrow \) fluid density.
- \( \nu \rightarrow \) kinematic viscosity coefficient.
As $H^2$ increases, the Lorentz force also increases, which reduces the velocity throughout the tube. The Lorentz force is a function of velocity, therefore the parabolic flow receives a greater flow force at the centre of tube. As the magnitude of the magnetic field increases, the force in the centre increases until the flow becomes almost constant through the tube except near walls, so the solution satisfies the no slip boundary condition. At moderate Hartmann numbers, the velocity profile is nearly flat. Very large Hartmann numbers (~10-10000) is also used for industrial and laboratory applications.

1.11 COMPOSITION OF BLOOD

The blood is composed with suspension of blood particles RBC, WBC and platelets in aqueous medium called plasma.

(i) Red Blood Cells (Erythrocytes)

Red blood cells (RBC) are biconcave with dimensions 2x8μm. Their density is 1.10 g/m$^3$ which is slightly more than plasma (1.03 g/m$^3$), so can be separated by centrifugation from plasma. The pigment that gives the cells their colour is called haemoglobin and it absorbs oxygen. Hematocrit is an important constituent of haemoglobin, which defines the volumetric fraction of RBC’s in the blood. At a normal physiological volume concentration of erythrocyte is 45%.

(ii) Plasma

A pale straw coloured liquid consisting of 90% water and 7% in the principal proteins albumin, globulin, lipoprotein and fibrinogen. Albumin and globulin are essential in maintaining cell viability. The lipoproteins carry lipids (fat) to the cells which gives the fuel of the body. Their concentration is 1/20$^{th}$ of the red cell concentration and diameter is very small of order 2.5μm.

(iii) White blood cells (Leukocytes)

These have an important function they form a defence system of the body, to fight against both infectious disease and foreign materials.

(iv) Blood Platelets
Formed in the red bone marrow and helps in the clotting of blood. Normally, they travel around in the blood with red and white blood cells in an inactivated state.

![Diagram of blood components](image)

**Figure 1.1(a)**

### 1.12 PHYSIOLOGICAL PROPERTIES OF BLOOD

(i) **Color**

Blood is red in color. Arterial blood is scarlet red because it contains more oxygen and venous blood is purple red because more of carbon dioxide.

(ii) **Volume**

The average volume of blood in normal adult is 5 lit. In new born it is 450ml. It increases during growth and reaches 5 lit at the time of Puberty. In adult females it is 4.5 lit.

(iii) **Reaction and pH**

Blood is slightly Alkaline and its pH in normal conditions is 7.4.

(iv) **Specific Gravity**

It is the ratio of the density of a substance to the density of a reference substance. The reference substance is generally taken as water. Temperature and pressure must be specified for sample and the reference. It is dimensional quantity. Mathematically, it is defined as

\[
SG = \frac{s_{\text{sample}}}{s_{\text{H}_2O}}
\]  

(1.24)
If \( SG = 1 \), then substance buoyant in water.

If \( SG > 1 \), then substance will be sink

If \( SG < 1 \), then substance will be float.

The specific gravity of some concerned constituents of present study are as follows:

<table>
<thead>
<tr>
<th>Constituent</th>
<th>Specific Gravity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Blood</td>
<td>1.052 to 1.061</td>
</tr>
<tr>
<td>Blood cells</td>
<td>1.092 to 1.101</td>
</tr>
<tr>
<td>Plasma</td>
<td>1.022 to 1.026</td>
</tr>
</tbody>
</table>

All of these values are referred to the density of water at \( 40^0\text{C} \).

(v) Viscosity

Viscosity is defined as the resistance of the fluid to deformation under shear stress. This describe the internal resistance to flow and can be considered as fluid friction. Primarily, it is determined by the bonds between the molecules of the fluid. Mathematically, viscosity is defined as the ratio of the shear stress to the velocity gradient i.e. shear rate. The viscosity of blood is \( 4 \times 10^{-2} \) dyne.s/cm\(^2\) (poise) which is nearly five times more than water. It is mainly due to red blood cells and plasma proteins.

1.13 FUNCTIONS OF BLOOD

(i) Nutrient Function

Nutritive substances like glucose, amino acids, lipids and vitamins derived from digested food are absorbed from gastrointestinal tract and carried by blood to different parts of the body for growth & production of energy.

(ii) Respiratory Function

The respiratory function of the blood is vital. A continues supply of oxygen is needed by living cells, in particular is brain, since deprivation is followed in minutes by unconsciousness and result a death. In a normal situation, man at rest uses about 250 milliliters of oxygen per minute. During physical exercise the oxygen requirement increases rapidly. All this oxygen is transport by the blood. It is done with the binding to hemoglobin of the red cells.

(iii) Excretory Function
Waste products formed in the tissues during various metabolic activities are removed by the blood and carried to the excretory organs like kidney, skin etc for excretion.

(iv) Transport of Hormones and Enzymes

The hormones which are secreted by ductless (endocrine) gland are released endocrine directly into the blood. The blood transports these hormones to their target organs or tissues. Blood also transports enzymes.

(v) Regulation of Water Balance

Water content of blood is freely interchangeable with interstitial fluid. This helps in regulation of water content in the body.

(vi) Regulation of Acid Base Balance

The plasma proteins and hemoglobin act as buffers and help in regulation of acid base balance.

(vii) Regulation of Body Temperature

Because of the high specific heat of blood, it is responsible for maintaining the thermoregulatory mechanism in the body, i.e. the balance between heat loss and heat gain in the body.

(viii) Storage Function

Water and some important substances like proteins, glucose, sodium and potassium are constantly required by the tissues. Blood serves as readymade source for these substances. These substances are taken from blood during the conditions like starvation, fluid loss, electrolyte loss, etc.

(ix) Defensive Function

Blood plays important role in defense of the body. The white blood cells are responsible defensive mechanism. Neutrophils and monocytes engulf the bacteria by phagocytosis. Lymphocytes are involved in phagocytosis. development of immunity. Eosinophils are responsible for detoxification, disintegration and removal of foreign proteins

1.14 THEORIES RELATED TO FLOW OF BLOOD
Modern approach of the cardiovascular system started with the research of William Harvey (1578-1657) who published all his inventions of circulation of blood in animals. Not microscopically, yet their observations were very reasonable and have wide applications in the conservation of mass.

Giovanni Borelli (1608-1679), called as father of bioengineering on his seminal contributions in the studies of muscles, joints, the cardiovascular system and many other aspects of the body concerned. He also gave the concept of Windkessel effect.

The Reverend Stephan Hales (1677-1746) gave the observations about the mechanics of the cardiovascular system including the first measurements of in-vivo blood pressure. The velocity, at which blood is ejected from the heart of 10-year old male, was measured experimentally by Hales.

The origin of quantitative mechanics in the cardiovascular system begins with Leonhard Euler (1707-1783). In 1755, he submitted an essay, in which he gave the one-dimensional equations of conservation of mass and momentum in a distensible tube which are given by

\[
\frac{ds}{dt} + \frac{ds \cdot vs}{dx} = 0
\]  

(1.25)

\[
2g \left( \frac{dp}{dz} \right) + \left( \frac{d}{dz} \left( \frac{dv}{dz} \right) \right) + \frac{dv}{dt} = 0
\]  

(1.26)

where, \(s\) is the cross-sectional area, \(v\) is the average velocity, \(p\) is pressure, \(g\) is the density of blood, \(t\) is time and \(z\) is the axial distance.

The next most important observation was given by Thomas Young (1773-1829) through his lecture ‘On the functions of the heart and the arteries’ delivered to the Royal Society in 1808. Also, he stated the correct formula for the wave speed in an artery but gave no derivation. Later on, he gave a derivation which is very hard to understand based on speed of sound in a compressible gas, some incomprehensible algebra and numerical guesses.

The next landmark in arterial mechanics was given by Jean Louis Poiseuille (1799-1869) through his law of flow in tubes. He conducted a very thorough investigation of flow in capillary tubes motivated by his observations of the mesenteric microcirculation of the frog. In this experiment, he claimed that volume flow rate \(Q\) varied as
\[ Q = \frac{kpD^4}{L} \]  

(1.27)

where, \( p \) the pressure drop along the tube, \( D \) and \( L \) are the diameter and length of the tube and \( k \) is a constant (depends on temperature and fluid flowing in the tube). At the same time, Hagen (1797-1884) conducted the similar experiments on the flow of water in cylindrical tubes with diameters 2.55, 4.01 and 5.91 mm. He suggested the law

\[ p = \frac{AQL + BQ^2}{D^4} \]  

(1.28)

where, \( A \) and \( B \) are constants (depends on temp.), term \( Q^2 \) is corresponding to the generation of kinetic energy in the fluid. At very low values of \( Q \), this relationship reduces to Poiseuille law. Both these formula were not incorporated the viscosity. The first Poiseuille’s law

\[ Q = \frac{\sqrt{4D^4}}{128\mu} \]  

(1.29)

where, \( \mu \) is the coefficient of viscosity. Wilhelm Eduard Weber studied the speed of waves in elastic tube theoretically and Ernst-Heinrich Weber gave the same observations experimentally. W.E. Weber derived the equation for the wave speed \( c \), which is given by

\[ c = \sqrt{\frac{R}{2k\mu}} \]  

(1.30)

where, \( \rho \) is the density of the fluid, \( k \) is the radial distensibility and \( R \) is the radius of the tube.

Moens (1846-1891) published a research paper on wave speed in arteries and Korteweg (1848-1941) published a theoretical study of the wave speed. Korteweg’s showed that the wave speed was determined both by the elasticity of the tube wall and the compressibility of the fluid. In case of blood and thin-walled tubes, this relation is known as Moens-Korteweg equation which is given by

\[ c = \sqrt{\frac{Eh}{2PB}} \]  

(1.31)
where, \( E \) is the Young’s modulus of the tube wall whose thickness is \( h \) and \( c \) the wave speed.

Otto Frank (1865-1944) gave a great contribution about quantitative physiology worked on the cardiovascular system. This first contribution to arterial mechanics was the formulation of the Windkessel effect mathematically. He gave the expression of change of volume during diastole as

\[
\frac{dV}{dt} = \frac{E}{w} \quad \text{and} \quad \frac{dp}{dv} = c
\]

(1.32)

where, \( V \) volume of the arterial flow, \( p \) is the pressure, \( w \) is the resistance of flow in the microcirculation, \( c \) is a constant. From these equations, he obtained an exponentially falling pressure

\[
p = p_0 e^{\frac{ct}{w}}
\]

(1.33)

where, \( p_0 \) is the pressure at the start of diastole.

Texon (1957, 1965) observed the location and progression of atherosclerotic lesions at curvatures, branches, bifurcations etc. to hemodynamic reduction of lateral pressure at those points, since a suction-action created by tensile force which stimulates the intima to proliferate and so a atherosclerotic plaque formed. However, a suction-action on the endothelium will exist only if the local pressure beneath the endothelium exceeds the intra luminal pressure.

Besides all these hypothesis, there are two well-known but controversial theories are famous, Fry’s high shear stress theory and Caro’s low shear stress theory. Fry (1968) resulted that deformation, swelling and eventual erosion of the endothelium may occur at sites where the local wall shear stress is relatively high, hence it may be responsible for local plaque formation. Fry conducted an in-vivo study on anesthetized, thoracotomized mongrel dogs and concluded that exposure of the endothelial surface to a time-averaged wall shear stress of approximately 380 dyne/cm\(^2\) could result effectively a destruction of endothelial surface. In the continuity, Fry (1969) found that exposure of the endothelial surface to time-average wall shear stresses below 380 dyne/cm\(^2\) could result in an increase in albumin flux into the intimal layer. This results an increase in wall permeability that implies an
increased flux of lipoproteins into or out of a vessel wall at stress exposure producing injury to the endothelial surface. Later, he discussed the nature of flow in arteries to correlate the regions of high shear stress with common sites of lesion development. It was also pointed out that local wall shear stress were relatively high at bifurcated walls and immediately downstream from the ostium to a branch. Bifurcations are one of the descending branches of the anterior heart wall. Later, it was found that this theory has a drawback on account of the mechanism of atherogensis which assumed as a damage and repair process of endothelium and ignored the fact that many atherosclerotic lesions occurs in the regions of low shear stress. Also, it cannot explain why high shear stress region in vivo are result of atherosclerotic lesions. Also, in-vivo, the critical wall shear stress causing endothelium structural change of about 400 dyne/cm$^2$ is not likely to occur in aorta under normal physiological conditions.

Caro’s (1969, 1971) low shear stress theory resulted that early atheroma occurs in the low shear regions not in the high shear regions. Caro et al. (1971) proposed a shear-dependent mass transfer theory. The theory considers simultaneous mechanism by which arterial wall cholesterol levels may be altered. It is based on a series of facts which are resulted to shear dependent diffusional efflux of cholesterol from the intima to the blood stream. This theory like the high shear stress theory have certain drawbacks. It does not explain why the cross wall mass transfer is enhanced in the low shear stress region. Caro et al. (1971) suggested that cholesterol accumulates in the low shear region due to its local diffusional efflux from wall to blood is inhibited by the reduced wall-blood concentration gradient. However, the in-vitro experimental data of Nerem (1990), explored that there is no obvious correlation between the wall uptake of albumin. The wall uptake albumin increases with the increase of wall shear stress, higher the shear stress, stronger the correlation.

In the series, Kleinstreuer (1996) and his group gave the combined aspects of the high and low shear stress theories. They postulated that very low oscillating wall shear initiates atherosclerotic lesions and both low and high shear stresses contribute to the growth of plaque formation. Consequently, a safe bandwidth of the Wall Shear Stress (WSS) with a lower value $\tau_{\text{min}}$ and a higher value $\tau_{\text{max}}$ has been demarked. When WSS fall outside the safe mode, there is a plaque formation and the magnitude of WSS beyond the band limits determines the growth rate of plaque. They applied this
idea to several aortic and carotid artery bifurcation and graft-bypass configurations and successfully determined the sites and growth patterns of atherosclerotic lesions and intimal hyperplasia respectively.
1.15 ATHEROSCLEROSIS

Atherosclerosis is the position in which an artery wall hardened due to the deposition of fatty materials such as cholesterol. The fatty substance commonly known as atheromas. The narrowing and any change of geometry in the artery can produce a great effect on the blood flow and thus on the artery itself. It is also known as arteriosclerotic vascular disease. It can be complicated as cardiac stroke/failure, brain stroke, kidney failure and formation of gangrene in the legs or toes.

1.15.1 Causes

In many books and literature on biofluid dynamics it has been accepted that the cause of atherosclerotic is not fully understood, but it is believed that blood flow through narrow artery experiences a high shear stress and hence recirculation which promotes the growth of the stenosis. Some researchers believe that atherosclerosis may be caused by the disorders such as diabetes and hypertension, viral infections, smoking, etc. Due to these factors an inflammatory reaction at the affected areas of the blood vessels occurs, which results the accumulation of fat cells and calcium and start forming a thickened layer progressively. This layer is known as plaque that obstructs the flow of blood.

1.15.2 Signs and Symptoms of Atherosclerosis

These involved very much on the blockage of blood vessel. Generally, atherosclerosis does not produce any features until the block results in a significant decrease in the blood flow. Blocks in the blood vessels of the legs is recognized frequently as cramps felt after walking, chest pain is the common symptom if the blockage in the blood vessel that supply the heart. The risks of kidney failure and brain stroke are increased if the responsible blood vessels are affected.

1.16 MODELS OF BLOOD FLOW

The circulatory system is a complex system of branching compliant tubes which works according to complex controller. So various models are used to study either single artery or the whole circulatory subsystem of specific organs. Because of compact and complexity of the human system, there are a multitude of variables which affect the functions, properties and response of the circulatory system. In real sense or experimentally it is impossible to include all the known variables in a single
system, so models are used to deal with a group of variables at a time and to study the interaction between the variables. Some commonly used models are described as

### 1.16.1 Fahraeus-Lindquist Effect

The idea of to correlate the vessel size to viscosity of blood was given by Fahraeus in 1929. He demonstrated that tube hematocrit decreases with decreasing tube diameter and approaches an asymptotic value at tube diameter larger that above 0.5mm. The phenomenon of the decrease in the apparent viscosity in small diameter tubes is known as Fahraeus Lindquist Effect. This effect is associated with the tendency of red blood cells to travel closer to the centre of the vessels. Thus, the more the decrease in vessel lumen, lesser the number of red blood cells that passes through resulting in a decrease in blood viscosity.

Two mathematical models have been developed to describe the Fahraeus-Lindquist effect

(i) Cell-free marginal layer model.

(ii) The sigma effect.

### (i) Cell-Free Marginal Layer Model

This model considered the steady flow of blood through a circular tube. The tube cross- section is divided into two regions, one is core region and second is the cell-free plasma region near the wall. The governing equations for both the regions are given by

\[
\frac{\Delta p}{L} = \frac{1}{r} \frac{d}{dr} \left( r \frac{dv_c}{dr} \right) \quad 0 \leq r \leq r - \square
\]

and

\[
\frac{\Delta p}{L} = \frac{1}{r} \frac{d}{dr} \left( r \frac{dv_p}{dr} \right) \quad R - \square \leq r \leq R
\]

For the solution of the problem, we impose no- slip boundary condition on the wall of plasma region, velocity gradient is zero at the center of the tube and velocity and shear stress are continuous at the interface between the two zones are taken. In this, the expression for volume flow is given by

\[
Q = \frac{B^4 \Delta p}{8 \square} \left[ 1 - \left( 1 - \frac{\square}{R} \right)^4 \left( 1 - \frac{\square}{R} \right) \right]
\]
Here, $v_c$ and $v_p$ are velocities of the blood in core and plasma region respectively and $\mu_c$ and $\mu_p$ are viscosity of blood in core and plasma region respectively. $\delta$ the thickness of plasma region and $R$ the radius of vessel.

(ii) The Sigma Effect

The sigma effect theory is based upon the assumption that the velocity profile is not continuous. Here, we consider that the tube diameter is so small that there is room for only $N$ red blood cells to move abreast.

Volume flow is written as:

$$Q = \frac{\Delta p}{2\mu} \int_0^R r^2 dr$$ \hspace{1cm} (1.37)

If we assume that the flow occurs in $N$ concentric laminas, each of thickness $\varepsilon$, then the radius $R$ of the tube will be $n\varepsilon$. In general $r=n\varepsilon$, $n=1, 2, \ldots, N$ and then $dr=\varepsilon \Delta n = \varepsilon$, $\Delta n=1$. With this assumption, flow rate is rewritten as

$$Q = \frac{\Delta p}{2\mu} \sum_{n=1}^{N} (n\varepsilon)$$

$$= \frac{\Delta p R^4}{8L} \left(1 + \frac{\varepsilon}{R}\right)$$ \hspace{1cm} (1.38)

1.16.2 Einstein Model

This gives the expression of viscosity of blood dependent on Haematocrit concentration. According to model

$$\mu = \mu_0 [1 + \beta h]$$ \hspace{1cm} (1.39)

where, $\mu_0$ the coefficient of viscosity of plasma, $\beta$ a constant dependent on particle shape, $h$ the haematocrit concentration i.e. volume fraction of the suspension occupied by particles. Generally, $\beta = 2.5$. Einstein showed $\beta = 2.5$ for spherical particles.

1.16.3 Casson’s Fluid Model

This model was originally introduced by Casson (1959) for the prediction of the flow behavior of pigment-oil suspensions. Oka was the first, who studied the flow characteristic of Casson fluids in tubes. He considered a generalized model for flow of
non-Newtonian fluids in tubes from which the Casson fluid model was derived as a special case.

In steady state, it is a nonlinear relation between shear stress and rates of strain (Fung, 1981) given by

\[
\frac{\gamma}{\gamma} = \frac{\gamma}{\gamma}^2 + \left[ - \frac{\partial \gamma}{\partial r} \right]^2 \quad \text{if} \quad \gamma \geq 0
\]

and

\[
\frac{\partial \gamma}{\partial r} = 0 \quad \text{if} \quad \gamma < 0
\]

where, \( \gamma \) denotes the yield stress and \( \gamma \) the Casson’s viscosity (viscosity at higher shear rate). The second equation corresponds to vanishing of velocity gradients in the region where the shear stress \( \gamma \) is less than the yield stress \( \gamma \), which implies the plug flow in the region \( \gamma \leq 0 \).

It describes the flow characteristic of blood more accurately at low shear rates for the flows through small blood vessels (Mc Donald, 1974). It is found to be applicable in developing models for blood haemodialysers.

1.16.4 Herschel-Bulkley Model

This model was introduced by Herschel and Bulkley in 1926 for non-Newtonian fluid. It gives the relationship between strain experienced by the fluid with stress in a non-linear complicated form. In this model, three parameters characterize the flow, \( \lambda \) (consistency), \( n \) (flow index) and \( \gamma \) (yield shear stress). The formula for this model is given by

\[
\frac{\gamma}{\gamma} = \left\{ \begin{array}{ll}
\frac{\gamma}{\gamma} & \gamma \leq \gamma \\
\frac{\gamma}{\gamma}^n + \frac{\gamma}{\gamma}^k & \gamma > \gamma
\end{array} \right.
\]

where, \( \lambda \) is a constant of proportionality, \( n \) is measure of degree of fluid shear-thinning or shear-thickening, \( \gamma \) gives the amount of stress that fluid may experience before it yields and begins to flow, \( \pi \) is the second invariant of the rate of strain tensor.
If \( n = 1 \) and \( \eta = 0 \), then flow is Newtonian fluid.

If \( n < 1 \) then the fluid is shear–thinning while for \( n > 1 \) produces a shear–thickening fluid.

The equation of model is also written as

\[
\tau = \eta \dot{\gamma} + \kappa
\]

(1.43)

where, \( \tau \) the shear stress, \( \dot{\gamma} \) the shear rate, \( \eta \) the yield stress, \( \kappa \) and \( n \) are model factors.

It is used to describe the rheological behavior of food product and biological fluids.

**1.17 BRIEF DESCRIPTION OF THE METHODS OF SOLUTIONS USED IN THE THESIS**

Since the Navier Stokes equations are non-linear in character, so there is not enough methods to solve all the problems exactly, for the solution of modeled problem, numerical solution are to be obtained with the help of some approximate methods.

Some of the methods are stated below

**1.17.1 Perturbation Method**

This theory is a mathematical tool in advanced sciences and engineering. In fact, it is the extension to the mathematical functions of the “guess, check and fix” method. This method is used to find an approximate solution to a problem which cannot be solved exactly. This theory is applicable on those problems which can be formulated by adding a ‘small’ term to the mathematical description of the exactly solvable problem. In this, the solution is obtained in terms of power series in some small parameter known as perturbation series. The first term of power series is the leading term and is the solution of the exactly solvable problem while the other terms describe the deviation in the solution, due to the deviation from the initial problem. First, we take a series in small parameter (here called \( \varepsilon \)) for the approximate solution \( u(x) \) as

\[
u(x) = u_0 + \eta \eta_1 + \eta \eta_2 + ....
\]

(1.44)

Here, \( u_0 \) is the known solution to the problem. Since the value of \( \varepsilon \) is very small, so the higher order terms in the series become smaller so can be avoided. Thus, for the approximate solution, only the first two terms are signified. Therefore
Because of the simplification introduced in every step of the problem, the solution obtained is deviated from the correct solution.

There are two types of perturbation method namely

(i) Regular perturbation (ii) Singular perturbation. In Regular perturbation, the order of differential equation is not reduced, it remains as such. While, in Singular perturbation method, the order of differential equation is reduced.

1.17.2 Laplace Transform Method

The Laplace transform is a well established mathematical method for the solution of ordinary differential equations. It is named in honor of the great French mathematician, astronomer and physicist Pierre Simon De Laplace (1749-1827). Like all transforms, the Laplace transform changes a given function \( u(t) \) by another function \( u(s) \) or \( \tilde{u} \). It finds various applications in physics, mathematics, signal processing, electrical engineering, optics, and control engineering. With the increasing complexity of applied mathematics and engineering problems, Laplace transform method helps in solving complex problems with a very simple approach. The Laplace transform of a function \( f(t) \) exists if the Laplace integral converges. The integral will converge if the function \( f(t) \) is piecewise continuous in every finite interval in its range and the function is of exponential order as \( t \) tends to infinity. A function \( f(t) \) is said to be of exponential order \( a > 0 \), if 
\[
\lim_{t \to \infty} te^{-at} f(t) \text{ exists or we can say as } t \to \infty, \text{ there exists a real number } M > 0 \text{ such that }
\]
\[
|e^{-at} f(t)| < M \quad \forall t > T
\]  

Now, we state the Laplace Transformation in two ways,

(1) Given \( f(t) \), the Laplace transform of \( f(t) \) is denoted by \( \tilde{f} \) or \( F(s) \) and it gives an average value of \( f(t) \) taken over all positive values of \( t \) such that the value \( F(s) \) represents an average of \( f(t) \) taken over all possible time intervals of lengths.

(2)  
\[
L\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t) dt \quad \text{for } s > 0
\]  

where, \( s \) is known as a parameter, which may be real or complex.
The following procedure is adopted for this method:

1. Applying Laplace transform on every term in the differential equation.
2. Taking the Laplace transform of each conditions which are associated with the problem.
3. After applying the given values of the given conditions as required in step (2), we will get an algebraic equation for \( F(s) \).
4. Then we solve the algebraic equation for \( F(s) \).
5. The solution of the differential equation is the inverse Laplace transform of \( F(s) \).

\[
f(t) = L^{-1}[F(s)]
\]

We determine the inverse Laplace Transform either by directly (using table) or by the use of the inversion theorem for the Laplace transformation, which states that

\[
f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) \, ds
\]

where, \( \gamma \) is to be so large that all singularities of \( f(s) \) lie to the left of the line \( (\gamma+i\infty, \gamma+i\infty) \). Further, by Cauchy’s Residue theorem for complex variable, we have

\[
f(t) = \text{sum of residues of } e^{st} F(s)
\]

\[
= \sum \text{residues of } e^{st} F(s)
\]

Table 1.1 (a) of Laplace Transform of Some Functions

<table>
<thead>
<tr>
<th>Functions</th>
<th>Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{s} ) ( s &gt; 0 )</td>
</tr>
<tr>
<td>( t^n )</td>
<td>( \frac{\Gamma(n+1)}{s^{n+1}} ) if ( n \in \mathbb{R} ), ( n &gt; -1 ) and ( s &gt; 0 )</td>
</tr>
<tr>
<td>( e^{at} )</td>
<td>( \frac{1}{s-a} ) ( s &gt; a )</td>
</tr>
<tr>
<td>( \sin at )</td>
<td>( \frac{a}{s^2 + a^2} ) ( s &gt; 0 )</td>
</tr>
<tr>
<td>( \cos at )</td>
<td>( \frac{s}{s^2 + a^2} ) ( s &gt; 0 )</td>
</tr>
<tr>
<td>( \sinh at )</td>
<td>( \frac{a}{s^2 - a^2} ) ( s &gt;</td>
</tr>
<tr>
<td>( \cosh at )</td>
<td>( \frac{s}{s^2 - a^2} ) ( s &gt;</td>
</tr>
</tbody>
</table>
1.17.3 Frobenius Method

It is named after Ferdinand Georg Frobenius. The Frobenius method extends the simple power series methods. The simple series expansion method is suitable when solutions of a differential equation are well-defined at the expansion point \( x = 0 \). The simple series method works effectively on many functions, there are some functions whose behavior cannot be predicted by the simple series method. e.g., the Bessel functions of second kind \( Y_0 \) and the functions involving negative or fractional powers. In that case, The Frobenius method can be used as an alternative of simple series method for the solution of different equation. This method is a mathematical tool to find an infinite series solution for a second order differential equation. This method tells us that we can seek a power series solution of the form

\[
\sum_{k=0}^{\infty} A_k (r - r_0)^k, \quad A_0 \neq 0 \quad \text{for homogeneous differential equation}
\]

\[
u(r) = D \sum_{k=0}^{\infty} A_k (r - r_0)^k - E \sum_{k=0}^{\infty} \sqrt[r]{\rho} (r - r_0)^{r+2} \quad \text{for non-homogeneous differential equation}
\]

where, \( A_k \) and \( \sqrt[r]{\rho} \) are the series constants to be determined by equation of the problem and \( D \) and \( E \) are arbitrary constant to be determined by the boundary condition. The following procedure can be adopted for this method

(1) Choose a value for \( r_0 \). If some conditions are given for \( u(r) \) at some point, then use that point for \( r_0 \), otherwise, select \( r_0 \) as convenient. We usually take \( r_0 = 0 \)

(2) Assume that the solution of the equation is of the form

\[
u(r) = \sum_{k=0}^{\infty} A_k r^k
\]

(1.50)

where, \( A_0 \) is an arbitrary constant.

(3) Substitute the values of \( u \) and its derivatives i.e.

\[
u = \sum_{k=0}^{\infty} A_k r^k
\]

(1.51)

\[
\frac{du}{dr} = \sum_{k=1}^{\infty} k A_k r^{k-1}
\]

(1.52)
\[
\frac{d^2 u}{dr^2} = \sum_{k=2}^{\infty} k(k-1)A_k r^{k-2}
\]  

(1.53)

in the given differential equation.

(4) Equate to zero the coefficients of various powers of \( r \) and obtain \( A_1, A_2, A_3, \ldots \).

Substituting the values of \( A_1, A_2, A_3, \ldots \) in the series assumed, we get the desired solution.

1.18 REVIEW OF LITERATURE

The flow in porous medium spreads a wide application in transport of macromolecules in aortic media, interstitial fluid flow in axisymmetric soft connective tissue, thermal therapy, blood flow through contracting muscles, heat transfer in muscle and skin tissue. Biological tissues contain dispersed cells separately by voids. Blood enter these tissues through vessels called arteries and perfuse to the tissue cells via blood capillaries, returned blood from the capillaries is accumulated in veins where the blood is pumped back to the heart. In some pathological situations, the distribution of fatty cholesterol and artery-clogging blood clots in the lumen of the coronary artery can be considered as equivalent to a fictitious porous medium. David et al. (2001) has developed a species transport model of platelet accumulation which included mechanisms of convection, shear-enhanced diffusion, near-wall platelet concentration and a kinetic model of platelet activation and aggregation for an initial quantitative estimate of the likelihood of occlusive thrombus in individual patients due to plaque erosion, artery spasm, incomplete angioplasty or plaque rupture. Xu et al. (2010) considered the blood clot as a porous medium to account for the transport property of blood flow in the extension of multiscale model by including a detailed submodel of surface-mediated control of blood coagulation (Xu et al., 2008; 2009).

It has been established that the biological systems in general are greatly affected by the application of external magnetic field. As per the investigations reported by Barnothy (1964-1969), the heart rate decreases by exposing biological systems to an external magnetic field. Korchevskii and Marcochnik (1965) have discussed the possibility of regulating the blood movement in human system by applying magnetic field. Keltner et al. (1990) reported an analysis of the pressure changes in vessels of the human vasculature under the action of strong magnetic fields. Their study
indicated that 15% Sodium Chloride solutions are retarded by transverse magnetic fields of 2.3 and 4.7 Tesla for fluxes below 0.5 l/min.

The use of catheters is of immense importance and has become a standard tool for understanding, diagnosis and treatment in modern medicine. When a catheter is inserted into the stenosed artery, the further increased impedance or frictional resistance to flow will alter the velocity distribution and so the pressure gradient of catheterized and uncatheterized artery will differ quantitatively. Kanai et al. (1996) established analytically that for each experiment, a catheter of an appropriate size is required in order to reduce the error due to the wave reflection at the tip of the catheter. The mean flow resistance increase during coronary artery catheterization in normal as well as stenosed arteries has been studied by Back et al. (1996). A number of theoretical studies of suspensions in general and blood flow in particular are given by (Jones (1966); Nuber (1967); Brunn (1975)), and experimental studies by Bugliarello and Heyden (1962), Bennet (1967), suggest the likely presence of slip (a velocity discontinuity) at the flow boundaries (or in their immediate neighborhood).

Sharma et al. (2011) showed the effect of heating and cooling on the temperature profiles inside the stenosed artery for understanding the effect of hyperthermia and cryosurgery treatment on tumor tissue by taking the blood as incompressible, fully developed and Newtonian fluid. Garcia and Riahi (2014) considered the blood flow and heat transfer through the inclined stenosed artery with or without a catheter. The hemodynamics indices like impedance, blood pressure force, heat flux at the artery and blood temperature are find out for both cases : with or without a catheter.

On the basis of experimental and theoretical observations Haynes (1960) proposed the marginal-zone theory that the later modified by Gordan (1970). This theory states that there is an annular zone of small thickness εR near the tube wall which is free from blood cells, called plasma zone of Newtonian in nature, while the central core, of radius R(1-ε) having uniform cell concentration. Further, Bugliarello and Sevilla (1970) and Cokelet (1972) have experimentally proved that for blood flowing through small vessels, there exists cell-poor plasma (Newtonian fluid) layer and a core region of suspension of almost all the erythrocytes. The two-fluid modeling of blood flow has been discussed and used to analyze blood rheology by a good number of
researchers. Shukla et al. (1980) applied a two-phase model to discuss the flow of blood through stenosis. Chaturani and Upadhya (1979, 1982) studied the flow of blood in small diameter tubes using the two-layered model of micropolar and couple stress fluids respectively. Chakravarty et al. (2004) considered a two-layer blood flow in a tapered flexible stenosed artery. The impact of taper angle, the wall deformation, the severity of the stenosis and the viscosity of peripheral layer on the hemodynamic indices resistive impedance, flux and wall shear stress to be studied. Sharan and Popel (2001) considered a two-phase model for the flow of blood in narrow tubes, assuming that the viscosity in the cell-free layer differs from the plasma due to the dissipation of energy near the wall caused by the motion near the cell-free layer.

Musad and Khan (2010) studied the effect on WSS in stenosed artery assuming blood as couple stress fluid with peripheral layer plasma and core layer, the suspension of erythrocytes and observed that WSS increases with the increase of height of stenosis, yield stress and core layer viscosity. Chaturani et al. (2001) obtained analytic solutions for the two-phase magnetic fluid for pulsatile flow of fluid.

Sinnott (2006) investigated the pulsatile blood flow in a bifurcation artery using Grid-Free method. In this they used a real carotid bifurcation arterial geometry derived from MRI and gave the comparison between the pulsatile and steady flow. Deshpande et al. (2009) simulated the subject specific blood flow in the human carotid artery bifurcation. They concluded that blood flow in the carotid bifurcation is associated with high wall shear stress and due to flow separation. The presence of even a mild stenosis may change the local flow considerably. They used the MRI and pulsed Doppler ultrasound techniques with CFD approach and found that this combined approach are superior than of medical imaging techniques.

1.19 SUMMARY OF THE THESIS

The present thesis entitled, “Numerical Simulation of Mathematical Models of Viscous Fluid Flow through Circular Tube” has been grouped in five chapters. Chapter 1, Introduction comprises fundamental concepts of fluid dynamics and biofluid mechanics for the sake of ready reference.
Chapters 2 to 5 included the problems in different frameworks for the clarity of the subject matter presented. Analytical solutions of the problem are derived and numerical values of the effects of various parameters involved in the problem are computed. In the end of thesis, the relevant references are provided in the alphabetical order. The chapter wise organization of the thesis is as follows:

Chapter 1 described the general introduction, a concise sketch of the development of fluid dynamics following the basic concepts, properties, definitions, types, physical parameters, methods and models of fluid dynamics and bio-fluid mechanics and their applications in various areas are discussed.

Chapter 2, entitled, “Pulsatile unsteady flow of blood through porous medium in a stenotic artery under the influence of transverse magnetic field”, is focused on study in an artery of circular cross-section and suffered with stenosis. The consideration of porous medium is due to the fact that artery clogging blood clots and cholesterol deposition in blood vessel form porous like medium. Such type of problem has many applications in the analyzing the hemodynamic of the blood during cardio-vascular diseases. This chapter deals with unsteady, pulsatile MHD blood flow through stenosed artery filled with porous medium. The governing equations are solved with Frobenius method and expression for velocity profiles, volumetric flow rate(Q), wall shear stress(WSS) and pressure gradient(P) are derived. Effects of various physical parameters Hartmann number H, Darcy number Da, Womersley number α and Hematocrit Hm in numerical form are computed. The analysis shows that WSS changes its sign twice in the region of stenosis nearby entry and exit of the stenosis. The occurrence of these variation suggests that there will be two region of circulation for the value of Hartmann number H> 2. The magnetic field strength also affecting the location of the circulation region, for weak magnetic field the circulation regions occurs closures to the extremes of the stenosis. While, on increasing strength of the magnetic field the circulation region moves towards the neck of the stenosis that may affect more adversely on the hemodynamic conditions of the flow.

Chapter 3 comprises two sections: Section-A dealt with the theoretical study of “Pulsatile blood flow through an inclined catheterized stenosed artery with slip on the wall in the presence of an external magnetic field” by considering the incompressible
Newtonian fluid model. The stenosis is considered axisymmetric, therefore the flow is of one dimensional along the axis of the vessel. A perturbation method is employed to solve the governing differential equations by using a small perturbation parameter $\varepsilon$. Analytic expression for the velocity, volumetric flow rate ($Q$), wall shear stress (WSS), wall shear stress gradient (WSSG), impedance ($\lambda$) have been derived and numerical results are presented graphically for different values of physical parameters of interest.

In Section -B, thermal effects of catheter on the flow behavior is analyzed to know the contribution on blood flow through stenosis when the catheter is subjected with controlled temperature. The effect of various non-dimensional parameters on temperature profiles and Nusselt number are presented and discussed. The study showed that Nusselt number decreases with the increase of Hartmann number $H$, Reynolds’s number $Re$ and Brinkman number. There is an increase in Nusselt number with the increase of catheter radius $k$ and Heat source parameter $N$.

Chapter 4 considered the problem on “Effect of Magnetic field and slip velocity on pulsatile two layered blood flow in artery”. The blood is considered as two-layered fluid and a slip velocity at the lumen wall. A static transverse magnetic field is considered on the artery that influenced the flow in the core region as it being RBC rich region. The governing equations are solved by using Womersley number as perturbation parameter. The effects of Hartmann number $H$, relative thickness of plasma to core region $\beta$, slip velocity at the wall of artery are assessed on the flow variables; velocity profiles, wall shear stress (WSS) and volumetric flow rate ($Q$) and discussed through graphs. The result reflected that there is a high jump in velocity of plasma region as compared to core region velocity introduced high velocity gradient on the arterial wall which may a cause of endothelium wall rupturing. The study also elaborates the effects of viscosity ratio $\beta$ on the core velocity. It is seen that with the increase of viscosity ratio $\left(\frac{\rho_p}{\rho}\right)$, the core velocity increases i.e. if the cell concentration in the blood reduces, the core velocity increases which is in good agreement with the physical results.
Chapter 5 focused on the “Effect of branch angle and magnetic field on the flow through a bifurcating vessel”. The blood is treated to be homogeneous, Newtonian fluid of constant density. The daughter arteries forming bifurcation are symmetrical about the axis of the trunk. A transverse static magnetic field is applied. The flow is treated to be unsteady and one-dimensional so that only the axial velocity components is nonvanishing. The modeled problem has been attempted to solve by Laplace Transform using the recurrence relation of Bessel’s function and modified Bessel’s function. The effect of various parameters on velocity, wall shear stress (WSS), wall shear stress gradient (WSSG) and volumetric flow rate (Q) has been computed. It is observe that the pattern of the velocity profiles and hemodynamic indices of the flowing blood alters in the daughter vessel from that in the parent one.