Appendix A

Classical Lamination Theory

The detailed steps in the formulation of equations pertaining to CLT and the assumptions made during the formulation are discussed in this section.

A.1. Basic assumptions

1. Each layer of the laminate is quasi-homogeneous and orthotropic.
2. The laminate is thin and its layers are in a state of plane stress. \( \sigma_z = \tau_{xz} = \tau_{yz} = 0 \)
3. All displacements are small compared with the thickness of the laminate.
4. Displacements are continuous throughout the laminate.
5. In-plane displacements vary linearly through the thickness of the laminate. Straight lines normal to the middle surface remain straight and normal to that surface after deformation.
6. Strain displacement and stress-strain relations are linear.
7. Normal distances from the middle surface remain constant, that is, the transverse normal strain \( \varepsilon_z = 0 \).
A.2. Strain displacement relations

Figure A.1 shows the position of the laminate in x-y plane. Enlarged views show section on the laminate normal to the y-axis, before and after deformation. The x-y plane is called as the mid-plane or reference plane and is located at equal distance from the top and bottom surfaces of the laminate. \( u_o \) and \( v_o \) are the reference plane displacements in the x and y directions respectively. \( w \) is the out of plane displacement in z direction. \( u_o, v_o \) and \( w \) are the functions of x and y only. i.e.,

\[
\begin{align*}
    u_o &= u_o(x, y) \\
    v_o &= v_o(x, y) \\
    w &= f(x, y)
\end{align*}
\]

(A.1)

The rotations of the x and y axes are

\[
\begin{align*}
    \alpha_x &= \frac{\partial w}{\partial x} \\
    \alpha_y &= \frac{\partial w}{\partial y}
\end{align*}
\]

(A.2)
Let B is a point of coordinate \( z_b \) and the in-plane displacement components of this point can be written as

\[
\begin{align*}
    u_b &= u_o - \alpha_x z_b \\
    v_b &= v_o - \alpha_y z_b
\end{align*}
\]

(A.3)

and in general
where \( z \) is the through the thickness coordinate of a general point of the cross section. For small displacements, the expressions for the strains can be written as

\[
\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} - z \frac{\partial^2 w}{\partial x^2}
\]

\[
\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_o}{\partial y} - z \frac{\partial^2 w}{\partial y^2}
\]

\[
\gamma_{xy} = \gamma_y = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y}
\]

The above expressions has two components; first one is the strain components on reference plane (mid-plane strains) and the other components is the curvatures. The strain components on reference plane can be written as

\[
\varepsilon_x^o = \frac{\partial u_o}{\partial x}
\]
\[ \varepsilon_y^o = \frac{\partial v_o}{\partial y} \] 
(A.6)

\[ \gamma_{xy}^o = \gamma_x^o = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \]

and the curvatures of the laminate can be written as

\[ k_x = -\frac{\partial^2 w}{\partial x^2} \]

\[ k_y = -\frac{\partial^2 w}{\partial y^2} \]

\[ k_{xy} = k_s = -\frac{2\partial^2 w}{\partial x \partial y} \]

The strains at any point in the laminate located at “z” distance from the mid-plane can be related to the reference plane strains and the laminate curvatures as follows

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_s
\end{bmatrix} = \begin{bmatrix}
\varepsilon_x^o \\
\varepsilon_y^o \\
\gamma_s^o
\end{bmatrix} + z \begin{bmatrix}
k_x \\
k_y \\
k_s
\end{bmatrix}
\] 
(A.8)
A.3. Stress-strain relations of a layer within a laminate

Consider an individual layer \( k \) in a laminate, whose mid-plane is at a distance \( z_k \) from the laminate reference plane. The stress-strain relations of an arbitrary layer referred to its material axes are

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{6k}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{21} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{6k}
\end{bmatrix}
\]

(A.9)
However, the direction of external load application may not always coincide with the lamina principle axis. To know the stresses along loading direction, the stress components referred to the principal material axes \((1,2)\) can be transformed to the loading axes \((x, y)\) by the following transformation relations.

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_s
\end{bmatrix}
= [T]^{-1}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_6
\end{bmatrix}
\]

(A.10)
where transformation matrix
\[
[T] = \begin{bmatrix}
 m^2 & n^2 & 2mn \\
 n^2 & m^2 & -2mn \\
 -mn & mn & m^2 - n^2
\end{bmatrix}
\]

where \( m = \cos \theta, n = \sin \theta \).

and after transformation to the laminate coordinate system, the stress-strain relations in layer “k” can be written as

\[
\begin{bmatrix}
 \sigma_x \\
 \sigma_y \\
 \tau_{x_k} \\
 \tau_{y_k}
\end{bmatrix} = \begin{bmatrix}
 Q_{xx} & Q_{xy} & Q_{xx} \\
 Q_{yx} & Q_{yy} & Q_{ys} \\
 Q_{xx} & Q_{sy} & Q_{ss}
\end{bmatrix} \begin{bmatrix}
 \varepsilon_x \\
 \varepsilon_y \\
 \gamma_{x_k} \\
 \gamma_{y_k}
\end{bmatrix}
\]  

(A.11)

where

\[
Q_{xx} = m^4 Q_{11} + n^4 Q_{22} + 2m^2 n^2 Q_{12} + 4m^2 n^2 Q_{66}
\]

\[
Q_{yy} = n^4 Q_{11} + m^4 Q_{22} + 2m^2 n^2 Q_{12} + 4m^2 n^2 Q_{66}
\]

\[
Q_{xy} = m^2 n^2 Q_{11} + m^2 n^2 Q_{22} + (m^4 + n^4)Q_{12} - 4m^2 n^2 Q_{66}
\]

\[
Q_{xx} = m^3 n Q_{11} - mn^3 Q_{22} - mn(m^2 - n^2)Q_{12} - 2mn(m^2 - n^2)Q_{66}
\]  

(A.12)

\[
Q_{ys} = mn^3 Q_{11} - m^3 n Q_{22} + mn(m^2 - n^2)Q_{12} + 2mn(m^2 - n^2)Q_{66}
\]

\[
Q_{ss} = m^2 n^2 Q_{11} - m^2 n^2 Q_{22} - 2m^2 n^2 Q_{12} + (m^2 - n^2)Q_{66}
\]
substituting the expressions given in equation A.8 in equation A.11, the stress in layer ‘k’, along the loading axis can be written as

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_s
\end{bmatrix}_k = \begin{bmatrix}
Q_{xx} & Q_{xy} & Q_{xs} \\
Q_{yx} & Q_{yy} & Q_{ys} \\
Q_{sx} & Q_{sy} & Q_{ss}
\end{bmatrix}_k \begin{bmatrix}
\epsilon_x^o \\
\epsilon_y^o \\
\gamma_s^o
\end{bmatrix} + z \begin{bmatrix}
Q_{xx} & Q_{xy} & Q_{xs} \\
Q_{yx} & Q_{yy} & Q_{ys} \\
Q_{sx} & Q_{sy} & Q_{ss}
\end{bmatrix}_k \begin{bmatrix} k_x \\
k_y \\
k_s
\end{bmatrix}
\]  

(A.13)

or, in brief

\[
\begin{bmatrix}
\sigma
\end{bmatrix}^k_{x,y} = \begin{bmatrix}
Q
\end{bmatrix}^k_{x,y} \begin{bmatrix}
\epsilon^o
\end{bmatrix}_{x,y} + z\begin{bmatrix}
Q
\end{bmatrix}^k_{x,y} \begin{bmatrix}
k
\end{bmatrix}_{x,y}
\]  

(A.14)

where

\[
\begin{bmatrix}
\epsilon^o
\end{bmatrix}_{x,y} = \text{reference plane strains}
\]

\[
\begin{bmatrix}
k
\end{bmatrix}_{x,y} = \text{curvatures of the laminate}
\]

\[
\begin{bmatrix}
Q
\end{bmatrix}^k_{x,y} = \text{transformed stiffness matrix of the } k^{th} \text{ layer.}
\]
A.3. Forces and Moments

The forces and moments considered for a layer “k” of a laminate is given in figure A.4. The stresses in layer can be written in terms of forces and moments as given below.

\[
N^k_x = \int_{-t/2}^{t/2} \sigma_x dz
\]

\[
N^k_y = \int_{-t/2}^{t/2} \sigma_y dz
\]  \hspace{1cm} (A.15)

\[
N^k_s = \int_{-t/2}^{t/2} \tau_s dz
\]

and
\[ M_x^k = \int_{-t/2}^{t/2} \sigma_x z \, dz \]

\[ M_y^k = \int_{-t/2}^{t/2} \sigma_y z \, dz \]

\[ M_z^k = \int_{-t/2}^{t/2} \tau_z z \, dz \]  

(A.16)

where

\[ Z = \text{through the thickness coordinate of a point in the cross section.} \]

\[ t = \text{layer thickness} \]

\[ N^k_x, N^k_y = \text{normal forces per unit length} \]

\[ N^k_s = \text{shear force per unit length} \]

\[ M^k_x, M^k_y = \text{bending moments per unit length.} \]

\[ M^k_z = \text{twisting moment per unit length} \]

In the case of multilayer laminate, the total force and moment resultants are obtained by summing the effects for all layers. Thus, for
the laminate with n number of layers as given in the figure A.5, the force and moment resultants can be given as

\[
\begin{bmatrix}
N_x \\
N_y \\
N_z
\end{bmatrix}
= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_x
\end{bmatrix}
\, dz
\]  

(A.17)

Figure A.5. General construction of a laminate.
and

\[
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_s
\end{bmatrix} \, dz
\]

(A.18)

where \(z_k\) and \(z_{k-1}\) are the \(z\) coordinates of the upper and lower surfaces of layer \(k\).

Substituting Equation A.13 for the layer stresses in equations A.17 and A.18, we obtain

\[
\begin{bmatrix}
N_x \\
N_y \\
N_z
\end{bmatrix} = \sum_{k=1}^{n} \left\{ \begin{bmatrix}
Q_{xx} & Q_{xy} & Q_{xz} \\
Q_{yx} & Q_{yy} & Q_{yz} \\
Q_{zx} & Q_{zy} & Q_{zz}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_s^0
\end{bmatrix} \int_{z_{k-1}}^{z_k} dz + \begin{bmatrix}
Q_{xx} & Q_{xy} & Q_{xz} \\
Q_{yx} & Q_{yy} & Q_{yz} \\
Q_{zx} & Q_{zy} & Q_{zz}
\end{bmatrix} \begin{bmatrix}
k_x \\
k_y \\
k_z
\end{bmatrix} \int_{z_{k-1}}^{z_k} z \, dz \right\}
\]

...............(A.19)

and

\[
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix} = \sum_{k=1}^{n} \left\{ \begin{bmatrix}
Q_{xx} & Q_{xy} & Q_{xz} \\
Q_{yx} & Q_{yy} & Q_{yz} \\
Q_{zx} & Q_{zy} & Q_{zz}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_s^0
\end{bmatrix} \int_{z_{k-1}}^{z_k} z \, dz + \begin{bmatrix}
Q_{xx} & Q_{xy} & Q_{xz} \\
Q_{yx} & Q_{yy} & Q_{yz} \\
Q_{zx} & Q_{zy} & Q_{zz}
\end{bmatrix} \begin{bmatrix}
k_x \\
k_y \\
k_z
\end{bmatrix} \int_{z_{k-1}}^{z_k} z^2 \, dz \right\}
\]

............... (A.20)
The above expressions can be combined to form one general expression that relates the in-plane forces and moments to reference plane strains and curvatures. The expression thus obtained is given below.

\[
\begin{bmatrix}
N_x \\
N_y \\
N_z \\
M_x \\
M_y \\
M_z
\end{bmatrix} =
\begin{bmatrix}
A_{xx} & A_{xy} & A_{xz} & B_{xx} & B_{xy} & B_{xz} \\
A_{yx} & A_{yy} & A_{yz} & B_{yx} & B_{yy} & B_{yz} \\
A_{zx} & A_{zy} & A_{zz} & B_{zx} & B_{zy} & B_{zz} \\
B_{xx} & B_{xy} & B_{xz} & D_{xx} & D_{xy} & D_{xz} \\
B_{yx} & B_{yy} & B_{yz} & D_{yx} & D_{yy} & D_{yz} \\
B_{zx} & B_{zy} & B_{zz} & D_{zx} & D_{zy} & D_{zz}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x^o \\
\varepsilon_y^o \\
\gamma_{sz}^o \\
k_x \\
k_y \\
k_z
\end{bmatrix}
\]  

(A.21)

Or, in brief,

\[
\begin{bmatrix}
N \\
M
\end{bmatrix} =
\begin{bmatrix}
A \cdot B \\
B \cdot D
\end{bmatrix}
\begin{bmatrix}
\varepsilon^o \\
k
\end{bmatrix}
\]  

(A.22)

where

\[
A_{ij} = \sum_{k=1}^{n} Q_{ij}^k (z_k - z_{k-1})
\]  

(A.23)

\(A_{ij}\) is the 3 x 3 matrix of extensional stiffnesses, or in-plane laminate moduli, relating in-plane loads to in-plane strains.

\[
B_{ij} = \frac{1}{2} \sum_{k=1}^{n} Q_{ij}^k (z_k^2 - z_{k-1}^2)
\]  

(A.24)
\( B_{ij} \) is the 3 x 3 matrix of coupling stiffnesses, or in-plane /flexure coupling laminate moduli, relating in-plane loads to curvatures and moments to in-plane strains. For symmetric laminates \( B_{ij} \) terms will be zero.

\[
D_{ij} = \frac{1}{3} \sum_{k=1}^{n} Q^j_k (z^3_k - z^3_{k-1})
\]

\( D_{ij} \) is the 3 x 3 matrix of bending or flexural laminate stiffness relating moments to curvatures. Where \( i,j = x,y,s \).

The load deformation relations given in equation A.22 can be inverted to express strains and curvatures as a function of applied loads and moments. In brief these can be written as

\[
\begin{bmatrix}
\varepsilon^0 \\
\vdots \\
k
\end{bmatrix} = \begin{bmatrix} a & b \\ a & b \\ \vdots & \vdots \\ c & d \end{bmatrix} \begin{bmatrix} N \\
\vdots \\
M
\end{bmatrix}
\]

(A.26)

here matrices \([a],[b],[c] \) and \([d] \) are the laminate compliance matrices obtained from the stiffness matrices as follows

\[
[a] = [A^{-1} - \{[B^*][D^{-1}]\}] [C^*]
\]
\[ [b] = [B^*][D^{-1}] \]  \hspace{1cm} (A.27)

\[ [c] = -[D^{-1}][C^*] \]

\[ [d] = [D^{-1}] \]

where

\[ [A^{-1}] = \text{inverse of matrix } [A] \]

\[ [B^*] = -[A^{-1}][B] \]  \hspace{1cm} (A.28)

\[ [C^*] = [B][A^{-1}] \]

\[ [D^*] = [D] - [(B)[A^{-1}]][B] \]

The use of above expressions in the formulation developed for the present work is explained in sections 3.1, 3.2 and 3.3.
APPENDIX B

Residual Stresses

Residual stresses are introduced in composite laminates during fabrication. These stresses are similar in nature to the hygrothermal stresses. During processing at elevated temperatures, there is a certain temperature level at which the composite material is assumed to be stress free. This temperature level may be taken as the glass transition temperature of the polymer matrix, or the melting temperature of the metal matrix. Residual stresses develop in the initially stress free laminate if the thermally anisotropic plies are oriented along different directions. Residual stresses are the functions of the many parameters, such as ply orientation and stacking sequence, curing process, fiber volume ratio, and other material and processing variables.

The procedure for elastic analysis of thermal residual stresses consists of the following steps:

Determination of the free thermal strains in each layer is done by introducing the difference $\Delta T$ between ambient and stress-free temperature in the equations below.

Residual strain referred to the longitudinal material axes of the lamina
\[ e_1 = \alpha_1 \Delta T \]

Residual strain referred to the transverse material axes of the lamina

\[ e_2 = \alpha_2 \Delta T \]

\[ e_\delta = 0 \]

The transformed residual strains referred to the x-y coordinate system are

\[ e_x = e_2 m^2 + e_\delta n^2 \]

\[ e_y = e_2 m^2 + e_\delta n^2 \]

\[ e_{xy} = e_x = 2(e_1 - e_2)mn \]

where \( m = \cos \theta, n = \sin \theta \)

The residual force resultants are defined by

\[
\begin{bmatrix}
N^{HT}_{x,y}
\end{bmatrix}
= \sum_{k=1}^{s}
\begin{bmatrix}
Q_{sx} & Q_{xy} & Q_{sy} \\
Q_{yx} & Q_{yy} & Q_{ys} \\
Q_{sx} & Q_{sy} & Q_{ss}
\end{bmatrix}
\begin{bmatrix}
e_x \\
e_y \\
e_{xy}
\end{bmatrix} t_k
\]

The residual moment resultants are defined as
The force-deformation and moment-deformation relations are identical to those derived for mechanical loading and defined as

$$
\begin{bmatrix}
N^{HT}
\end{bmatrix}_{x,y} = \begin{bmatrix}
N_x^{HT} \\
N_y^{HT} \\
N_z^{HT}
\end{bmatrix} = \sum_{k=1}^{n} \begin{bmatrix}
Q_{xx} & Q_{xy} & Q_{xz} \\
Q_{yx} & Q_{yy} & Q_{yz} \\
Q_{zx} & Q_{zy} & Q_{zz}
\end{bmatrix} \begin{bmatrix}
e_x \\
e_y \\
e_z
\end{bmatrix} \bar{z}_k t_k
$$

(B.3)

The above relations B.2 and B.3 can also be presented in a combined concise form as

$$
\begin{bmatrix}
\bar{N}
\end{bmatrix} = \begin{bmatrix}
\bar{N}
\end{bmatrix}_{x,y} = \begin{bmatrix}
\bar{N}
\end{bmatrix}_{x,y} + \begin{bmatrix}
\bar{N}^{HT}
\end{bmatrix}_{x,y}
$$

(B.4)

$$
\begin{bmatrix}
\bar{M}
\end{bmatrix} = \begin{bmatrix}
\bar{M}
\end{bmatrix}_{x,y} = \begin{bmatrix}
\bar{M}
\end{bmatrix}_{x,y} + \begin{bmatrix}
\bar{M}^{HT}
\end{bmatrix}_{x,y}
$$

where \( \begin{bmatrix}
\bar{N}
\end{bmatrix} \) and \( \begin{bmatrix}
\bar{M}
\end{bmatrix} \) are total force and moment resultants equal to the respective sums of their mechanical and residual components. In the absence of mechanical loading \( \begin{bmatrix}
\bar{N}
\end{bmatrix}_{x,y} \) and \( \begin{bmatrix}
\bar{M}
\end{bmatrix}_{x,y} \) are zero. Then

$$
\begin{bmatrix}
\bar{N}
\end{bmatrix} = \begin{bmatrix}
\bar{N}^{HT}
\end{bmatrix}_{x,y} \quad \text{and} \quad \begin{bmatrix}
\bar{M}
\end{bmatrix} = \begin{bmatrix}
\bar{M}^{HT}
\end{bmatrix}_{x,y}
$$

The force-deformation and moment-deformation relations are identical to those derived for mechanical loading and defined as
The reference plane strains and curvatures of the laminate are obtained from

\[
\begin{bmatrix}
\varepsilon^0 \\
\vdots \\
k
\end{bmatrix} = \begin{bmatrix} a:b \\ \vdots \\ c:d \end{bmatrix} \begin{bmatrix} N \\ \vdots \\ M \end{bmatrix}
\]  \hspace{1cm} (B.6)

The total strains in layer \( k \) at a distance \( z \) from the reference plane are

\[
\begin{bmatrix}
\varepsilon_{sc} \\
\varepsilon_{yc} \\
\gamma_{se}^k
\end{bmatrix} = \begin{bmatrix} \varepsilon^0_s \\
\varepsilon^0_y \\
\lambda^0_s
\end{bmatrix} + z \begin{bmatrix} k_x \\
k_y \\
k_z
\end{bmatrix}
\]  \hspace{1cm} (B.7)

The stresses in layer \( k \) referred to the laminate coordinate axes (x, y) can be obtained by

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_s
\end{bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\
Q_{yx} & Q_{yy} & Q_{ys} \\
Q_{sx} & Q_{sy} & Q_{ss}
\end{bmatrix} \begin{bmatrix}
\varepsilon_{sc} \\
\varepsilon_{yc} \\
\gamma_{se}^k
\end{bmatrix}
\]  \hspace{1cm} (B.8)
These stresses can then be transformed to the lamina axes (1, 2) by the transformation relation

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_6
\end{bmatrix} = [T] \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_s
\end{bmatrix}
\]

(B.9)

The importance of residual stresses and the incorporation of these residual stresses in present formulation is discussed in section 3.4
APPENDIX C

Tsai-Hill Theory

Theories of failures will explain the condition to identify failure in a lamina. The conditions will be put forward in forms of equations explaining the relation between various stress components to strengths. Various failure theories are available to identify a failure in composite lamina. Literature survey has shown that Tsai-Hill criterion is the simplest and accurate to predict the failure. The important steps in this failure theory are explained below.

For a two dimensional stress state referred to the principal stress directions, the von Mises yield criterion has the following form

\[ \sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 = \sigma_{yp}^2 \]  

(C.1)

where \( \sigma_{yp} \) is the yield stress.

This criterion is modified for the case of anisotropic ductile materials and proposed the following form is proposed.

\[ A \sigma_1^2 + B \sigma_2^2 + C \sigma_1 \sigma_2 + D \tau_0^2 = 1 \]  

(C.2)
where \( A, B, C \) and \( D \) are material parameters characteristic of the state of anisotropy in the material. This criterion is adapted to orthotropic materials, such as unidirectional lamina with transverse isotropy. The parameters in Equation C.2 can be related to the basic strength parameters of the lamina by conducting real or imaginary elementary experiments.

By assuming a uni-axial longitudinal loading, the failure can be expected to occur when \( \sigma_1'', \sigma_2 = \tau_6 = 0 \). \( F_1 \) is the longitudinal strength. Substituting these terms in Equation C.2 yields

\[
A = \frac{1}{F_1^2} \quad \text{(C.3)}
\]

By assuming a uni-axial transverse loading, the failure can be expected to occur when \( \sigma_2'' = F_2, \sigma_1 = \tau_6 = 0 \). \( F_2 \) is the transverse strength. Substituting these terms in Equation C.2 yields

\[
B = \frac{1}{F_2^2} \quad \text{(C.4)}
\]
By assuming the in-plane shear loading, the failure can be expected to occur when \( \sigma_1 = \sigma_2 = 0, \ \tau^u = F_6 \), \( F_6 \) is the shear strength. Substituting these terms in Equation C.2 yields

\[
D = \frac{1}{F_6^2} \tag{C.5}
\]

The subscript “\( u \)” indicates ultimate value of stress at failure.

A bi-axial test is needed to determine the parameter \( C \) that accounts for the interaction between normal stress \( \sigma_1 \) and \( \sigma_2 \). It is assumed that under equal bi-axial normal loading \( \sigma_1 = \sigma_2 \neq 0, \ \tau^u = 0 \), the material will fail according to maximum stress criterion. This means, the failure will occur when the transverse stress \( \sigma_2 \) reaches the transverse strength value, \( F_2 \). The value of \( F_2 \) is used here, since it is much lower than the longitudinal strength \( F_1 \). Equation C.2 then yields

\[
C = \frac{1}{F_1^2} \tag{C.6}
\]
When the values of terms A, B, C and D are substituted in Equation C.2, the Tsai-Hill criterion for a two dimensional state of stress can be written as

$$\frac{\sigma_1^2}{F_1^2} + \frac{\sigma_2^2}{F_2^2} + \frac{\tau_{12}^2}{F_{12}^2} - \frac{\sigma_1 \sigma_2}{F_1^2} = 1$$  \hspace{1cm} (C.7)

In the above equation no distinction is made between tensile and compressive strengths. However, the appropriate strength values are to be used in Equation C.7 according to the nature of normal stresses \(\sigma_1\) and \(\sigma_2\). Thus

\[ F_1 = F_{1t} \text{ when } \sigma_1 > 0 \text{; } F_{1c} \text{ when } \sigma_1 < 0 \]

\[ F_2 = F_{2t} \text{ when } \sigma_2 > 0 \text{; } F_{2c} \text{ when } \sigma_2 < 0 \]

(C.8)
List of Publications


