CHAPTER 3

SPECTRAL ANALYSIS USING NONPARAMETRIC METHODS

3.1 Introduction:

In this section, we consider nonparametric techniques of spectrum estimation. These methods are based on the idea of estimating the autocorrelation sequence of a random process from a set of measured data, and then taking the Fourier transform to obtain an estimate of the power spectrum. We begin with the periodogram, a nonparametric method first introduced by Schuster in 1898 in his study of periodicities in sunspot numbers [64,65][47, 49]. As we will see, although the periodogram is easy to compute, it is limited in its ability to produce an accurate estimate of the power spectrum, particularly for short data records. We will then examine a number of modifications to the periodogram that have been proposed to improve its statistical properties. These include the modified periodogram, Bartlett’s method, Welch’s method, and the Blackman-Tukey method.

3.2 The Periodogram

The power spectrum of a wide-sense stationary random process is the Fourier transform of the autocorrelation sequence,
\[ P_x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_x(k)e^{-j\omega k} \quad (3.1) \]

Therefore, spectrum estimation is, in some sense, an autocorrelation estimation problem. An autocorrelation ergodic process and an unlimited amount of data, the autocorrelation sequence may, in theory, be determined using the time-average

\[ r_x(k) = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n+k)x^*(n) \quad (3.2) \]

However, if \( x(n) \) is only measured over a finite interval, say \( n = 0, 1, \ldots, N - 1 \), then the autocorrelation sequence must be estimated using, for example, Eq. (3.2) with a finite sum,

\[ \hat{r}(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n+k)x^*(n) \quad (3.3) \]

In order to ensure that the values of \( x(n) \) that fall outside the interval \([0, N - 1]\) are excluded from the sum, Eq. (3.3) will be rewritten as follows:

\[ \hat{r}(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} x(n+k)x^*(n) \quad ; \quad k = 0, 1, \ldots, N - 1 \quad (3.4) \]

With the values of \( \hat{r}_x(k) \) for \( k < 0 \) defined using conjugate symmetry, \( \hat{r}_x(-k) = \hat{r}_x^*(k) \), and \( \hat{r}_x(k) \) set equal to zero for \( |k| \geq N \). Taking the discrete-time Fourier transform of \( \hat{r}_x(k) \) leads to an estimate of the power spectrum known as the periodogram,

\[ P_{\text{per}}(e^{j\omega}) = \sum_{k=-N+1}^{N-1} \hat{r}_x(k)e^{-j\omega k} \quad (3.5) \]
Although defined in terms of the estimated autocorrelation sequence $\hat{r}_x(k)$, it will be more convenient to express the periodogram directly in terms of the process $x(n)$ and this may be done as follows. Let $x_N(n)$ be the finite length signal of length $N$ that is equal to $x(n)$ over the interval $[0, N - 1]$, and is zero otherwise,

$$x_N(n) = \begin{cases} x(n) & ; \ 0 \leq n < N \\ 0 & ; \ \text{otherwise} \end{cases} \quad (3.6)$$

Thus, $x_N(n)$ is the product $x(n)$ with a rectangular window $w_k(n)$,

$$x_N(n) = W_R(n)x(n) \quad (3.7)$$

In terms of $x_N(n)$, the estimated autocorrelation sequence may be written as follows:

$$\hat{r}_x(k) = \frac{1}{N} \sum_{n=-\infty}^{\infty} x_N(n+k)x_N^*(n) = \frac{1}{N} x_N(k) \ast x_N^*(-k) \quad (3.8)$$

Taking the Fourier transform and using the convolution theorem, the periodogram becomes

$$P_{per}(e^{j\omega}) = \frac{1}{N} X_N(e^{j\omega})X_N^*(e^{j\omega}) = \frac{1}{N} \left| X_N(e^{j\omega}) \right|^2 \quad (3.9)$$

Where $X_N(e^{j\omega})$ is the discrete-time Fourier transform of the $N$-point data sequence $x_N(n)$,

$$X_N(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_N(n)e^{-j\omega n} = \sum_{n=0}^{N-1} x(n)e^{-j\omega n} \quad (3.10)$$

Thus, the periodogram is proportional to the squared magnitude of the DTFT of $x_N(n)$, and may be easily computed using a DFT as
$$x_N(n) \rightarrow X_N(k) \rightarrow \frac{1}{N} |X_N(k)|^2 = P_{\text{per}}(e^{j2\pi k/N})$$

As we now show, the periodogram has an interesting interpretation in terms of filter banks. Let \(h_i(n)\) be an FIR filter of length \(N\) that is defined as

$$h_i(n) = \frac{1}{N} e^{j\omega_0 i} w_R(n) = \begin{cases} \frac{1}{N} e^{j\omega_0 i} ; & 0 < n < N \\ 0 ; & \text{otherwise} \end{cases}$$

(3.11)

The frequency response of this filter is

$$H_i(e^{j\omega}) = \sum_{n=0}^{N-1} h_i(n) e^{-j\omega n} = e^{-j(\omega - \omega_0)} \frac{\sin[N(\omega - \omega_0)/2]}{N \sin[\omega - \omega_0]/2}$$

(3.12)

and, is a band pass filter with a center frequency \(\omega_0\), and a bandwidth that is approximately equal to \(\Delta \omega = 2\pi/N\). If a WSS random process \(x(n)\) is filtered with \(h_i(n)\), then the output process is

$$Y_i(n) = x(n) * h_i(n) = \sum_{k=n-N+1}^{n} x(k) h_i(n-k) = \frac{1}{N} \sum_{k=n-N+1}^{n} x(k) e^{j(n-k)\omega_0}$$

(3.13)

Since \(\left|H_i(e^{j\omega})\right|_{\omega = \omega_0} = 1\), then the power spectrum of \(x(n)\) and \(y(n)\) are equal at frequency \(\omega_0\), \(P_x(e^{j\omega_0}) = P_y(e^{j\omega_0})\)

Furthermore, if the bandwidth of the filter is small enough so that the power spectrum of \(x(n)\) may be assumed to be approximately constant over the pass band of the filter.

The magnitude of the frequency response of the band pass filter used in the filter bank interpretation of the periodogram is calculated
Power in $y_1(n)$ will be approximately

$$E\left\{ |y_1(n)|^2 \right\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_x(e^{j\omega})|H_1(e^{j\omega})|^2 d\omega = \frac{\Delta_0}{2\pi} P_x(e^{j\omega}) = \frac{1}{N} P_x(e^{j\omega})$$

and,

$$P_x(e^{j\omega}) \approx NE\left\{ |y_1(n)|^2 \right\}$$

Thus, if we are able to estimate the power in $y_1(n)$, then the power spectrum at frequency $\omega_i$ may be estimated as follows:

$$\hat{P}_x(e^{j\omega_i}) \approx \hat{N}E\left\{ |y_1(n)|^2 \right\}$$

One simple yet very crude way to estimate the power is to use a one-point sample average, $\hat{E}\left\{ |y_1(n)|^2 \right\} = |y_1(N-1)|^2$

From Eq. (3.13) we see that this is equivalent to

$$|y_1(N-1)|^2 = \frac{1}{N} \left\{ \sum_{k=0}^{N-1} x(k) e^{-j\omega_i} \right\}^2$$

Therefore,

$$\hat{P}_x(e^{j\omega_i}) = N|y_1(N-1)|^2 = \frac{1}{N} \left\{ \sum_{k=0}^{N-1} x(k) e^{-j\omega_i} \right\}^2$$

that is equivalent to the periodogram.
It was shown that the periodogram is proportional to the squared magnitude of the DTFT of the finite length sequence $x_N(n)$. Therefore, from a computational point of view, the periodogram is simple to evaluate. In this section, we look at the performance of the periodogram. Ideally, as the length of the data record increases, the periodogram should converge to the power spectrum of the process, $\hat{P}_{\text{per}}(e^{j\omega})$ is a function of the random variables $x(0),...,x(N)$, it is necessary to consider convergence in a statistical sense. Therefore, in this section, we will look at mean-square convergence of the periodogram [31, 32] i.e., we will be interested in whether or not

$$
\lim_{N \to \infty} \mathbb{E}\left[\left(\hat{P}_{\text{per}}(e^{j\omega})\right)^2\right] = 0
$$

In order for the periodogram to be mean-square convergent, it is necessary that it be asymptotically unbiased

$$
\lim_{N \to \infty} \mathbb{E}\left[\hat{P}_{\text{per}}(e^{j\omega})\right] = P_x(e^{j\omega}) \tag{3.18}
$$

and have a variance that goes to zero as the data record length $N$ goes to infinity,

$$
\lim_{N \to \infty} \mathbb{E}\left[\hat{P}_{\text{per}}(e^{j\omega})\right] = 0 \tag{3.19}
$$
In other words, $P_{\text{per}}(e^{j\omega})$ must be a consistent estimate of the power spectrum. First, we consider the bias of the periodogram.

### 3.2.1 Periodogram Bias

To compute the bias of the periodogram, we begin by finding the expected value of $\hat{r}_x(k)$. From Eq. (3.4) it follows that the expected value of $\hat{r}_x(k)$ for $k = 0, 1, \ldots N-1$ is

$$E\left\{ \hat{r}_x(k) \right\} = \frac{1}{N} \sum_{n=0}^{N-1-k} E\{x(n+k)x^*(n)\} = \frac{1}{N} \sum_{n=0}^{N-1-k} r_x(k) = \frac{N - K}{N} r_x(k)$$

And, for $k \geq N$, the expected value is zero. Using the conjugate symmetry of $\hat{r}_x(k)$ we have

$$E\left\{ \hat{r}_x(k) \right\} = w_B(k) r_x(k) \quad (3.20)$$

Where

$$w_B(k) = \begin{cases} \frac{N - |k|}{N} & ; |k| \leq N \\ 0 & ; |k| > N \end{cases} \quad (3.21)$$

is a Bartlett (triangular) window. Therefore, $\hat{r}_x(k)$ is a biased estimate of the auto correlation. Since the Bartlett window is applied to the autocorrelation sequence, $w_B(k)$ is referred to as a lag window. This is in contrast to a data window that is applied to $x(n)$.

Using Eq. (3.20) it follows that the expected value of the periodogram

$$E\left\{ P_{\text{per}}(e^{j\omega}) \right\} = E\left\{ \sum_{k=\frac{N}{N+1}}^{N} \hat{r}_x(k)e^{jk\omega} \right\} = \sum_{k=\frac{N}{N+1}}^{N} E(r_x(k)e^{jk\omega})$$
\[ \sum_{k=\infty}^{\infty} r_x(k)w_B(k) e^{j\omega k} \]  \hspace{1cm} (3.22)

Since \( \text{E}\left( \hat{P}_{\text{per}}(e^{j\omega}) \right) \) is the Fourier transform of the product \( r_x(k)w_B(k) \), using the frequency convolution theorem we have

\[ \text{E}\left( \hat{P}_{\text{per}}(e^{j\omega}) \right) = \frac{1}{2\pi} P_x(e^{j\omega}) W_B(e^{j\omega}) \]  \hspace{1cm} (3.23)

Where \( W_B(e^{j\omega}) \) is the Fourier transform of the Bartlett window,

\[ w_B(k), \quad W_B(e^{j\omega}) = \frac{1}{N} \sin\left(\frac{N\omega}{2}\right)^2 \]  \hspace{1cm} (3.24)

Thus, the expected value of the periodogram is the convolution of the power spectrum \( P_x(e^{j\omega}) \) with the Fourier transform of a Bartlett window and, therefore, the periodogram is a biased estimate. However, since \( W_B(e^{j\omega}) \) converges to an impulse as \( N \) goes to infinity, the periodogram is asymptotically unbiased

\[ \lim_{N \to \infty} \text{E}\left( \hat{P}_{\text{per}}(e^{j\omega}) \right) = P_x(e^{j\omega}) \]  \hspace{1cm} (3.25)

To illustrate the effect of the lag window, \( W_B(k) \), on the expected value of the periodogram, consider a random process consisting of a random phase sinusoid in white noise \( x(n) = A \sin(n\omega + \phi + v(n) \]

Where \( \phi \) is random variable that is uniformly distributed over the interval \( [-\pi, \pi] \), and \( v(n) \) is white noise with a variance \( \sigma_v^2 \). The power spectrum of \( x(n) \) is

\[ P_x(e^{j\omega}) = \sigma_v^2 + \frac{1}{2} \pi A^2 [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \]
Therefore, it follows from Eq. (3.23) that the expected value of the periodogram is

\[
\hat{P}_{\text{per}}(e^{j\omega}) = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega})
\]

\[
= \sigma_v^2 + \frac{1}{4} A^2 \left[ W_B(e^{j(\omega_0 - \omega)}) + W_B(e^{j(\omega_0 + \omega)}) \right]
\]

(3.26)

There are two effects that should be noted in this example. First, is the spectral smoothing that is produced by \( W_B(e^{j\omega}) \), which leads to a spreading of the power in the sinusoid over a band of frequencies that has a bandwidth of approximately \( 4\pi / N \). The second effect is the power leakage through the side lobes of the window, which creates secondary spectral peaks at frequencies \( \omega_k \approx \omega_c \pm \frac{2\pi}{N} k \). As we know that, it is possible for these side lobes to mask low-level narrowband components.

In addition to biasing the periodogram, the smoothing that is introduced by the Bartlett window also limits the ability of the periodogram to resolve closely-spaced narrowband components in \( x(n) \). Consider, for example, a random process consisting of two sinusoids in white noise \( x(n) = A_1 \sin(n\omega_1 + \phi_1) + A_2 \sin(n\omega_2 + \phi_2) + \nu(n) \)

Where \( \phi_1 \) and \( \phi_2 \) are uncorrelated uniformly distributed random variables and where \( \nu(n) \) is white noise with a variance of \( \sigma_v^2 \). The power spectrum of \( x(n) \) is

\[
P_x(e^{j\omega}) = \sigma_v^2 + \frac{1}{2} \pi A_1^2 \delta(\omega - \omega_1) + \nu_0(\omega + \omega_1) + \frac{1}{2} \pi A_2^2 \delta(\omega - \omega_2) + \delta(\omega + \omega_2)
\]

and the expected value of the periodogram is
\[
E\left[\hat{P}_{\text{per}}(e^{j\omega})\right] = \frac{1}{2\pi} P_X(e^{j\omega}) * W_B(e^{j\omega})
\]

\[
= \sigma_v^2 + \frac{1}{4} A_1^2 \left[ W_B(e^{j(\omega - \omega_1)}) + W_B(e^{j(\omega + \omega_1)}) \right]
+ \frac{1}{4} A_2^2 \left[ W_B(e^{j(\omega - \omega_2)}) + W_B(e^{j(\omega + \omega_2)}) \right] \quad (3.27)
\]

Since the width of the main lobe of \( W_B(e^{j\omega}) \) increases as the data record length decreases, for a given data record length, \( N \), there is a limit on how closely two sinusoids or two narrowband processes may be located before they can no longer be resolved. One way to define this resolution limit is to set \( \Delta\omega \) equal to the width of the main lobe of the spectral window, \( W_B(e^{j\omega}) \), as its “half-power” of 6dB point. For the Bartlett window, \( \Delta\omega = 0.89(2\pi/N) \), which means that the resolution of the periodogram is

\[
\text{Res}\left[\hat{P}_{\text{per}}(e^{j\omega})\right] = 0.89 \frac{2\pi}{N} \quad (3.28)
\]

It is important to note, however, that Eq. (3.28) is nothing more than a rule of thumb that should only be used as a guideline in determining the amount of data that is necessary for a given resolution. There is nothing sacred, for example, with the proportionality constant 0.89(2\pi). On the other hand, what is important is the fact that the resolution is inversely in Eq. (3.28) and is somewhat arbitrary, one generally finds that it is difficult to resolve details in the spectrum that are much finer than this.
3.2.2 Variance of the Periodogram. We have seen that the periodogram is an asymptotically unbiased estimate of the power spectrum. In order for it to be a consistent estimate, it is necessary that the variance go to zero as $N \to \infty$. Unfortunately, it is difficult to evaluate the variance of the periodogram for an arbitrary process $x(n)$ since the variance depends on the fourth-order moments of the process. However, as we show next, the variance may be evaluated in the special case of white Gaussian noise.

Let $x(n)$ be a Gaussian white noise process with variance $\sigma_x^2$. Using Eq. (8.9) the periodogram may be expressed as follows:

$$\hat{P}_{\text{per}}(e^{j\omega}) = \frac{1}{N} \left\{ \sum_{k=0}^{N-1} x(k)e^{-jk\omega} \right\}^2 = \frac{1}{N} \left\{ \sum_{k=0}^{N-1} x(k)e^{-jk\omega} \right\} \left\{ \sum_{l=0}^{N-1} x^*(l)e^{jl\omega} \right\}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} x(k)x^*(l)e^{-j(k-l)\omega} \quad (3.29)$$

Therefore, the second-order moment of the periodogram is

$$\mathbb{E}\left[ \hat{P}_{\text{per}}(e^{j\omega_1})\hat{P}_{\text{per}}(e^{j\omega_2}) \right]$$

$$= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \mathbb{E}\left[ x(k)x^*(l)x(m)x^*(n) \right] e^{j(k-l)\omega_1}e^{-jm\omega_2} \quad (3.30)$$

Which depends on the fourth-order moments of $x(n)$. Since $x(n)$ is Gaussian, we may use the moment factoring theorem to simplify these moments [17,44]. For complex Gaussian random variables, the moment factoring theorem is

$$\mathbb{E}\left[ x(k)x^*(l)x(m)x^*(n) \right] = \mathbb{E}\left[ x(k)x^*(l) \right]\mathbb{E}\left[ x(m)x^*(n) \right]$$

$$+ \mathbb{E}\left[ x(k)x^*(n) \right]\mathbb{E}\left[ x(kmx^*(l)) \right] \quad (3.31)$$
Substituting Eq. (3.31) into Eq. (3.30), the second-order moment of the periodogram becomes a sum of two terms. The first term contains products of $E[x(k)x^*(l)]$ with $E[x(m)x^*(n)]$. For white noise, these terms are equal to $\sigma_x^4$ when $k = 1$ and $m = n$, and they are equal to zero otherwise. Thus, the first term simplifies to

$$\frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \sigma_x^4 = \sigma_x^4 \quad (3.32)$$

The second term, on the other hand, contains products of $E[x(k)x^*(n)]$ with $E[x(m)x^*(l)]$. Again, for white noise, these terms are equal to $\sigma_x^4$ when $k = n$ and $l = m$, and they are equal to zero otherwise.

Therefore, the term becomes

$$\frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \sigma_x^4 e^{-jk(\omega_1 - \omega_2)} e^{j(k-l)\omega_1} = \frac{\sigma_x^4}{N^2} \sum_{k=0}^{N-1} e^{-jk(\omega_1 - \omega_2)} \sum_{l=0}^{N-1} e^{j(l)(\omega_1 - \omega_2)}$$

$$= \frac{\sigma_x^4}{N^2} \left[ 1 - e^{-jN(\omega_1 - \omega_2)} \right] \left[ 1 - e^{-jN(\omega_1 - \omega_2)} \right]$$

$$= \frac{\sigma_x^4}{N^2} \left[ \sin N(\omega_1 - \omega_2 / 2) \right]^2$$

Combining Eq. (3.32) and Eq. (3.33) it follows that

$$E\left( \hat{P}_{\text{per}}(e^{j\omega_1}) \hat{P}_{\text{per}}(e^{j\omega_2}) \right) = \sigma_x^4 \left\{ 1 + \left[ \frac{\sin N(\omega_1 - \omega_2 / 2)}{N \sin(\omega_1 - \omega_2 / 2)} \right]^2 \right\} \quad (3.34)$$

Since

$$\text{Cov}\left( \hat{P}_{\text{per}}(e^{j\omega_1}), \hat{P}_{\text{per}}(e^{j\omega_2}) \right) = E\left( \hat{P}_{\text{per}}(e^{j\omega_1}) \hat{P}_{\text{per}}(e^{j\omega_2}) \right) - E\left( \hat{P}_{\text{per}}(e^{j\omega_1}) \right) E\left( \hat{P}_{\text{per}}(e^{j\omega_2}) \right)$$
and $E\left[ \hat{P}_{\text{per}}(e^{j\omega}) \right] = \sigma_x^2$, then the covariance of the periodogram is

$$\text{Cov}\left[ \hat{P}_{\text{per}}(e^{j\omega_1}) \hat{P}_{\text{per}}(e^{j\omega_2}) \right] = \sigma_x^4 \left\{ 1 + \left[ \frac{\sin N(\omega_1 - \omega_2 / 2)}{N \sin(\omega_1 - \omega_2 / 2)} \right]^2 \right\}$$

(3.35)

Finally, setting $\omega_1 = \omega_2$ we have, for the variance,

$$\text{Var}\left[ \hat{P}_{\text{per}}(e^{j\omega}) \right] = \sigma_x^4$$

(3.36)

Thus, the variance does not go to zero as $N \to \infty$, and the periodogram is not a consistent estimate of the power spectrum. In fact, since $P_x(e^{j\omega} = \sigma_x^2)$ then the variance of the periodogram of white Gaussian noise is proportional to the square of the power spectrum,

$$\text{Var}\left[ \hat{P}_{\text{per}}(e^{j\omega}) \right] = P_x^2(e^{j\omega})$$

(3.37)

The analysis given above for the variance of the periodogram assumes that $x(n)$ is white Gaussian noise. Although the statistical analysis of a nonwhite Gaussian process is much more difficult, we may derive an approximate expression for the variance as follows. Recall that a random process $x(n)$ with power spectrum $P_x(e^{j\omega})$ may be generated by filtering unit variance white noise $\nu(n)$ with a linear shift-invariant filter $h(n)$ that has a frequency response $H(e^{j\omega})$ with

$$|H(e^{j\omega})|^2 = P_x(e^{j\omega})$$

(3.38)
As defined in Eq. (3.7), if \( x_N(n) \) and \( v_N(n) \) are sequences of length \( N \) that are formed by windowing \( x(n) \) and \( v(n) \), respectively, then the periodograms of these processes are

\[
\hat{P}_{per}(e^{j\omega}) = \frac{1}{N} |X_N(e^{j\omega})|^2 \tag{3.39}
\]

\[
\hat{P}_{per}(e^{j\omega}) = \frac{1}{N} |V_N(e^{j\omega})|^2 \tag{3.40}
\]

Although \( x_N(n) \) is not equal to the convolution of \( v_N(n) \) with \( h(n) \), if \( N \) is large compared to the length of \( h(n) \) so that the transient effects are small, then \( x_N(n) \approx h(n) \ast v_N(n) \)

Since

\[
|X_N(e^{j\omega})|^2 \approx |H(e^{j\omega})|^2 |V_N(e^{j\omega})|^2 = P_x(e^{j\omega}) P_{per}(e^{j\omega}) \tag{3.41}
\]

Substituting Eqs. (3.39) and (3.40) into Eq. (3.41) we have

\[
\hat{P}_{per}(e^{j\omega}) \approx \hat{P}_x(e^{j\omega}) \hat{P}_{per}(e^{j\omega})
\]

Therefore,

\[
\text{Var}\left[ \hat{P}_{per}(e^{j\omega}) \right] \approx P_x(e^{j\omega}) \text{Var}\left[ \hat{P}_{per}(e^{j\omega}) \right] \]

and, since the variance of the periodogram of \( v(n) \) is equal to one, then we have
Thus, assuming that N is large, the variance of the periodogram of Gaussian random process is proportional to the square of its power spectrum.

The second-order moment and covariance of the periodogram may be similarly generalized for nonwhite Gaussian noise. For the second-order moment, Eq. (3.34) becomes

\[
E\left[ \hat{P}_{\text{per}}(e^{j\omega_1}) \hat{P}_{\text{per}}(e^{j\omega_2}) \right] \approx P_x(e^{j\omega_1}) P_x(e^{j\omega_2}) \left[ 1 + \frac{\sin N(\omega_1 - \omega_2 / 2)}{N\sin(\omega_1 - \omega_2 / 2)} \right]^2
\]

[3.43]

and, for the covariance, Eq. (3.35) becomes

\[
\text{Cov}\left[ \hat{P}_{\text{per}}(e^{j\omega_1}) \hat{P}_{\text{per}}(e^{j\omega_2}) \right] \approx P_x(e^{j\omega_1}) P_x(e^{j\omega_2}) \frac{\sin N(\omega_1 - \omega_2 / 2)^2}{N\sin(\omega_1 - \omega_2 / 2)^2}
\]

(3.44)

Note that for large N, the term in brackets is approximately zero provided \( \omega_1 - \omega_2 \gg 2\pi / N \), which implies that there is little correlation between one frequency and another.

The performance of the nonparametric power spectrum estimation methods is studied on the following nonuniformly sampled data sequences (noisy signals);
• Stationary White Gaussian noise.

• Narrowband sinusoidal components in white Gaussian noise

• Narrowband sinusoidal components in broadband noise

Consider the case for a white Gaussian noise w(n). The White noise is having normal distribution with zero mean and unit variance. The white Gaussian noise can be modeled as \( N(0, 1) \) and the amplitude of white noise is always constant dc value. It consists of all the frequency components ranging from \(-\infty\) to \(+\infty\). The signals that are not sampled with equal sampling intervals are known as nonuniform samples sequences. The Nyquist rate (twice the highest frequency component of the signal) is less than the sampling rate for nonuniformly sampled data sequences.

### 3.2.3 Simulation Results and analysis of white noise:

The Periodograms of a simulated stationary White Gaussian w(n) for different lengths \( N=64, 256 \) and 512 are as illustrated from Figures 3.2 to 3.10. The plots suggest the performance characteristics of the periodogram for different number of realizations and the simulations are run by Montecarlo techniques. The spectral characteristics reveal that the periodogram, spectrum estimation for power is not a consistent estimate as the variance of the spectral estimate does not converge to a narrow value even though the bias approaches to a true value for the large number of observation of data samples.
Figure 3.2 White Gaussian noise of zero mean and unit variance for $N=64$.

Figure 3.3 Monte carlo simulations of 50 realizations for periodogram.

Figure 3.4 Average periodogram of 50 realizations.
Figure 3.5 White Gaussian noise of zero mean and unit variance for N=256.

Figure 3.6 Monte carlo simulations of 50 realizations for periodogram.

Figure 3.7 Average periodogram of 50 realizations.
Figure 3.8 White Gaussian noise of zero mean and unit variance for N=512.

Figure 3.9 Monte carlo simulations of 50 realizations for periodogram.

Figure 3.10 Average periodogram of 50 realizations.
The periodogram estimate is studied by generating different lengths of stationary white Gaussian noise $w(n)$. White Gaussian noise is having zero mean and unit variance and is modeled as $N(0,1)$. The periodogram estimates fluctuate so erratically that it is impossible to conclude from its observations that a signal has a flat spectrum. Furthermore, the size of the fluctuations are not reduced by increasing the length of the segments of $w(n)$. In this context, we could not expect the periodogram to converge to the true power spectrum for large number of observation of data segment lengths. We could also observe from the results of table 3.1 that the mean value is reduced to true value but the variance is not reduced as the number of samples $N$ is increased. This illustrates that the periodogram is not a consistent estimator for the power spectrum estimation as the variance does not approaches to small quantities for the large number of observation of data samples. The plots suggest the performance characteristics of the periodogram for different number of realizations and the simulations are run by Montecarlo techniques. The spectral characteristics reveal that the periodogram, spectrum estimation for power is not a consistent estimate as the variance of the spectral estimate does not converge to a narrow value even though the bias approaches to a true value for the large number of observation of data samples.
3.2.4 Simulation Results and Analysis of two sinusoids in white noise: Consider two sinusoids observed in white Gaussian noise. The amplitudes of the two sinusoids are 1 and 1 respectively and the white noise is having zero mean and unit variance. The nonuniform sampled signal is \[ y(n) = \cos(0.4\pi n) + \cos(0.45\pi n) + w(n) \]

The resolution performance of the periodogram is as shown in the Figure 3.10 to Figure 3.14

**Table 3.1 Statistical values for Periodogram**

<table>
<thead>
<tr>
<th>No. of samples</th>
<th>64</th>
<th>256</th>
<th>512</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.8876</td>
<td>0.9912</td>
<td>0.9957</td>
</tr>
<tr>
<td>Variance</td>
<td>0.8643</td>
<td>1.1701</td>
<td>1.1912</td>
</tr>
<tr>
<td>MSE</td>
<td>0.8834</td>
<td>1.0645</td>
<td>1.1896</td>
</tr>
</tbody>
</table>

**Figure 3.11 Monte Carlo simulations of 50 runs for the periodogram**
Figure 3.12 The spectral peaks are not resolved at 0.4 and 0.45 frequencies for 
\( N=32 \) samples in the Averaged periodogram.

Figure 3.13 Monte Carlo simulations of 50 runs for the periodogram.

Figure 3.14 The spectral peaks are now resolved at 0.4 and 0.45 frequencies for \( N=64 \) samples in the Averaged periodogram.
The frequencies in the periodogram are not resolved for $N=32$ number of samples but as expected for $N=64$, it is possible to separate the frequencies. The windowing technique helps only to smooth the periodogram but not for resolution of the frequencies. The properties of the periodogram are summarized in Table 3.2.

### Table 3.2 Properties of the Periodogram

| Periodogram       | $\hat{P}_{per}(e^{j\omega}) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n)e^{-jn} \right|^2$ |
|-------------------|-----------------------------------------------------------------------------------|
| Bias              | $E\left[ \hat{P}_{per}(e^{j\omega}) \right] = \frac{1}{2\pi} P_x(e^{j\omega})*W_B(e^{j\omega})$ |
| Resolution        | $\Delta\omega = 0.89 \frac{2\pi}{N}$                                               |
| Variance          | $\text{Var}\left[ \hat{P}_{per}(e^{j\omega}) \right] \approx P_x^2(e^{j\omega})$       |

### 3.3 The Modified Periodogram

#### 3.3.1 Introduction: we saw that the periodogram is proportional to the squared magnitude of the Fourier transform of the windowed signal $x(n) = x(n)w_R(n)$,

$$\hat{P}_{per}(e^{j\omega}) = \frac{1}{N} \left| X_N(e^{j\omega}) \right|^2 = \frac{1}{N} \left| \sum_{n=-\infty}^{\infty} x(n)w_R(n)e^{-jn\omega} \right|^2$$

(3.45)
Instead of applying a rectangular window $x(n)$, Eq. (3.45) suggests the possibility of using other data windows. Consider the effect of the data window on the bias of the periodogram.

Using Eq. (3.45), the expected value of the periodogram is

$$
\mathbb{E}\left[ \hat{P}_{\text{per}}(e^{j\omega}) \right] = \frac{1}{N} \mathbb{E}\left[ \left\{ \sum_{n=-\infty}^{\infty} x(n) w_R(n) e^{-jn\omega} \right\} \left\{ \sum_{m=-\infty}^{\infty} x(m) w_R(m) e^{-jm\omega} \right\}^* \right]
$$

$$
= \frac{1}{N} \mathbb{E}\left[ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(n) x^*(m) w_R(m) w_R(n) e^{-j(n-m)\omega} \right]
$$

$$
= \frac{1}{N} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} r_x(n-m) w_R(m) w_R(n) e^{-j(n-m)\omega} \quad (3.46)
$$

With the change of variables, $k = n - m$, Eq. (3.46) becomes

$$
\mathbb{E}\left[ \hat{P}_{\text{per}}(e^{j\omega}) \right] = \frac{1}{N} \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} r_x(k) w_R(n) w_R(n-k) e^{-jk\omega}
$$

$$
= \frac{1}{N} \sum_{k=-\infty}^{\infty} r_x(k) w_B(k) e^{-jk\omega} \quad (3.47)
$$

Where

$$
w_B(k) = w_R(k) \ast w_R(-k) = \sum_{n=-\infty}^{\infty} w_R(n) w_R(n-k)
$$
is a Bartlett window. Using the frequency convolution theorem, it follows that the expected value of the periodogram is

\[
E\left\{ \hat{P}_{\text{per}}(e^{j\omega}) \right\} = \frac{1}{2\pi N} P_x(e^{j\omega}) \ast \left| W_R(e^{j\omega}) \right|^2
\]  \hspace{1cm} (3.48)

Where

\[
W_R(e^{j\omega}) = \frac{\sin(N\omega/2)}{\sin(\omega/2)} e^{-j(n-1)\omega/2}
\]

is the Fourier transform of the rectangular data window, \( w_R(n) \). Therefore, the amount of smoothing in the periodogram is determined by the window that is applied to the data. Although a rectangular window has a narrow main lobe compared to other window and, therefore, produces the least amount of spectral smoothing, it has relatively large side lobes that may lead to masking of weak narrowband components. The periodogram of a process that is windowed with a general window \( w(n) \) is called a modified periodogram and is given by

\[
\hat{P}_M(e^{j\omega}) = \frac{1}{NU} \left| \sum_{n=-\infty}^{\infty} x(n)w(n)e^{-j\omega n} \right|^2
\]  \hspace{1cm} (3.49)

Where \( N \) is the length of the window and

\[
U = \frac{1}{N} \sum_{n=0}^{N-1} |w(n)|^2
\]  \hspace{1cm} (3.50)
is a constant that, as we will see, is defined so that $\hat{P}_M(e^{j\omega})$ will be asymptotically unbiased.

Let us now evaluate the performance of the modified periodogram. It follows from the derivation of Eq. (3.48) that the expected value of $\hat{P}_M(e^{j\omega})$ is

$$E\left[\hat{P}_M(e^{j\omega})\right] = \frac{1}{2\piNU} P_x(e^{j\omega}) * |W(e^{j\omega})|^2$$  \hspace{1cm} (3.51)$$

where $W(e^{j\omega})$ is the Fourier transform of the data window. Note that

$$U = \frac{1}{N} \sum_{n=0}^{N-1} |w(n)|^2 = \frac{1}{2\pi N} \int_{-\pi}^{\pi} |W(e^{j\omega})|^2 d\omega$$  \hspace{1cm} (3.52)$$

Then

$$\frac{1}{2\piNU} \int_{-\pi}^{\pi} |W(e^{j\omega})|^2 d\omega = 1$$

and, with an appropriate window, $|W(e^{j\omega})|^2 / NU$ will converge to an impulse of unit area as $N \to \infty$, and the modified periodogram will be asymptotically unbiased. (Note that if $w(n)$ is a rectangular window, then $U = 1$ and the modified periodogram reduces to the periodogram).

Since the modified periodogram is simply the periodogram of a windowed data sequence, the variance of $\hat{P}_M(e^{j\omega})$ will be approximately the same as that for the periodogram, i.e.,

$$\text{Var}\left[\hat{P}_M(e^{j\omega})\right] \approx \frac{P^2_x(e^{j\omega})}{x}$$  \hspace{1cm} (3.53)$$
Therefore, the modified periodogram is not a consistent estimate of the power spectrum and the data window offers no benefit in terms of reducing the variance. What the window does provide, however, is a trade-off between spectral resolution (main lobe width) and spectral masking (sidelobe amplitude). For example, with the resolution of $\hat{P}_M(e^{j\omega})$ defined to be the 3dB bandwidth of the data window. The properties of the modified periodogram are summarized in Table 3.3.

**Table 3.3 Properties of the Modified Periodogram**

| Modified Periodogram | $\hat{P}_M(e^{j\omega}) = \frac{1}{NU} \left| \sum_{n=-\infty}^{\infty} w(n)x(n)e^{-j\omega} \right|^2$ |
|----------------------|------------------------------------------------------------------|
| Bias                 | $E\left[\hat{P}_M(e^{j\omega})\right] = \frac{1}{2\pi} P_x(e^{j\omega})^* | W(e^{j\omega})^2$ |
| Resolution           | Window dependent                                                 |
| Variance             | $\text{Var}\left[\hat{P}_M(e^{j\omega})\right] \approx P_x^2(e^{j\omega})$ |

As we observed from the Table 3.3 the power spectrum estimation using the modified periodogram technique provides low variance and better resolution compared to the periodogram. This method provides the resolution based on the appropriate window selected with the estimated correlation sequence values.
3.3.2 Simulation Results and Analysis: Consider the case of three sinusoids in white Gaussian noise. The three sinusoids are having amplitudes of 1, 1, and 0.25 at the respective frequencies of 0.4, 0.45, and 0.9. The white Gaussian noise is having zero mean and unit variance and the noisy signal is given by

\[ y(n) = \cos(0.4\pi n) + \cos(0.45\pi n) + 0.25 \cos(0.9\pi n) + w(n) \]

An ensemble of 100 realizations of \( y(n) \) is generated using the length \( N=128 \) samples. The periodogram and the modified periodograms are observed for the case of hamming window and the results are shown from the Figure 3.15 to Figure 3.18. We can observe that in periodogram at the frequencies of 0.4, 0.45 and 0.9 the existence of spurious peaks especially near the two close frequencies. In modified periodogram because of hamming window is used the spurious peaks have been suppressed. The peak corresponding to frequency 0.9 is sufficiently enhanced. This enhancement is clearly at the expense of frequency resolution. The variance for the modified periodogram is still not reduced.
Figure 3.15 Monte Carlo simulations of 100 realizations for periodogram.

Figure 3.16 Average periodogram of 100 realizations using Hamming window.
Figure 3.17 Monte Carlo simulations of 100 realizations for modified periodogram.

Figure 3.18 Average modified periodogram of 100 realizations Hamming window.
3.4 Bartlett’s Method: Periodogram Averaging

3.4.1 Introduction: In this section, we look at Bartlett’s method of periodogram averaging, which, unlike either the periodogram or the modified periodogram, produces a consistent estimate of the power spectrum [5]. The motivation for this method comes from the observation that the expected value of the periodogram converges to \( P_x(e^{j\omega}) \) as the data record length \( N \) goes to infinity,

\[
\lim_{N \to \infty} E\left\{ \hat{P}_{\text{per}}(e^{j\omega}) \right\} = P_x(e^{j\omega})
\]  

(3.55)

Therefore, if we can find a consistent estimate of the mean, \( E\left\{ \hat{P}_{\text{per}}(e^{j\omega}) \right\} \), then this estimate will be a consistent estimate of \( P_x(e^{j\omega}) \). We saw that in sample mean, how averaging a set of uncorrelated measurements of a random variable \( x \) yields a consistent estimate of the mean, \( E\{x\} \). This suggests that we consider estimating the power spectrum of a random process by periodogram averaging. Thus, let \( x_i(n) \) for \( i = 1, 2, \ldots, K \) be \( K \) uncorrelated realizations of a random process \( x(n) \) over the interval \( 0 \leq n < 1 \). With \( P_{\text{per}}(e^{j\omega}) \) the periodogram of \( x_i(n) \),

\[
\hat{P}_{\text{per}}(e^{j\omega}) = \frac{1}{L} \left| \sum_{n=0}^{L-1} x_i(n) e^{-j\omega n} \right|^2 ; i = 1, 2, \ldots, K
\]  

(3.56)
the average of these periodograms is

\[ \hat{P}_x(e^{j\omega}) = \frac{1}{K} \sum_{i=1}^{K} \hat{P}_{\text{per}}^{(i)}(e^{j\omega}) \]  

(3.57)

Evaluating the expected value of \( \hat{P}_x(e^{j\omega}) \) we have

\[ E \left[ \hat{P}_{\text{per}}(e^{j\omega}) \right] = E \left[ \hat{P}_{\text{per}}^{(i)}(e^{j\omega}) \right] = \frac{1}{2\pi N} p_x(e^{j\omega}) * W_B(e^{j\omega}) \]  

(3.58)

where \( W_B(e^{j\omega}) \) is the Fourier transform of a Bartlett window, \( w_B(k) \), that extends from \(-L\) to \(L\). Therefore, as with the periodogram, \( \hat{P}_x(e^{j\omega}) \) is asymptotically unbiased. In addition, with our assumption that the data records are uncorrelated, it follows that the variance of \( \hat{P}_x(e^{j\omega}) \) is

\[ \text{Var} \left[ \hat{P}(e^{j\omega}) \right] = \frac{1}{K} \text{Var} \left[ \hat{P}_{\text{per}}^{(i)}(e^{j\omega}) \right] \approx \frac{1}{K} p_x^2(e^{j\omega}) \]  

(3.59)

Which goes to zero as \( K \) goes to infinity. Therefore, \( \hat{P}_x(e^{j\omega}) \) is a consistent estimate of the power spectrum provided that both \( K \) and \( L \) are allowed to go to infinity. However, the difficulty with this approach is that uncorrelated realizations of a process are generally not available. Instead, one typically only has a single realization of length \( N \). Therefore, Bartlett proposed that \( x(n) \) be partitioned into
K nonoverlapping sequences of length $L$ where $N = KL$. The Bartlett estimate is then computed as in Eq.(3.56) and Eq.(3.57) with

$$x_i(n) = x(n + iL) \quad n = 0,1,\ldots,L - 1$$
$$i = 0,1,\ldots,K - 1$$

Thus, the Bartlett estimate is

$$\hat{P}_B(e^{j\omega}) = \frac{1}{N} \sum_{i=0}^{K-1} \sum_{n=0}^{L-1} x(n + iL)e^{-j\omega n}$$  \hspace{1cm} (3.60)

Based on our analysis of the periodogram and the modified periodogram, we may easily evaluate the performance of Bartlett’s methods as follows. First, as in Eq.(3.58), the expected value of Bartlett’s estimate is

$$E\{\hat{P}_B(e^{j\omega})\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega})$$ \hspace{1cm} (3.61)

Therefore, $\hat{P}_B(e^{j\omega})$ is asymptotically unbiased. Second, since the periodograms used in $\hat{P}_B(e^{j\omega})$ are computed using sequences of length $L$, then the resolution is

$$\text{Res}\{\hat{P}_B(e^{j\omega})\} = 0.89 \frac{2\pi}{L} = 0.89K \frac{2\pi}{N} \hspace{1cm} (3.62)$$

which is $K$ times larger than the periodogram. Finally, since the sequences $x_i(n)$ are generally correlated with one another, unless $x(n)$ is white noise, then the variance reduction will not be as large as that
is given in Eq. (3.59). However, the variance will be inversely proportional to K and, assuming that the data sequences are approximately uncorrelated, for large N the variance is approximately

\[
\text{Var}\left(\hat{P}(e^{j\omega})\right) \approx \frac{1}{K} \text{Var}\left(\hat{P}_{\text{per}}(e^{j\omega})\right) \approx \frac{1}{K} \text{Var}_x(e^{j\omega})
\] (3.63)

Thus, if both K and L are allowed to go to infinity as \(N \to \infty\), then \(\hat{P}_B(e^{j\omega})\) will be a consistent estimate of the power spectrum. In addition, for a given value of N, Bartlett’s method allows one to trade a reduction in spectral resolution for a reduction in variance by simply changing the values of K and L.

Table 3.4 summarizes the properties of Bartlett’s method.

<table>
<thead>
<tr>
<th>Power spectral estimate</th>
<th>(\hat{P}<em>B(e^{j\omega}) = \frac{1}{N} \sum</em>{i=0}^{K-1} \sum_{n=0}^{L-1} x(n + iL)e^{-jn\omega}^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias</td>
<td>(E\left(\hat{P}_B(e^{j\omega})\right) = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega}))</td>
</tr>
<tr>
<td>Resolution</td>
<td>(\Delta\omega = 0.89K \frac{2\pi}{N})</td>
</tr>
<tr>
<td>Variance</td>
<td>(\text{Var}\left(\hat{P}_B(e^{j\omega})\right) \approx \frac{1}{K} \text{Var}_x(e^{j\omega}))</td>
</tr>
</tbody>
</table>

3.4.2 Simulations Results and Analysis: Consider the case of three sinusoids in white Gaussian noise. The three sinusoids are having amplitudes of 1, 1, and 0.25 at the respective frequencies of 0.4, 0.45
and 0.9. The white Gaussian noise is having zero mean and unit variance and the noisy signal is given by

\[ y(n) = \cos(0.4\pi n) + \cos(0.45\pi n) + 0.25 \cos(0.9\pi n) + w(n) \]

The simulated Bartletts estimate for signal \( y(n) \) of length \( N=128 \) is as shown in the Figure from 3.19 to 3.24.

![Figure 3.19 Barlett estimate for 50 realizations for K=1 segment.](image-url)
Figure 3.20 Barlett estimate for 50 realizations for K=4 segments.

Figure 3.21 Barlett estimate for 50 realizations for K=8 segments.
Figure 3.22 Barlett estimate for 50 realizations for $K=16$ segments.

Figure 3.23 Average 50 realizations of Barlett estimate for $K=1$ segment.
Figure 3.24 Average 50 realizations of Barlett estimate for K=4 segments.

Figure 3.25 Average 50 realizations of Barlett estimate for K=8 segments.
Figure 3.26 Average 50 realizations of Barlett estimate for \( K=16 \) segments.

The Barlett estimate is performed on the signal \( y(n) \) for different number (\( K=1, 4, 8, \) and 16) of nonoverlapping of segments. We can observe that the variance has consistently reduced as the number of segments is increased and the vertical scale is also reduced. However, this reduction of variance has come at the cost of widening of the spectral peaks. The bias and the variances for different number of nonoverlapping of segments are as shown in the Table 3. 5.

<table>
<thead>
<tr>
<th>N=128</th>
<th>K=1</th>
<th>K=4</th>
<th>K=8</th>
<th>K=16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean of ( \hat{P}_b(e^{j\omega}) )</td>
<td>0.6493</td>
<td>0.6477</td>
<td>0.6462</td>
<td>0.6459</td>
</tr>
<tr>
<td>Variance of ( \hat{P}_b(e^{j\omega}) )</td>
<td>15.3114</td>
<td>3.3240</td>
<td>1.6475</td>
<td>0.8558</td>
</tr>
</tbody>
</table>
3.5 Welch Method using linear overlapping of samples:

3.5.1 Introduction: Welch proposed two modifications to Barlett’s method [59]. The first is to allow the sequences \( x_i(n) \) to overlap, and the second is to allow a data window \( w(n) \) to be applied to each sequence, thereby producing a set of modified periodograms that are to be averaged. Assuming that successive sequences are offset by \( D \) points and that each sequence is \( L \) points long then the \( i \)th sequence is given by

\[
x_i(n) = x(n + iD); \quad n = 0,1,\ldots,L - 1
\]

Thus, the amount of overlap between \( x_i(n) \) and \( x_{i+1}(n) \) is \( L - D \) points, and if \( K \) sequences cover the entire \( N \) data points, then

\[
N = L + D(K - 1)
\]

For example, with no overlap \( (D = L) \) we have \( K = N/L \) sections of length \( L \) as in Bartlett’s method. On the other hand, if the sequences are allowed to overlap by 50% \( (D = L/2) \) then we may form \( K = 2 \frac{N}{L} - 1 \) sections of length \( L \), thus maintaining the same resolution (section length) as Bartlett’s method while doubling the number of modified periodograms that are averaged, thereby reducing the variance. However, with a 50% overlap we could also form

\[
K = \frac{N}{L} - 1
\]
with Welch’s method may be written explicitly in terms of \( x(n) \) as follows

\[
\hat{P}_B(e^{j\omega}) = \frac{1}{KLU} \sum_{i=0}^{K-1} \sum_{n=0}^{L-1} w(n)x(n+iD)e^{-jn\omega}^2
\]  

(3.64)

or, more succinctly, in terms of modified periodograms as

\[
\hat{P}_B(e^{j\omega}) = \frac{1}{K} \sum_{i=0}^{K-1} P_M(e^{j\omega})
\]  

(3.65)

Therefore, the expected value of Welch’s estimate is

\[
E\left\{ \hat{P}_W(e^{j\omega}) \right\} = E\left\{ \hat{P}_M(e^{j\omega}) \right\} = \frac{1}{2\pi L\omega} P_x(e^{j\omega})^* \left| W(e^{j\omega}) \right|^2
\]  

(3.66)

where \( W(e^{j\omega}) \) is the Fourier transform of the \( L \)-point data window, \( w(n) \), used in Eq. (3.64) to form the modified periodograms. Thus, as with each of the previous periodogram-based methods, Welch’s method is an asymptotically unbiased estimate of the power spectrum. The resolution, however, depends on the data window. As with the modified periodogram, the resolution is defined to be the 3 \( \text{dB} \) bandwidth of the data window. The variance, on the other hand, is more difficult to compute since, with overlapping sequences, the modified periodograms cannot be assumed to be uncorrelated. Nevertheless, it has been shown that, with a Bartlett window and a 50% overlap, the variance is approximately]
Comparing Eqs. (3.67) and (3.63) we see that, for a given number of sections $K$, the variance of the estimate using Welch’s method is larger than that for Bartlett’s method by a factor of $9/8$. However, for a fixed amount of data, $N$, and a given resolution (sequence length $L$) with a 50% overlap twice as many sections may be averaged in Welch’s method. Expressing

$$
\text{Var}\left[ \hat{P}_W(e^{j\omega}) \right] \approx \frac{9}{8K} P_x^2(e^{j\omega})
$$

(3.67)

**Table 3.6 Properties of Welch’s Method**

<table>
<thead>
<tr>
<th>Welch Estimate</th>
<th>$\hat{P}<em>B(e^{j\omega}) = \frac{1}{KLU} \sum</em>{i=0}^{K-1} \left[ \sum_{n=0}^{L-1} w(n)x(n+iD)e^{-j\omega n} \right]^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bias</td>
<td>$E\left[ \hat{P}_W(e^{j\omega}) \right] = \frac{1}{2\pi LU} P_x(e^{j\omega}) \ast</td>
</tr>
<tr>
<td>Variance</td>
<td>$\text{Var}\left[ \hat{P}_W(e^{j\omega}) \right] \approx \frac{9}{16N} P_x^2(e^{j\omega})$</td>
</tr>
<tr>
<td>Resolution</td>
<td>Window dependent</td>
</tr>
</tbody>
</table>

Assuming 50% overlap and a Bartlett window.

The variance in terms of $L$ and $N$, we have (assuming a 50% overlap)

$$
\text{Var}\left[ \hat{P}_W(e^{j\omega}) \right] \approx \frac{9L}{16N} P_x^2(e^{j\omega})
$$

(3.68)
Since \( N/L \) is the number of sections that are used in Bartlett’s method, it follows from Eq. (3.63) that

\[
\text{Var}\left(\hat{P}_W(e^{j\omega})\right) \approx \frac{9}{16} \text{Var}\left(\hat{P}_W(e^{j\omega})\right).
\]

Although it is possible to average more sequences for a given amount of data by increasing the amount of overlap, the computational requirements increase in proportion with \( K \). In addition, since an increase in the amount of overlap increases the correlation between the sequences \( x_1(n) \), there are diminishing returns when increasing \( K \) for a fixed \( N \). Therefore the amount of overlap is typically either 50\% or 75\%.

### 3.5.2 Simulation Results and analysis:

Consider the Welch’s method for the random process

\[
y(n) = \cos(0.4\pi n) + \cos(0.45\pi n) + 0.25 \cos(0.9\pi n) + w(n)
\]

for \( N=1024 \) number of samples and the window employed for each segment is Hamming Window and 50\% overlapping of samples is allowed.
Figure 3.27 Realization of Welch estimate using Monte Carlo simulations for 

\[ K=1 \text{ segment.} \]

Figure 3.28 Realization of Welch estimate using Monte Carlo simulations for 

\[ K=4 \text{ segments.} \]

Figure 3.29 Realization of Welch estimate using Monte Carlo simulations for 

\[ K=8 \text{ segments.} \]
Figure 3.30 Realization of Welch estimate using Monte Carlo simulations for 

$K=16$ segments.

Figure 3.31 Average 50 realizations of Welch estimate for $K=1$ segments.
Figure 3.32 Average 50 realizations of Welch estimate for K=4 segments.

Figure 3.33 Average 50 realizations of Welch estimate for K=8 segments.
Figure 3.34 Average 50 realizations of Welch estimate for K=16 segments.

Table 3.7 Mean and Variances for the Welch estimate.

<table>
<thead>
<tr>
<th>Length of the data sequence N=1024</th>
<th>Number of segments K=1</th>
<th>Number of segments K=4</th>
<th>Number of segments K=8</th>
<th>Number of segments K=16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean of $\hat{P}_w(e^{j\omega})$</td>
<td>0.2462</td>
<td>0.3242</td>
<td>0.2794</td>
<td>0.2609</td>
</tr>
<tr>
<td>Variance of $\hat{P}_w(e^{j\omega})$</td>
<td>1.8624</td>
<td>0.8435</td>
<td>0.3155</td>
<td>0.1447</td>
</tr>
</tbody>
</table>
As per the values of Table 3.7 and from Figures 3.28 to 3.33, it is concluded that for the large number of observation of data samples the mean approaches to the true value and variances is further reduced greatly. Hence Welch estimate is asymptotically consistent estimate for the estimation of power spectrum. Consider again the Welch estimate of the data sequence y(n) with different segment lengths and different percentage of overlapping of samples. The lengths of y(n) are 64, 128, 256, and 1024 respectively for the segment lengths 2, 4, 8, and 32.

The statistical properties of Welch’s method depend on the amount of overlap that is used and on the type of data window. With a 50% overlap and a Bartlett window, the variability for large $N$ is approximately

$$V_W = \frac{9}{8} \frac{1}{K} = \frac{9}{16} \frac{L}{N}$$

Since the 3 dB bandwidth of a Bartlett window of length $L$ is $1.28(2\pi/L)$ the resolution is

$$\Delta \omega = 1.28 \frac{2\pi}{L}$$

and the figure of merit becomes

$$M_W = 0.72 \frac{2\pi}{N}$$
Table 3.8 Mean and variances Welch estimate for length N=64.

<table>
<thead>
<tr>
<th>N=64</th>
<th>%overlapping of samples.</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>K=2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.7861</td>
<td>0.1281</td>
</tr>
<tr>
<td></td>
<td>47.6</td>
<td>0.5639</td>
<td>0.1500</td>
</tr>
<tr>
<td></td>
<td>77</td>
<td>0.5112</td>
<td>0.1071</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.5558</td>
<td>0.2007</td>
</tr>
<tr>
<td></td>
<td>K=3</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.7862</td>
<td>0.0552</td>
</tr>
<tr>
<td></td>
<td>58.8235</td>
<td>0.5218</td>
<td>0.0270</td>
</tr>
<tr>
<td></td>
<td>88.2353</td>
<td>0.5589</td>
<td>0.1197</td>
</tr>
<tr>
<td></td>
<td>92.5926</td>
<td>0.5006</td>
<td>0.1898</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.5552</td>
<td>0.2214</td>
</tr>
<tr>
<td></td>
<td>K=4</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.6691</td>
<td>0.1417</td>
</tr>
<tr>
<td></td>
<td>43.4783</td>
<td>0.5381</td>
<td>0.0409</td>
</tr>
<tr>
<td></td>
<td>64.5161</td>
<td>0.5703</td>
<td>0.0869</td>
</tr>
<tr>
<td></td>
<td>91.8367</td>
<td>0.6830</td>
<td>0.0898</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.4910</td>
<td>0.1280</td>
</tr>
</tbody>
</table>
Table 3.9 Mean and variances Welch estimate for length $N=128$.

<table>
<thead>
<tr>
<th>N=128</th>
<th>%overlapping of samples.</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>0.5428</td>
</tr>
<tr>
<td></td>
<td></td>
<td>37.9747</td>
<td>0.5874</td>
</tr>
<tr>
<td></td>
<td></td>
<td>67.7083</td>
<td>0.5013</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>0.6051</td>
</tr>
<tr>
<td>K=2</td>
<td></td>
<td>0</td>
<td>0.7612</td>
</tr>
<tr>
<td></td>
<td></td>
<td>48.3871</td>
<td>0.6494</td>
</tr>
<tr>
<td></td>
<td></td>
<td>83.3333</td>
<td>0.6952</td>
</tr>
<tr>
<td></td>
<td></td>
<td>91.7431</td>
<td>0.7182</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>0.6666</td>
</tr>
<tr>
<td>K=3</td>
<td></td>
<td>0</td>
<td>0.6831</td>
</tr>
<tr>
<td></td>
<td></td>
<td>64.5161</td>
<td>0.6994</td>
</tr>
<tr>
<td></td>
<td></td>
<td>88.4211</td>
<td>0.7025</td>
</tr>
<tr>
<td></td>
<td></td>
<td>96.4912</td>
<td>0.6337</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>0.4910</td>
</tr>
</tbody>
</table>

<p>| K=4   |                           | 0     | 0.6831   | 0.1215   |
|       |                           | 64.5161 | 0.6994   | 0.3020   |
|       |                           | 88.4211 | 0.7025   | 0.5011   |
|       |                           | 96.4912 | 0.6337   | 0.0898   |
|       |                           | 100   | 0.4910   | 0.3336   |</p>
<table>
<thead>
<tr>
<th>N=512</th>
<th>%overlapping of samples.</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>K=2</td>
<td>0</td>
<td>0.6704</td>
<td>1.0671</td>
</tr>
<tr>
<td></td>
<td>23.4483</td>
<td>0.6683</td>
<td>0.8269</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.6372</td>
<td>0.9552</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.5998</td>
<td>1.0214</td>
</tr>
<tr>
<td>K=3</td>
<td>0</td>
<td>0.6254</td>
<td>0.5892</td>
</tr>
<tr>
<td></td>
<td>53.0303</td>
<td>0.6291</td>
<td>0.9462</td>
</tr>
<tr>
<td></td>
<td>81.0811</td>
<td>0.6292</td>
<td>0.9840</td>
</tr>
<tr>
<td></td>
<td>97.9592</td>
<td>0.6841</td>
<td>1.5844</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.5682</td>
<td>1.1465</td>
</tr>
<tr>
<td>K=4</td>
<td>0</td>
<td>0.6694</td>
<td>0.6064</td>
</tr>
<tr>
<td></td>
<td>33.9181</td>
<td>0.7065</td>
<td>0.9505</td>
</tr>
<tr>
<td></td>
<td>88.4211</td>
<td>0.7025</td>
<td>0.5011</td>
</tr>
<tr>
<td></td>
<td>93.4579</td>
<td>0.5780</td>
<td>0.9555</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.6535</td>
<td>1.6345</td>
</tr>
</tbody>
</table>
Figure 3.35 Welch estimate for

\[ y(n) = \cos(0.4\pi n) + \cos(0.45\pi n) + 0.25\cos(0.9\pi n) + w(n) \] of lengths N=128,256,512

and 1024 of different number segments with different % of linear overlaps.
3.6 Blackman-Tukey Method: Periodogram Smoothing

3.6.1 Introduction: The methods of Bartlett and Welch are designed to reduce the variance of the periodogram by averaging periodograms and modified periodograms, respectively. Another method for decreasing the statistical variability of the periodogram is periodogram smoothing, often referred to as the Blackman-Tukey method after the pioneering work of Blackman and Tukey in spectrum analysis [70][6]. To see how periodogram smoothing may reduce the variance of the periodogram, recall that the periodogram is computed by taking the Fourier transform of a consistent estimate of the autocorrelation sequence. However, for any finite data record of length $N$, the variance of $\hat{r}_x(k)$ at $k = N - 1$ is

$$\hat{r}_x(N - 1) = \frac{1}{N} x(N - 1)x(0)$$

Since there is little averaging that goes into the formation of the estimates of $r_x(k)$ for $|k| \approx N$, no matter how large $N$ becomes, these estimates will always be unreliable. Consequently, the only way to reduce the variance of the periodogram is to reduce the variance of these estimates or to reduce the contribution that they make to the periodogram. In the methods of Bartlett and Welch, the variance of the periodogram is decreased by reducing the variance of the autocorrelation estimate by averaging. In the Blackman-Tukey method, the variance of the periodogram is reduced by applying a
window to $\hat{r}_x(k)$ in order to decrease the contribution of the unreliable estimates to the periodogram. Specifically, the Blackman-Tukey spectrum estimate is

$$
\hat{P}_{BT}(e^{j\omega}) = \frac{1}{2\pi} \sum_{k=-M}^{M} \hat{r}_x(k)w(k)e^{-jk\omega}
$$

(3.69)

Where $w(k)$ is a lag window that is applied to the autocorrelation estimate. For example, if $w(k)$ is a rectangular window extending from $-M$ to $M < N - 1$, then the estimates of $r_x(k)$ having the largest variance are set to zero and, consequently, the power spectrum estimate will have a smaller variance. What is trade off for this reduction in variance, however, is a reduction in resolution since a smaller number of autocorrelation estimates are used to form the estimate of the power spectrum.

Using the frequency convolution theorem, the Blackman-Tukey spectrum may be written in the frequency domain as follows:

$$
\hat{P}_{BT}(e^{j\omega}) = \frac{1}{2\pi} \hat{P}_{\text{per}}(e^{j\omega}) \ast W(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{P}_{\text{per}}(e^{ju})W(e^{j(\omega-u)})du
$$

(3.70)

Therefore, the Blackman-Tukey estimate smoothes the periodogram by convolving with the Fourier transform of the autocorrelation window, $W(e^{j\omega})$. Although there is considerable flexibility in the choice of the window that may be used, $w(k)$ should be conjugate symmetric so that $W(e^{j\omega})$ is real-valued, and the window should have
a nonnegative Fourier transform, that $\hat{P}_{BT}(e^{j\omega})$ is guaranteed to be nonnegative.

To analyze the performance of the Blackman-Tukey method, we will evaluate the bias and the variance (the resolution is window-dependent). The bias may be computed by taking the expected value of Eq. (3.70) as follows:

$$E\left\{ \hat{P}_{BT}(e^{j\omega}) \right\} = \frac{1}{2\pi} E\left\{ \hat{P}_{per}(e^{j\omega}) \right\} * W(e^{j\omega})$$

(3.71)

Substituting Eq.(3.23) for the expected value of the periodogram we have

$$E\left\{ \hat{P}_{BT}(e^{j\omega}) \right\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega}) * W(e^{j\omega})$$

(3.72)

or, equivalently,

$$E\left\{ \hat{P}_{BT}(e^{j\omega}) \right\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega}) * W(e^{j\omega})$$

(3.73)

If we let $w_{BT}(k) = w_B(k)w(k)$ be the combined window that is applied to the autocorrelation sequence $r_x(k)$, using the frequency convolution theorem we have

$$E\left\{ \hat{P}_{BT}(e^{j\omega}) \right\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W(e^{j\omega})$$

(3.74)
If we assume that \( M \ll N \) so that \( w_B(k)w(k) = w(k) \), then we have

\[
E\left[ \hat{P}_{BT}(e^{j\omega}) \right] \approx \frac{1}{2\pi} P_x(e^{j\omega}) \ast W(e^{j\omega})
\]

(3.75)

where \( W(e^{j\omega}) \) is the Fourier transform of the lag window \( w(k) \).

Evaluating the variance of the Blackman-Tukey spectrum estimate requires a bit more work. Since

\[
\text{Var}\left[ \hat{P}_{BT}(e^{j\omega}) \right] = E\left[ \hat{P}_{BT}^2(e^{j\omega}) \right] - E^2\left[ \hat{P}_{BT}(e^{j\omega}) \right]
\]

(3.76)

We begin by finding the mean-square value \( E\left[ \hat{P}_{BT}(e^{j\omega}) \right] \). From Eq. (3.70) we have

\[
\hat{P}_{BT}^2(e^{j\omega}) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \hat{P}_{per}(e^{j\omega}) \hat{P}_{per}(e^{j\omega}) \hat{P}_{per}(e^{j\omega}) \hat{P}_{per}(e^{j\omega}) W(e^{j(\omega-u)}) W(e^{j(\omega-v)}) dv du
\]

Therefore, the mean-square value is

\[
E\left[ \hat{P}_{BT}(e^{j\omega}) \right] = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} E\left[ \hat{P}_{per}(e^{j\omega}) \hat{P}_{per}(e^{j\omega}) \right] W(e^{j(\omega-u)}) W(e^{j(\omega-v)}) dv du
\]

By using the approximation given in Eq. (3.43) for

\[ E\left[ \hat{P}_{per}(e^{j\omega}) \hat{P}_{per}(e^{j\omega}) \right] \]

leads to an expression for the mean-square value that contains two terms. The first term is
\[
\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} P_x(e^{ju}) P_x(e^{jv}) W(e^{j(\omega-u)}) W(e^{j(\omega-v)}) \, du \, dv
\]

\[
= \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} P_x(e^{ju}) W(e^{j(\omega-u)}) \, du \right]^2 = E^2 \left\{ \hat{P}(e^{j\omega}) \right\}.
\] (3.77)

that is cancelled by the second term in Eq. (3.76). Therefore, the variance is

\[
\text{Var}\left\{ \hat{P}_{\text{BT}}(e^{j\omega}) \right\} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} P_x(e^{ju}) P_x(e^{jv})
\]

\[
\times \left[ \frac{\sin N(u-v)/2}{N \sin(u-v)/2} \right]^2 \frac{1}{N} \left[ \frac{\sin(N\omega/2)}{\sin(\omega/2)} \right]^2
\]

\[
\times W_B(e^{j(\omega-u)}) W_B(e^{j(\omega-v)}) \, du \, dv
\] (3.78)

Since

\[
W_B(e^{j\omega}) = \frac{1}{N} \left[ \frac{\sin(N\omega/2)}{\sin(\omega/2)} \right]^2
\]

is the discrete-time Fourier transform of a Bartlett window, then \( w_B(k) \) approaches a constant as \( N \to \infty \) and \( W_B(e^{j\omega}) \) converges to an impulse. Therefore, if \( N \) is large then the term in brackets may be approximated by an impulse of area \( 2\pi / N \),

\[
\left[ \frac{\sin N(u-v)/2}{N \sin(u-v)/2} \right]^2 \approx \frac{2}{N} \delta(u-v)
\]

Thus, for large \( N \), the variance of the Blackman-Tukey estimate is approximately
If $M$ is large enough so that we may assume that $P_X(e^{j\omega})$ is constant across the main lobe of $W(e^{j(\omega-u)})$, then $P_X^2(e^{j\omega})$ may be pulled out of the integral, \[
abla \left[ \hat{P}_{BT}(e^{j\omega}) \right] \approx \frac{1}{2\pi N} \int_{-\pi}^{\pi} P_X^2(e^{j\omega}) W^2(e^{j(\omega-u)}) du \]

Finally, using Parseval’s theorem we have

\[
\nabla \left[ \hat{P}_{BT}(e^{j\omega}) \right] = P_X^2(e^{j\omega}) \frac{1}{N} \sum_{k=-M}^{M} w^2(k) \quad (3.79)
\]

provided $N \gg M \gg 1$. Thus, from Eqs. (3.75) and (3.79) we again see the trade-off between bias and variance. For a small bias, $M$ should be large in order to minimize the width of the main lobe of $W(e^{j\omega})$ whereas $M$ should be small in order to minimize the sum in Eq. (8.79). Generally, it is recommended that $M$ have a maximum value of $M = N/5[26]$. The properties of the Blackman-Tukey method are summarized in Table 3.11

### 3.6.2 Simulation Results and Analysis:

Blackman & Tukey power spectra estimate for $y(n) = \cos(0.4\pi n) + \cos(0.45\pi n) + 0.25\cos(0.9\pi n) + w(n)$ of lengths N=128,256,512 and 1024 of different number of segments with different percentage of linear overlaps.
Figure 3.36 Blackman & Tukey power spectrum estimate for

\[ y(n) = \cos(0.4 \pi n) + \cos(0.45 \pi n) + 0.25 \cos(0.9 \pi n) + w(n) \] of lengths \( N=128, 256, 512 \)

and 1024 of different number segments with different % of linear overlaps.
### Table 3.11 Properties of the Blackman-Tukey Method

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>BT- Periodogram</strong></td>
<td>$\hat{P}<em>B(e^{j\omega}) = \sum</em>{k=-M}^{M} r_x(k)w(k)e^{-jk\omega}$</td>
</tr>
<tr>
<td><strong>Bias</strong></td>
<td>$E\left{ \hat{P}_B(e^{j\omega}) \right} \approx \frac{1}{2\pi} P_x(e^{j\omega}) * W(e^{j\omega})$</td>
</tr>
<tr>
<td><strong>Variance</strong></td>
<td>$\text{Var}\left{ \hat{P}<em>B(e^{j\omega}) \right} \approx P_x(e^{j\omega}) \frac{1}{N} \sum</em>{k=-M}^{M} w^2(k)$</td>
</tr>
<tr>
<td><strong>Resolution</strong></td>
<td>Window dependent</td>
</tr>
</tbody>
</table>

### 3.7 Conclusions:

An important issue in the selection of a spectrum estimation technique is the performance of the estimator. In comparing one nonparametric method to another, there is a trade-off between resolution and variance. This section summarizes the performance of each nonparametric technique in terms of two criteria. The first is the variability of the estimate,

$$V = \frac{\text{Var}\left\{ \hat{P}_x(e^{j\omega}) \right\}}{E^2 \left\{ \hat{P}_x(e^{j\omega}) \right\}}$$

which is a normalized variance. The second measure is an overall figure of merit that is defined as the product of the variability and the resolution $M = V\Delta\omega$. 
This figure of merit should be as small as possible. However, as we will soon discover, this figure of merit is approximately the same for all of the nonparametric methods that we have considered.

**Periodogram.** We have shown that the periodogram is asymptotically unbiased and that, for large \( N \), the variance is approximately equal to \( P_x^2(e^{j\omega}) \). Asymptotically, therefore, the variability of the periodogram is equal to one

\[
V_{\text{per}} = \frac{P_x^2(e^{j\omega})}{P_x^2(e^{j\omega})} = 1
\]

Thus, since the resolution of the periodogram is

\[
\Delta \omega = 0.89 \frac{2\pi}{N}
\]

then the overall figure of merit is

\[
M_{\text{per}} = 0.89 \frac{2\pi}{N}
\]

which is inversely proportional to the data record length, \( N \).

**Bartlett’s Method.** In Bartlett’s method, a reduction in variance is achieved by averaging periodograms. With \( N = KL \), if \( N \) is large then the variance is approximately

\[
\text{Var}\left(\hat{P}_B(e^{j\omega})\right) \approx \frac{1}{K} P_x^2(e^{j\omega})
\]
and the variability is

\[ V_B = \frac{1}{K} \frac{P_X^2(e^{j\omega})}{P_X^2(e^{j\omega})} = \frac{1}{K} \]

Since the resolution is \( \Delta \omega = 0.89(2\pi K / N) \) then the figure of merit is

\[ M_B = 0.89 \frac{2\pi}{N} \]

which is the same as the figure of merit for the periodogram.

**Welch’s Method.** The statistical properties of Welch’s method depend on the amount of overlap that is used and on the type of data window. With a 50% overlap and a Bartlett window, the variability for large \( N \) is approximately

\[ V_W = \frac{9}{8} \frac{1}{K} = \frac{9}{16} \frac{L}{N} \]

Since the 3 dB bandwidth of a Bartlett window of length \( L \) is \( 1.28(2\pi / L) \) the resolution is

\[ \Delta \omega = 1.28 \frac{2\pi}{L} \]

and the figure of merit becomes

\[ M_W = 0.72 \frac{2\pi}{N} \]

**Blackman-Tukey Method.** Since the variance and resolution of the Blackman-Tukey method depend on the window that is used, suppose
w(k) is a Bartlett window of length $2M$ that extends from $k = -M$ to $k = M$. Assuming that $N >> M >> 1$, the variance of the Blackman-Tukey estimate is approximately

$$\text{Var}\left\{ \hat{P}_\text{BT}(e^{j\omega}) \right\} \approx P_x(e^{j\omega}) \frac{1}{N} \sum_{k=-M}^{M} \left( 1 - \frac{|k|}{M} \right)^2 \approx P_x(e^{j\omega}) \frac{2M}{3N}$$

Therefore, the variability is

$$V_{\text{BT}} = \frac{2M}{3N}$$

Since the 3 dB bandwidth of a Bartlett window of length $2M$ is equal to $1.28(2\pi / 2M)$, then the resolution is

$$\Delta \omega = 0.64 \frac{2\pi}{M}$$

and the figure of merit becomes

$$M_{\text{BT}} = 0.43 \frac{2\pi}{N}$$

which is slightly smaller than the figure of merit for Welch’s method.

Table 3.7 provides a summary of the performance measures presented above for the periodogram-based spectrum estimation techniques discussed in this section. What is apparent from this table is that each technique has a figure of merit that is approximately the same, and that these figures of merit are inversely proportional to the length of the data sequence, $N$. Therefore, although each method differs in
its resolution and variance, the overall performance is fundamentally limited by the amount of data that is available. In the following sections, we will look at some entirely different approaches to spectrum estimation in the hope of being able to find a high-resolution spectrum estimate with a small variance that works well on short data records. Overall it shows that the periodogram has a large variance on the estimate of its spectral density coefficients and also the variance of the estimate not improved by the consideration of large data. The Welch estimate is asymptotically consistent estimate for 50% overlapping of samples. But the variance does not reduce to a minimum quantity as the percentage of linear overlapping of the samples increases. Hence this is a drawback of the Welch estimate. Blackman & Tukey estimate is also insignificant for high data computations. Hence to overcome these effects the following techniques are proposed.

1) Power spectrum estimation using nonlinear overlapping of samples.

2) Power spectrum estimation using prewhitening and post coloring technique.

3) Power spectrum using averaged windowed least squares estimation. These algorithms are explained in the chapter 4.