Chapter III

Combined Forced and Natural Convection in a Vertical Channel Filled with Saturated Porous Medium Under Various Physical Conditions

Extensive exploration and research in natural, forced and mixed convection in saturated porous media finds applications in geothermal technology, cooling of nuclear fuel, insulation of high temperature gas cooled reactor vessels, thermal energy storage tanks, regenerative heat exchangers, petroleum reservoirs and chemical catalytic reactors which in turn have tremendous impact on our energy and ecology problems.

Vertical channel is the most fundamental configuration in heat transfer processes. Study of mixed convection in a vertical channel filled with porous medium has relevance in the modelling of cooling processes in modern electronic devices, nuclear reactors, and electrically heated vertical heat exchangers. Rectangular coolant channels are employed in heat exchange devices. In order to maintain good performance and reliability, the temperature of these devices must be carefully controlled. Fibrous materials enclosed between two vertical plates are used in engineering applications as means to decrease heat flux and corresponding heat losses. The phenomenon of convection through porous medium also plays a
major role in the performance of double walls filled with fibrous or granular insulators, for use in energy efficient buildings. The extensive survey of literature in chapter I indicates that study of mixed convection in vertical channels filled with porous media has received little attention. This fact gave us an impetus to analyse combined natural and forced convection in a vertical channel filled with porous medium.

Chapter III of the thesis in its first 3 sections describes the influence of various physical conditions namely, time variation of wall temperature with time, inclusion of heat generating source and viscous heating of the channel, on laminar convection in a vertical channel filled with porous medium, where permeability is a constant. In the fourth section the characteristics of laminar convection in a vertical pipe filled with porous medium is analysed. Section 5 analyses the conditions for existence of multiple solutions for natural convection. In section 6 a refined analysis of the problems discussed in sections 1 and 2 is presented.

Section 3.1 is devoted to the study of mixed laminar convection in a vertical channel filled with porous medium under a pressure gradient. The walls of the channel are maintained at a uniform temperature gradient and two physical situations namely, steady heating of ascending cold fluid and
steady cooling of ascending hot fluid are considered. The governing equations of the problem are solved both numerically and analytically and the influence of various parameters entering into the problem are discussed in detail.

In section 3.2 a numerical analysis of laminar convection in a vertical channel filled with porous medium is presented in the presence of viscous dissipation and Darcy dissipation. The governing equations are solved using an implicit finite difference scheme developed by Keller (1971).

Unsteady laminar convection in a uniformly heated vertical channel filled with porous medium is investigated, in the presence of Darcy and viscous dissipations and a heat generating source in section 3.3. Unsteadiness is introduced by varying the wall temperature with time and the governing equations are solved numerically using an implicit finite difference scheme.

Section 3.4 deals with convection in a vertical pipe filled with porous medium which is an extension of Morton's (1960) work in the absence of porous medium. The governing equations of the problem are solved numerically for steady heating of ascending cold fluid and steady cooling of ascending hot fluid, using Runge-Kutta-Gill method.

In section 3.5, the conditions for the existence of multiple solutions, for natural convection in a vertical channel filled with porous medium, for Darcy flow model is
investigated in the presence of dissipation terms.

In section 3.6, a refined analysis of mixed convection is presented by taking into account the thermal conductivities of the solid and fluid constituting the porous media and incorporating the Kozney-Blake expression connecting porosity and permeability.

3.1 **MIXED CONVECTION IN THE ABSENCE OF DISSIPATIONS**

Laminar convection in a vertical channel in the absence of porous medium has been extensively studied by many authors [Ostrach (1952, 1954) and Sparrow et al. (1959)]. Several investigators Tao (1960) and Lauber et al. (1966) have analysed combined free and forced convection in a vertical channel with identical heating of two walls. Morton (1960) presented an exact solution for laminar convection in a uniformly heated vertical pipe in the presence of a pressure gradient.

By analysing the cooling effectiveness of porous material in a coolant passage, Koh (1974) concluded that heat transfer can be enhanced by insertion of solid matrices in porous media. Later, Koh (1975) experimentally verified the same for low permeabilities. Kaviany (1984) has investigated laminar flow through a porous channel bounded by isothermal parallel plates maintained at constant temperature.

In this section mixed convection in a vertical channel filled with saturated porous medium is analysed for two
physical situations namely, steady heating of ascending cold fluid and steady cooling of ascending hot fluid. The solutions of the governing equations are obtained by five methods—regular parameter perturbation method, singular parameter perturbation method, Runge-Kutta-Gill method, matrix method and Keller Box method. The influence of various parameters on the flow and heat transfer characteristics are analysed in detail.

3.1(a) Mathematical formulation and boundary conditions:

The physical model is shown in fig (3.1.1). The walls of the channel are maintained at a uniform temperature gradient \( (\Gamma/a) \) in the vertical direction. The wall temperature \( T_w \) is given by \( T_w = T_0 + \Gamma (x/a) \), where \( x \) is the coordinate in the vertical direction; \( T_0 \) is the temperature of the wall at the origin and "a" is the half width of the channel. The flow field and the temperature field are symmetrical about the centre line of the channel at \( y = 0 \). Since thermal and geometrical symmetry exists, the temperature gradient at the symmetry axis vanishes. In general a finite length channel introduces mathematical difficulties which preclude the possibility of an exact solution to the governing equations. For this reason, the limiting case of very long channel has been made for exact mathematical solutions after Ostrach (1952). The flow is steady, fully developed (a channel whose height is large
FIG. 3.1.1. PHYSICAL MODEL

\[ T_w = T_0 + \Gamma \left( \frac{x}{a} \right) \]
compared to the spacing between the walls) and velocity is a function of "y" only. The porosity and permeability of the medium are constant on account of the isotropic nature of the medium.

Incorporating the assumptions in section 2.3 the governing steady state equations of motion based on volume averaged principles, after Vafai and Tien (1981) in the absence of inertial and dissipation effects, from equations (2.1.7) and (2.1.8) reduce to the form,

$$\frac{\mu}{\varepsilon} \frac{\partial^2 u}{\partial y^2} - \frac{\mu u}{k} = \frac{\partial P}{\partial x} - \rho g$$ ...

(3.1.1)

$$u \frac{\partial T}{\partial x} = \alpha_e \frac{\partial^2 T}{\partial y^2}$$ ...

(3.1.2)

where,

- \(k\) - permeability of the medium
- \(\mu\) - dynamic viscosity of the fluid
- \(\frac{\mu}{\varepsilon}\) - effective viscosity
- \(\varepsilon\) - porosity of the porous medium
- \(u\) - velocity component in the x direction
- \(\rho\) - density of the fluid

\(\alpha_e\) - effective thermal diffusivity, \(\alpha_e = \frac{\lambda_e}{\rho \cdot C_p}\), \(C_p\) - specific heat at constant pressure

\(\lambda_e\) - effective thermal conductivity

In equation (3.1.1), the first term on L.H.S represents viscous resistance and the second term represents Darcy
The boundary conditions of the problem are:
\[ \frac{\partial u}{\partial y} = 0, \quad \frac{\partial T}{\partial y} = 0 \text{ at } y = 0. \]

\[ u = 0, \quad T = T_v \text{ at } y = a \]

Boundary condition \[ \frac{\partial T}{\partial y} = 0, \] arises due to the coexistence of geometric and thermal symmetry.

Using Boussinesq approximation and incorporating the following non-dimensional variables namely,
\[ x = aX, \quad y = aY, \quad u = \alpha U/a \text{ and } \theta = \Gamma \Theta \]
the dimensionless form of momentum and energy equations and their respective boundary conditions are:
\[ \frac{d^2 U}{d Y^2} - \sigma^2 U = -N + R \theta \Gamma \quad \ldots (3.1.6) \]
\[ \frac{d^2 \theta}{d Y^2} = -U \quad \ldots (3.1.7) \]
\[ \frac{d u}{d Y} = 0 \text{ at } Y = 0; \quad U = 0 \text{ at } Y = 1 \quad \ldots (3.1.8) \]
\[ \frac{d \theta}{d Y} = 0 \text{ at } Y = 0, \quad \theta = 0 \text{ at } Y = 1 \]

where \( \sigma^2 = a^2 e/k \); \( a \) is the half width of the channel; \( k \) is the permeability of the porous medium; \( \nu = \mu/\rho \); \( \nu_e = \nu/\epsilon \)
\( \theta = T_v - T(y) \) and \( \Gamma = \pm 1 \)

Following Morton, (1960)
\[ N = \left( - \frac{a^3}{\alpha v_e} \right) \left[ 1 + \frac{1}{\rho} \frac{\partial P}{\partial x} \right] + g \] is the dimensionless pressure gradient and \( R_e = \frac{(g \beta a^3 | \Gamma |)}{\alpha v_e} \) is the Rayleigh number.

Rayleigh number is positive when \( \Gamma = +1 \) (steady heating of cold fluid or upflow heated) and negative when \( \Gamma = -1 \) (steady cooling of ascending hot fluid or upflow cooled). For positive Rayleigh number pressure and buoyancy oppose each other whereas for negative Rayleigh number they reinforce each other.

3.1(b) Solution Techniques: To provide good design heat transfer information over a wide range of conditions, there is need for flexible analytical and numerical methods.

Equations (3.1.6) and (3.1.7) are coupled equations with homogeneous boundary conditions. They are solved numerically and analytically for the boundary conditions (3.1.8).

Analytical methods of solutions:

1) Regular parameter perturbation method: In this method, choosing \( R_e \) as the perturbation parameter solutions \( U \) and are expanded up to second order in \( R_e \) as:

\[ U = U_0 + R_e U_1 + R_e^2 U_2 + \ldots \] (3.1.9)

\[ \theta = \theta_0 + R_e \theta_1 + R_e^2 \theta_2 + \]
substituting (3.1.9) in (3.1.6) and (3.1.7) the solutions satisfying (3.1.8) up to second order in $R_a$ are given by,

$$U = \frac{N}{\sigma^2} \left( 1 - \frac{\cosh \sigma Y}{\cosh \sigma} \right) + R_a \left[ -\frac{2N}{\sigma^6} \frac{\cosh \sigma Y}{\cosh \sigma} + \frac{N}{2\sigma^4} (Y^2 - 1) \right] + o(R_a^2) \quad (3.1.10)$$

$$\Theta = -\frac{N \sigma^2 Y^2}{2\sigma^4} + \frac{N}{\sigma^4} \frac{\cosh \sigma Y}{\cosh \sigma} + \frac{N}{\sigma^2} \left( \frac{1}{2} - \frac{1}{\sigma^2} \right) + R_a \left[ \frac{2N}{\sigma^6} \frac{\cosh \sigma Y}{\cosh \sigma} - \frac{N \sigma^4 Y^4}{24\sigma^4} + \frac{N}{24\sigma^4} - \frac{N}{4\sigma^4} + \frac{N \sigma^2 Y^2}{4\sigma^4} - \frac{N \sigma^2 Y^2}{\sigma^6} - \frac{2N}{\sigma^6} + \frac{N}{\sigma^6} \right] + o(R_a^2) \quad (3.1.11)$$

Equations (3.1.10) and (3.1.11) are evaluated for small values of $R_a$ and $\sigma$ and it is seen that with increase in $\sigma$ there is decrease in the values of $U$ and $\Theta$ (Table 3.1.1). The presence of porous medium offers resistance to flow.

It is of practical interest to find mass flow rate and friction factor. The mass flow rate is a quantitative manifestation of the effect of permeability on the flow. If $M_p$ denotes mass flow rate per channel width in the presence of porous medium, then $M_p$ is given by

$$M_p = 2 \lambda \rho_0 \int_0^1 (U_0 + R_a U_1) \, dY \quad (3.1.12)$$
From (3.1.12) expression for \( \frac{M_p}{M_r} \) is obtained as,

\[
\frac{M_p}{M_r} = \frac{(1-\tanh \sigma)/\sigma^2 + R_a \left[ (-2N/\sigma^2) \tanh \sigma + 1/6\sigma^4 - 1/\sigma^4 + 2/\sigma^6 \right]}{1/3 + R_a \left( -\frac{272}{5040} \right)}
\]  

.. (3.1.13)

Here \( M_r \) denotes the mass flow rate in the absence of porous medium. It is seen that \( (M_p/M_r) \) decreases with increase in \( \sigma \) but increases with increase in Rayleigh number.

In engineering applications the pressure drop \( \Delta p \) associated with the flow of fluid inside a channel is a quantity of practical interest. The pressure drop \( \Delta p \) is also related to the friction factor. The pressure drop \( \Delta p \) over length \( L \) is related to the friction factor "f" by the following expression:

\[
\Delta p = f \frac{L}{D} \frac{\rho U_m^2}{2}
\]

where \( U_m \) is the mean velocity, \( D \) the diameter, and \( L \) the length. If \( M \) is the flow rate through the channel, the pumping power required to get the fluid through the channel against the pressure drop can be calculated as \( M \Delta p \).

The dimensionless ratio \( \frac{C_{fp}}{C_f} \) (where \( C_{fp} \) is the friction factor in the presence of porous medium and \( C_f \) is the friction factor in the absence of porous medium) is evaluated by using the following expression for friction factor \( C_f \).
In equation (3.1.14) \( D \) is the equivalent diameter given by \( D = 4a \).

The resultant dimensionless ratio \( C_{fp}/C_f \) is obtained as:

\[
\frac{C_{fp}}{C_f} = \frac{-\sigma^2 \left( \frac{1}{3} - R_a \frac{272}{5040} \right)}{\left( 1 - \tanh \sigma \right) + R_a \left( \frac{-2}{\sigma^4} \tanh \sigma - \frac{2}{6\sigma^2} + \frac{2}{\sigma^4} \right)} \quad ..(3.1.15)
\]

Equation (3.1.15) is evaluated for various values of \( \sigma \) and \( R_a \). It is found that \( C_{fp}/C_f \) increases with increase in \( \sigma \) and decreases with increase in Rayleigh number.

The important quantity of interest in heat transfer problems is the rate of heat transfer \( q = \alpha_e (d\theta/dy) \) \([\alpha_e = (\lambda_e/\rho C_p)\) thermal diffusivity]. Following Morton (1960) the dimensionless heat transfer rate- Nusselt number, is defined as:

\[
Nu = \frac{q_{total} D}{\lambda_e \theta_m}, \text{ where } D = 4a \text{ and } \theta_m \text{ is the difference between wall temperature and mean temperature across the channel and } \theta_m = \int_0^1 \theta \, dy.
\]

The Nusselt number is evaluated as
For very small values of \( R_a \) upto 5, regular parameter perturbation method is effective in calculating important physical quantities such as Nusselt number, mass flow rate and friction factor.

### ii) Singular parameter perturbation:

The boundary layer type solutions for large \( \sigma \) are obtained by singular perturbation method treating \( 1/\sigma = \eta \) as the perturbing parameter. Using this method solutions are obtained for \( R_a < < \sigma^2 \) and \( R_a \sim O(\sigma^2) \) by two approaches.

(a) \( R_a < < \sigma^2 \)

By combining (3.1.6) and (3.1.7) a single equation for \( U \) is obtained as:

\[
\frac{d^4 U}{d \gamma^4} - \sigma^2 \frac{d^2 U}{d \gamma^2} + R_a \Gamma = 0 \quad ..(3.1.17)
\]

Defining \( \eta = 1/\sigma \), equation (3.1.17) reduces to

\[
\eta^2 \frac{d^4 U}{d \gamma^4} - \frac{d^2 U}{d \gamma^2} + R_a \eta^2 \Gamma U = 0 \quad ..(3.1.18)
\]

Equation (3.1.18) has to be solved using the following boundary conditions:
$U = 0 \text{ at } |Y| = 1 \quad \ldots (3.1.19a)$

\[
\frac{d^2 U}{dY^2} - \sigma^2 U = -N \text{ at } |Y| = 1
\]

\[
\frac{dU}{dY} = 0 \text{ at } Y = 0 \quad \ldots (3.1.19b)
\]

\[
\frac{d^3 U}{dY^3} - \sigma^2 \frac{dU}{dY} = 0 \text{ at } Y = 0
\]

**Outer solution:**

When $\eta \to 0$ or $\sigma \to \infty$, equation (3.1.17) reduces to

\[
\frac{d^2 U_{\text{out}}}{dY^2} = 0 \quad \ldots (3.1.20)
\]

Solution of (3.1.20) satisfying the outer boundary condition (3.1.19b) is

\[
U_{\text{outer}} = A \quad \ldots (3.1.21)
\]

**Inner solution:** The inner solution of (3.1.18) is obtained by defining the stretching variable,

\[
Y^* = \frac{1-|Y|}{\eta} \quad \ldots (3.1.22)
\]

Substituting (3.1.22) in (3.1.18) and neglecting $\sigma \eta^2 U$ term the following equation is obtained

\[
\frac{d^4 U_{\text{inner}}}{dY^4} - \frac{d^2 U_{\text{inner}}}{dY^2} = 0 \quad \ldots (3.1.23)
\]

The general solution of (3.1.23) is
\[
U_{\text{inner}} = C + D |Y'| + Ee|Y'| + Fe^{-|Y'|} \quad \text{(3.1.24)}
\]

As \( Y' \rightarrow \infty \), \( Ee^{-|Y'|} \rightarrow 0 \). Therefore, neglecting \( Ee^{-|Y'|} \) term (3.1.24) is transformed as

\[
U_{\text{inner}} = C + D |Y'| + Fe^{-|Y'|} \quad \text{(3.1.25)}
\]

The constants \( C \) and \( D \) in (3.1.24) are determined using the inner boundary conditions namely,

\[
U_{\text{inner}} = 0 \quad \text{at } |Y'| = 0
\]

\[
\frac{d^2U_{\text{inner}}}{dY'^2} - U_{\text{inner}} = -N \eta^2 \quad \text{at } |Y'| = 0 \quad \text{(3.1.26)}
\]

The inner solution of (3.1.18) satisfying (3.1.26) is obtained as:

\[
U_{\text{inner}} = \frac{N}{\sigma^2} + D |Y'| - \frac{N}{\sigma^2} e^{-|Y'|} \quad \text{(3.1.27)}
\]

The constant \( D \) in term (3.1.27) which is linear is evaluated using the outer boundary conditions,

\[
\frac{dU}{dY'} = 0 \quad \text{at } |Y'| = 1/\eta \quad \text{(3.1.28)}
\]

The inner solution is given by

\[
U_{\text{inner}} = \frac{N}{\sigma^2} \left[ 1 - e^{-|Y'|} - e^{-|Y'|} \right] \quad \text{(3.1.29)}
\]

To determine the constant \( A \) the matching principle after Julian Cole (1968) viz,
is used, and \( A \) is given by

\[
A = \frac{N}{\sigma^2} \left[ 1 - e^{-1/\eta} \left| Y^* \right| - e^{-|Y^*|} \right] \tag{3.1.31}
\]

As \( Y^* \to \infty, \quad e^{-Y^*} \to 0 \) and \( e^{-1/\eta|Y^*|} \) is negligible.

Hence,

\[
A = \frac{N}{\sigma^2} \tag{3.1.32}
\]

The composite solution is

\[
U = U_{\text{outer}} + U_{\text{inner}} - \text{common term}
\]

\[
U = \frac{N}{\sigma^2} + \frac{N}{\sigma^2} \left[ 1 - e^{-1/\eta} \left( \frac{1-|Y|}{\eta} \right) - e^{-(1-|Y|)/\eta} \right] - \frac{N}{\sigma^2}
\]

\[
= \frac{N}{\sigma^2} \left[ 1 - e^{-1/\eta} \left| Y^* \right| - e^{-|Y^*|} \right] \tag{3.1.33}
\]

The important feature of this equation is that it is independent of Rayleigh number. Further for large values of \( \sigma \), \( \frac{d^2U}{dY^2} = 0 \), or for low permeable media flow equation reduces to Darcy model. It is to be noted that equation for \( \Theta \) is not of boundary layer type. The expression for \( \Theta \) from (3.1.7) and (3.1.33) is
\[
\frac{N}{\sigma^2} \left[ \frac{y^2}{2} - \frac{e^{-\sigma} y^2}{2} + \frac{e^{-\sigma} y^3}{2} - \frac{e^{-\sigma} e^{\sigma y}}{\sigma^2} \right] + \frac{N}{\sigma^2} \left[ \frac{1}{2} - \frac{e^{-\sigma}}{2} - \frac{1}{2 \sigma^2} - \frac{e^{-2\sigma}}{\sigma^2} \right] \\
\frac{N Y}{\sigma^2} \frac{e^{-\sigma} e}{6} - \frac{1}{2 \sigma^2} + \frac{e^{-2\sigma}}{2 \sigma^2} \quad \ldots(3.1.34)
\]

b) \( R_a \sim \sigma^2 \ (R_a/\sigma^2 = 0(1)) \)

Dividing (3.1.6) by \( \sigma^2 \) through out leads to

\[
\frac{1}{\sigma^2} \frac{d^2 U}{d Y^2} - U = - \frac{N}{\sigma^2} + \left( \frac{R_a}{\sigma^2} \right) \theta \Gamma \quad \ldots(3.1.35)
\]

For \( \sigma > 1 \) and \( R_a \) comparable with \( \sigma^2 \) the momentum equation valid in the outer region is

\[
- U = - \frac{N}{\sigma^2} + \left( \frac{R_a}{\sigma^2} \right) \theta \quad \ldots(3.1.36)
\]

Assuming \( \Gamma = 1 \) and substituting (3.1.36) in (3.1.7) results in,

\[
\frac{d^2 \theta}{d Y^2} = - \frac{N}{\sigma^2} + \left( \frac{R_a}{\sigma^2} \right) \theta \quad \ldots(3.1.37)
\]

Solution of (3.1.37) satisfying the boundary conditions

\[
\frac{d \theta}{d Y} = 0 \text{ at } Y = 0
\]

\[\theta = 0 \text{ at } Y = 1\] is given by
The above solution is uniformly valid in the entire domain.

Solution for velocity distribution is obtained for the outer and inner regions separately and then uniformly valid solution is constructed.

The outer solution for \( U \) is obtained by incorporating (3.1.38) in (3.1.36) and is given by

\[
U_{\text{outer}} = \frac{N}{\sigma^2} \left[ 1 - \left( 1 - \frac{e^{\sqrt{R_a}/\sigma^2} Y + e^{-\sqrt{R_a}/\sigma^2} Y}{e^{\sqrt{R_a}/\sigma^2} + e^{-\sqrt{R_a}/\sigma^2}} \right) \right] \tag{3.1.39}
\]

This solution satisfies the boundary condition \( \frac{dU}{dY} = 0 \) at \( Y = 0 \) in the outer region.

To obtain the inner solution of equation (3.1.36) satisfying the boundary condition \( U = 0 \) at \( Y = 1 \) stretching variable \( \frac{Y'}{Y} = \frac{(1-|Y|)}{\eta} \) is used. Incorporating the stretching
variable equation (3.1.6) reduces to

\[ \frac{1}{\eta^2} \frac{d^2 U_{inner}}{d Y'^2} - \frac{1}{\eta^2} U_{inner} = -N + R_a \Theta \Gamma \]  \hspace{1cm} \text{(3.1.40)}

Since the solution for \( \Theta \) is already used to obtain the outer solution for \( U \) the term involving \( \Theta \) in the expression (3.1.40) is neglected. Hence the equation for \( U \) valid in the inner region is

\[ \frac{d^2 U_{inner}}{d Y'^2} - U_{inner} = -N \eta^2 \]  \hspace{1cm} \text{(3.1.41)}

The boundary condition for \( U_{inner} \) is

\[ U_{inner} = 0 \text{ at } |Y'| = 0 \]  \hspace{1cm} \text{(3.1.42)}

and further \( U_{inner} \) should be finite as \( Y' \to \infty \).

Therefore the solution of \( U_{inner} \) satisfying the boundary condition (3.1.42) is

\[ U_{inner} = \frac{N}{\sigma^2} \left[ 1 - e^{-|Y'|} \right] \]  \hspace{1cm} \text{(3.1.43)}

It is interesting to note that in this case no matching condition is necessary since the requirements that \( U_{inner} \) should be finite as \( Y' \to \infty \), eliminates the determination of one constant.

The composite solution is

\[ U = U_{inner} + U_{outer} - \text{common terms} \] or
The composite satisfies both the conditions $U = 0$ at $Y = 1$ and
\[ \frac{dU}{dY} = 0 \] at $Y = 0$ except for a transcendentally small term.

The expression (3.1.44) is valid for all values of positive Rayleigh number and for large values of $\sigma$. For very small values of $\sigma$ the expressions are not valid even for small values of $R_a$. The boundary layer thickness is of the order of $\frac{1}{\sigma}$ for small values of $\sigma$ and is of the order of $1/\sigma^2$ for large values of $\sigma$.

Having obtained $"U"$ and $\Theta$, $\text{Nu}$, $\frac{M_p}{N_p}$ and $\frac{C_f\rho}{C_l}$ can be calculated using (3.1.16), (3.1.13) and (3.1.15) respectively.

Numerical Methods:

Perturbation methods are valid for small values $R_a$, for very large values of $\sigma$ and for values of $R_a$ of the order of $\sigma$.

In order to obtain solutions for arbitrary values of the parameters, semi-numerical method and numerical methods, are utilised.

i) Semi-numerical method: In this method equation (3.1.17) is solved using the following boundary conditions.
\[ U = 0 \text{ at } Y = \pm 1 \]  

\[
\frac{d^2U}{dY^2} - \sigma^2 U = -N \text{ at } Y = \pm 1
\]  

General solution of (3.1.17) satisfying (3.1.45) is

\[ U = A e^{m_1 Y} + B e^{m_2 Y} + C e^{m_3 Y} + D e^{m_4 Y} \]  

(3.1.46)

where \(m_1, m_2, m_3\) and \(m_4\) are roots of the indicial equations and constants \(A, B, C, D\) are evaluated from (3.1.45). From (3.1.45) and (3.1.46) an augmented matrix \(PX = Q\) is obtained where,

\[
X = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \quad Q = \begin{bmatrix} 0 \\ 0 \\ -N \end{bmatrix} \quad \text{and} \quad P = \text{coefficient matrix}
\]

To evaluate \(A, B, C\) and \(D\) the matrix equation is solved with SIMQ subroutine in Dec 10 for different values of parameters \(R\) and \(N\). Substituting the values of \(A, B, C\) and \(D\) value of \(U\) is obtained from equation (3.1.46) and \(\theta\) is evaluated from the value of \(U\). Important physical quantities such as Nusselt number, mass flow rate and skin friction are evaluated as mentioned previously.

The above method yields solution for both small and large values of the porous parameter and Rayleigh number, unlike perturbation methods which impose some limitations on
the values of the physical parameters.

(ii) Runge-Kutta-Gill Method: The governing equations (3.1.6) and (3.1.7) are solved numerically using the self starting Runge-Kutta-Gill method with Newton and Raphson modification after Sen et al. (1976) for various values of $\sigma$, $N$ and $R_s$. The above method directly evaluates $\Theta$, $\frac{d\Theta}{dY}$, $U$ and $\frac{dU}{dY}$ and the step size can be controlled. In this method the function $f(X,Y)$ must be evaluated for different values of $Y$ in every step. This repeated evaluation of functional values takes much computer time. The above method converges only for values of $\sigma$ less than 40. The integrals appearing in numerical and semi-numerical methods are evaluated using trapezoidal rule after John Mc Cormick et al. (1971). The values of $U$, $\frac{dU}{dY}$, $\Theta$ and $\frac{d\Theta}{dY}$ are utilised to calculate Nusselt number, mass flow rate and friction factor.

(iii) Keller Box Method: In this method the differential equations (3.1.6) and (3.1.7) are converted into first order system and the difference equations are obtained using central difference scheme. The resulting set of algebraic equations are solved using block tri-diagonal scheme after Keller (1971). The values of velocity, temperature and their respective gradients thus obtained, are used to calculate important physical quantities.

For steady cooling of ascending hot, fluid, solutions are obtained by incorporating a negative Rayleigh number.
3.1(c) Discussion of the methods of solution.

The convection problem has been solved using five methods. In this section the limitation of various methods are discussed. The regular parameter perturbation solutions are valid for small values of $R_a$. For small values of $R_a$ and various values of $\sigma$, it agrees with the solutions obtained by other methods as indicated in Table 3.1.1. Singular perturbation solutions are obtained for very large values of $\sigma$ for two situations namely $R_a < < \sigma^2$ and $R_a \sim O (\sigma^2)$. For $R_a < < \sigma^2$, $R_a$ has negligible effect on the velocity and temperature profiles. For $R_a \sim O (\sigma^2)$ the velocity and temperature profiles depend on Rayleigh number and for large values of $\sigma$ the results agree with the values obtained using numerical method as seen in Table 3.1.2. The semi-numerical method yields solutions for arbitrary values of parameters whereas the Runge-Kutta-Gill method fails to converge for $\sigma > 30$. The problem is solved using an implicit finite difference scheme known as the Keller Box method. The above method yields solutions for values of $\sigma > 100$. The method is fast converging though the algebra is complex. The solutions obtained by the five methods are compared in Table 3.1.1. There is close agreement among the five methods and the Keller Box method is found to be the best suited, since it yields solutions for arbitrary values of the parameters.
### TABLE 3.1.1
VALUES OF \( \theta \) AND \( \eta \) FOR VARIOUS VALUES OF \( \sigma \) OBTAINED BY DIFFERENT METHODS FOR \( \mathbb{R}_3 = 5 \) & \( N = 10 \).

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>Regular perturbation</th>
<th>( \theta )</th>
<th>( \eta )</th>
<th>( \theta )</th>
<th>( \eta )</th>
<th>Numerical method</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.3494</td>
<td>0.1603</td>
<td>-</td>
<td>-</td>
<td>0.3614</td>
<td>0.1640</td>
</tr>
<tr>
<td>10</td>
<td>0.0975</td>
<td>0.0460</td>
<td>-</td>
<td>-</td>
<td>0.0977</td>
<td>0.0461</td>
</tr>
<tr>
<td>25</td>
<td>0.0159</td>
<td>0.0076</td>
<td>0.1600</td>
<td>0.0076</td>
<td>0.0159</td>
<td>0.0077</td>
</tr>
<tr>
<td>50</td>
<td>0.0040</td>
<td>0.0020</td>
<td>0.0039</td>
<td>0.0019</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>70</td>
<td>0.0020</td>
<td>0.0009</td>
<td>0.0020</td>
<td>0.0009</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>Keller Box method</th>
<th>( \theta )</th>
<th>( \eta )</th>
<th>( \theta )</th>
<th>( \eta )</th>
<th>Seminumerical method</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.3663</td>
<td>0.1713</td>
<td>0.3614</td>
<td>0.1640</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.0990</td>
<td>0.4805</td>
<td>0.0971</td>
<td>0.0460</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>0.0162</td>
<td>0.0078</td>
<td>0.0159</td>
<td>0.0076</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.0036</td>
<td>0.0018</td>
<td>0.0039</td>
<td>0.0019</td>
<td></td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>0.0021</td>
<td>0.0009</td>
<td>0.0020</td>
<td>0.0009</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3.1(d) Results and discussions:

The present investigation brings out the influence of porous parameter $\sigma$, Rayleigh number and normalised pressure on laminar convection in a vertical channel filled with porous medium. In steady heating of ascending cold fluid buoyancy and pressure oppose each other. In steady cooling of ascending hot fluid buoyancy and pressure reinforce each other.

The influence of porous parameter on flow and heat transfer characteristics is investigated first. The effect of $\sigma$ on velocity and temperature are shown in Figures (3.1.2 & 3.1.3) respectively. Both velocity and temperature decrease with an increase in $\sigma$ because the porous material offers resistance to the flow. Fig (3.1.4) shows that Nusselt number increases with an increase in $\sigma$ for small values of $Ra$ which is in conformity with the results of Kaviany (1985). The ratio of mass flow rate $(M_p/M_f)$ decreases with an increase in $\sigma$ and ratio of friction factor $C_{fp}/C_f$ increases with an increase in $\sigma$ as illustrated in Figs (3.1.5) and (3.1.6). The above results are in qualitative agreement with those of Rudraiah and Nagaraj (1977).

The influence of Rayleigh number on velocity distribution is indicated in Fig (3.1.7). For small values of $\sigma$ and large values of $Ra$, there is a slight increase in velocity near the wall which can be explained on the basis of
FIG. 3.1.2. VELOCITY DISTRIBUTION VS Y

$R_a = 5, N = 1, T_w > T_o$
FIG. 3.1.4. Nusselt number Vs $R_a$

$T_W > T_0$

$\alpha$

0

5

50

70
temperature dependence of viscosity after Lyutikas (1968). Though the variation of viscosity with temperature is not considered, viscosity does decrease in a fluid as temperature increases considerably. Subsequently, for large values of Rayleigh number, velocity increases by overcoming viscous resistance at the wall. For higher values of $\sigma$ this rise in velocity is not considerable. Nusselt number and ratio of mass flow rate increase with increase in $R_a$ as indicated in Figs (3.1.4 & 3.1.5).

Since normalised pressure $N$ appears as a linear term, its effect is also linear on laminar convection as seen in Table 3.1.3.

For steady cooling of ascending fluid, pressure and buoyancy reinforce each other and hence the magnitude of velocity and temperature are higher than in the case of steady heating of ascending fluid. The effects of $\sigma$, on $\text{Nu}$, $\text{Mp}$ and $C_{f_p}/C_f$ are similar to the case of steady heating of ascending cold fluid. For this situation an increase in Rayleigh number does not lead to a rise in velocity near the wall. For small values of $\sigma$ and $R_a > 100$ there is flow reversal as illustrated in Fig (3.1.8) and an adverse temperature gradient is created due to which heat is transferred from the fluid to the wall. This behaviour is in conformity with the results of Ostrach (1954) in the absence of porous media. Ostrach found that for steady cooling, flows
### TABLE 3.1.2

**Comparison of values of \( U \) obtained by singular perturbation method \((r_a/\sigma^2 = 0) \) and matrix method for \( \sigma = 50, \ n = 10 \)**

<table>
<thead>
<tr>
<th>( y )</th>
<th>( r_a = 1000 )</th>
<th>( r_a = 5 )</th>
<th>( r_a = 1000 )</th>
<th>( r_a = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.003314</td>
<td>0.003399</td>
<td>0.003312</td>
<td>0.003399</td>
</tr>
<tr>
<td>0.2</td>
<td>0.003341</td>
<td>0.003399</td>
<td>0.003420</td>
<td>0.003399</td>
</tr>
<tr>
<td>0.6</td>
<td>0.003556</td>
<td>0.003399</td>
<td>0.003556</td>
<td>0.003399</td>
</tr>
<tr>
<td>0.8</td>
<td>0.003748</td>
<td>0.00374</td>
<td>0.003740</td>
<td>0.00374</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

### TABLE 3.1.3

**Values of \( U \) and \( \theta \) for various values of \( n \) for \( \sigma = 20, \ r_a = 100 \)**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( U )</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0022196</td>
<td>0.001126</td>
</tr>
<tr>
<td>10</td>
<td>0.0221960</td>
<td>0.012690</td>
</tr>
<tr>
<td>100</td>
<td>0.2219600</td>
<td>0.112690</td>
</tr>
<tr>
<td>1000</td>
<td>2.2219600</td>
<td>1.126900</td>
</tr>
</tbody>
</table>
Fig. 3.1.5: \( \frac{M_p}{M_f} \) vs \( Ra \)

- \( M_p - Mass \) flow rate with porous media
- \( M_f - Mass \) flow rate without porous media

Units: \( Ra \)
\( C_{fp} \) - Friction factor with porous media

\( C_f \) - Friction factor in absence of porous media

**FIG.3.1.6.** \( \frac{C_{fp}}{C_f} \) Vs \( \sigma \)
FIG. 3.1.7. VELOCITY DISTRIBUTION Vs Y

\[ \sigma = 5, N = 1, T_w > T_0 \]
**FIG. 3.1.8. VELOCITY DISTRIBUTION Vs Y**

\[ U \times 10^{-3} \]

\[ Y \]

\[ \sigma = 10, N = 1, T_W < T_0 \]

- \( R_\alpha = 100 \)
- \( R_\alpha = 50 \)
- \( R_\alpha = 5 \)
- \( R_\alpha = 5000 \)
- \( R_\alpha = 1000 \)
### Table 3.1.4

**Variation of $\theta$ with Rayleigh Number for Upflow Cooled Situation at $\gamma=0.2$**

<table>
<thead>
<tr>
<th>$R_a$</th>
<th>$\theta$</th>
<th>$R_a$</th>
<th>$Nu$</th>
<th>$Nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.42654</td>
<td>5</td>
<td>10.6952</td>
<td>10.6964</td>
</tr>
<tr>
<td>20</td>
<td>0.58702</td>
<td>50</td>
<td>10.9840</td>
<td>10.0960</td>
</tr>
<tr>
<td>50</td>
<td>1.62440</td>
<td>100</td>
<td>11.4476</td>
<td>9.4976</td>
</tr>
<tr>
<td>100</td>
<td>-0.37884</td>
<td>1000</td>
<td>18.2440</td>
<td>9.1964</td>
</tr>
</tbody>
</table>
on either side of a critical value of Rayleigh number are of completely different character. The values of Rayleigh number for which flow shows reversal increases with increase in $\sigma$. It is seen from Table 3.1.4 and Fig (3.1.8) that, for steady cooling of ascending hot fluid $\theta$ and $U$ increase with increase in $Ra$, indicating that pressure and buoyancy reinforce each other. Table 3.1.5 shows that heat transfer decreases with increase in $Ra$.

The results of the present section are of interest in transport processes occurring in electronic devices and nuclear reactors. A study of temperature distribution and the associated heat flux distribution will aid in an assessment and evaluation of cooling rate needed for packed bed reactors and also in the optimum design of these devices. The heat transfer coefficient obtained will be useful to estimate the cooling rate and life span of a reservoir.

3.2 MIXED CONVECTION IN THE PRESENCE OF DARCY AND VISCOUS DISSIPATIONS

Heat dissipation is encountered in packed bed chemical reactors and also during storage of agricultural products due to metabolism of the product. In almost all transport processes viscous and Darcy dissipations are neglected. This is valid at 300 K, 1 atmospheric pressure, and at terrestrial gravity, for most gases and for low Prandtl number liquids. However, in high gravity appliances such as in gas turbine
blade cooling, where body force is as large as $10^4 g$
dissipation effects become important. The propagation of
waves in a porous medium has small free decay time and hence
has considerable dissipation effects. This necessitates the
inclusion of dissipative effects. There are many engineering
applications where the viscous dissipations become important
such as high speed flow through small conduits, and extrusion
processes. Hence to analyse the effects of dissipation in
such practical applications, mixed convection in a vertical
channel filled with porous medium is investigated in the
presence of viscous and Darcy dissipations.

Rudraiah and Nagaraj (1977) have investigated natural
convection through vertical porous stratum with boundaries
maintained at constant temperature. They have obtained
solutions for the momentum and energy equations including
Darcy and viscous dissipations and found that increase in
porous number decreases the influence of Darcy and viscous
dissipations. Oberbeck convection through vertical porous
stratum in the presence of dissipation terms was studied by
Rudraiah, Venkatachallapa and Mala Shetty (1982).
Chandrasekhara (1984) investigated the effects of lateral
mass flux, on velocity and temperature in Ekman boundary
layers in a saturated porous medium in the presence of
dissipation terms. His investigations revealed that Darcy
dissipation counteracts viscous dissipation and reverses the

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direction of heat transfer.

The present section is devoted to numerical analysis of steady mixed convection in a vertical channel filled with porous medium in the presence of Darcy and viscous dissipations. The volume averaged Brinkman model is used and the governing equations are solved numerically using an implicit finite difference scheme developed by Keller (1971) for steady heating of ascending cold fluid and steady cooling of ascending hot fluid.

3.2(a) Mathematical formulation:

The problem being an extension of the analysis in section 3.1, the physical model and the assumptions involved are the same with the additional feature that the fluid within the channel is heated by both Darcy and viscous dissipations.

The pertinent equations of the problem are:

\[
\frac{\mu}{\varepsilon} \frac{\partial^2 u}{\partial y^2} - \frac{\mu}{k} u = \frac{\partial p}{\partial x} - \rho g
\]  

..(3.2.1)

\[
u \frac{\partial T}{\partial x} = \frac{\alpha_e}{\varepsilon} \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\varepsilon \rho_0 C_p} \left( \frac{du}{dy} \right)^2 + \frac{\mu}{kC_p \rho_0} u^2
\]  

..(3.2.2)

where \( k, \varepsilon, \mu, \mu/\varepsilon, u, \rho, \alpha_e, \lambda_e \) and \( C \) are given by (3.1.3).

The second term on the right hand side of (3.2.2) represents the viscous dissipation and the third term represents the Darcy dissipation.
Incorporating the dimensionless variables from (3.1.5) the governing equations of the problem in dimensionless form are

\[
\frac{d^2 U}{d Y^2} - \sigma^2 U = - N + R_a \Theta \Gamma \quad \quad \quad \quad (3.2.3)
\]

\[
\frac{d^2 \Theta}{d Y^2} = - \Theta + \frac{1}{Pr} \left( \frac{E}{R_a^2} \right) \left( \frac{dU}{dY} \right)^2 + \left( \frac{\sigma^2}{Fr} \right) \left( \frac{E}{R_a^2} \right) U^2 \quad \quad \quad (3.2.4)
\]

In the above equations, \( R_a, N, \) and \( \sigma \) are the same as in section 3.1 and \( E, R_a \) and \( Pr \) are given by

\[
E = \frac{U^2}{C_p \Gamma} \quad \text{(Eckert number)}
\]

\[
Re = \frac{U a}{v_e} \quad \text{(Reynolds number)}
\]

\[
Pr = \frac{v_e}{\alpha_e} \quad \text{(Prandtl number)}
\]

Equations (3.2.3) and (3.2.4) satisfy the boundary conditions (3.1.8).

3.2 (b) **Analysis:**

The coupled equations (3.2.3) and (3.2.4) are non-linear. Equations (3.2.3) and (3.2.4) in the absence of Darcy and viscous dissipations reduce to those of Chandrasekhara and Radha Narayanan (1989). Further, the above equations in the absence of porous media reduce to those
equations analysed by Grief and Habib (1971) in the absence of radiation. Since both $U$ and $\theta$ are coupled a closed form analytical solution is a complicated task and is likely to have limitations on the values of various parameters.

Equations (3.2.3) and (3.2.4) under the boundary conditions (3.1.8) are solved numerically for $E/R^2_a=1$ using an implicit finite difference scheme which is described in detail in chapter II.

In order to assess the accuracy of the method the velocity and temperature values are compared with Chandrasekhara et al. (1989) and Grief et al. (1971) and presented in Table 3.2.1. The results are found to be in good agreement. The method yields the values of $U$, $dU/dY$, $\theta$, $d\theta/dY$ from which quantities of physical interest like Nusselt number, ratio $M_{pd}/M_p$ ($M_{pd}$ is the mass flow rate in the presence of dissipation and $M_p$ is the mass flow rate in the absence of dissipation) and ratio of friction factor $C_{fpd}/C_f$ ($C_{fpd}$ is the friction factor in the presence of dissipation terms and $C_f$ is the friction factor in the absence of dissipation.) are calculated for various values of $\sigma$, $R_a$ and $Pr$.

For steady cooling of ascending hot fluid the governing equations are solved by incorporating a negative Rayleigh number, and the important physical quantities are evaluated as in previous section.
### TABLE 3.2.1

**COMPARISON OF VELOCITY AND TEMPERATURE FOR N = 1**

<table>
<thead>
<tr>
<th>Present Analysis</th>
<th>Grief et al. (1971)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 0$, $R_a = 16$</td>
<td>$\sigma = 0$, $R_a = 16$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$y$</th>
<th>$U$</th>
<th>$\theta$</th>
<th>$U$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.1146</td>
<td>0.059</td>
<td>0.125</td>
<td>0.059</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1130</td>
<td>0.056</td>
<td>0.115</td>
<td>0.056</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1060</td>
<td>0.048</td>
<td>0.109</td>
<td>0.048</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0910</td>
<td>0.035</td>
<td>0.100</td>
<td>0.037</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Present Analysis</th>
<th>Chandrasekhar et al. (1989)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 50$, $R_a = 5$</td>
<td>$\sigma = 50$, $R_a = 5$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$y$</th>
<th>$U$</th>
<th>$\theta$</th>
<th>$U$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000364</td>
<td>0.000185</td>
<td>0.000389</td>
<td>0.000190</td>
</tr>
<tr>
<td>0.2</td>
<td>0.000365</td>
<td>0.000177</td>
<td>0.000390</td>
<td>0.000185</td>
</tr>
<tr>
<td>0.4</td>
<td>0.000369</td>
<td>0.000156</td>
<td>0.000395</td>
<td>0.000160</td>
</tr>
<tr>
<td>0.6</td>
<td>0.000377</td>
<td>0.000119</td>
<td>0.000399</td>
<td>0.000120</td>
</tr>
</tbody>
</table>
3.2(c) Results and discussions.

The results of the present analysis are presented in figures (3.2.1 to 3.2.4) and Tables 3.2.2. to 3.2.4. The parameters governing the above problem are Pr, N, Ra and φ. First the effect of these parameters on steady heating of ascending cold fluid is discussed.

The effect of dissipation terms for moderate values of Ra is to increase the velocity slightly as seen in Fig (3.2.1), and increase the magnitude of temperature significantly as illustrated in Table 3.2.2. Fig (3.2.1) also indicates that for very high Ra, dissipation increases the velocity considerably.

It is observed that in the absence of dissipations, increase in porous parameter φ retards the flow and in turn reduces the temperature. But the presence of dissipations increase the magnitude of velocity and temperature. Hence, by incorporating dissipations convective part of momentum transfer is increased. In the absence of dissipations, heat transfer rate is negative indicating heat is transferred from the fluid to the wall. However, in the presence of dissipations heat transfer rate is positive. The presence of dissipation reverses the direction of heat transfer. The above behaviour is in conformity with that observed by Chandrasekhara (1984). In the presence of dissipations, velocity and temperature increase with increase in Rayleigh
### Table 3.2.2

**Comparison of Values of \( U \) and \( \theta \) with and without Dissipation**

**Terms for \( n = 5000, R_a = 5, \, Pr = 7 \).**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( U )</th>
<th>( \theta )</th>
<th>( U )</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.131029</td>
<td>-48.2332</td>
<td>0.020144</td>
<td>-7.2190</td>
</tr>
<tr>
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<td>0.020139</td>
<td>-6.9290</td>
</tr>
<tr>
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<td>-40.4147</td>
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<td>-6.0617</td>
</tr>
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<td>0.6</td>
<td>0.128837</td>
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<td>0.020090</td>
<td>-4.6162</td>
</tr>
<tr>
<td>0.8</td>
<td>0.127149</td>
<td>-17.1910</td>
<td>0.020050</td>
<td>-2.5949</td>
</tr>
<tr>
<td>1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

**Without Dissipation**

\( \sigma = 200 \quad \sigma = 500 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( U )</th>
<th>( \theta )</th>
<th>( U )</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.124992</td>
<td>0.062494</td>
<td>0.020040</td>
<td>0.0100</td>
</tr>
<tr>
<td>0.2</td>
<td>0.124993</td>
<td>0.059900</td>
<td>0.020000</td>
<td>0.0096</td>
</tr>
<tr>
<td>0.4</td>
<td>0.124994</td>
<td>0.052494</td>
<td>0.020000</td>
<td>0.0094</td>
</tr>
<tr>
<td>0.6</td>
<td>0.124995</td>
<td>0.039950</td>
<td>0.020000</td>
<td>0.0064</td>
</tr>
<tr>
<td>0.8</td>
<td>0.124997</td>
<td>0.022496</td>
<td>0.020000</td>
<td>0.0036</td>
</tr>
<tr>
<td>1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
number as seen in Fig 3.2.1 and Table 3.2.3 respectively, whereas in the absence of dissipations, magnitude of velocity and temperature decrease with increase in Rayleigh number. The above behaviour is due to the fact that Darcy dissipation and viscous dissipation introduce temperature reversal, and in the momentum equation buoyancy and pressure reinforce each other.

Table 3.2.2 shows that in the presence of dissipation the influence of normalised pressure gradient on temperature profile is considerable even for very high values of $\sigma$ of the order of 500. The velocity profile, Nusselt number and friction factor are not affected much by the normalised pressure gradient in the presence of dissipations.

From Fig (3.2.3) it is evident that Nusselt number in the absence of dissipation decreases with increase in $\sigma$ for $\sigma > 70$. However, in the presence of dissipation Nusselt number steadily increases with increase in $\sigma$ for $R_\alpha > 5$. Fig (3.2.3) also shows that values of Nusselt number in the presence of dissipation are lesser compared to their values in the absence of dissipation, for $R_\alpha < 100$.

The ratio $M_{pd}/M_p$ decreases with increase in $\sigma$ as seen in Fig (3.2.4).

Table 3.2.2 documents the fact that effect of dissipation is dominant only when the dimensionless pressure is large.
### Table 3.2.3

VALUES OF $\theta$ WITH AND WITHOUT DISSIPATION TERMS FOR $N = 100$

AT $y = 0.2$

<table>
<thead>
<tr>
<th>$\sigma$ (W/TB)</th>
<th>WITH DISSIPATION</th>
<th>WITHOUT DISSIPATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 50$</td>
<td>$\theta$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.019168</td>
<td>0.004798</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.064116</td>
<td>0.007220</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.072220</td>
<td>0.018872</td>
</tr>
</tbody>
</table>

| $\sigma = 100$ | $\theta$         | $\theta$            |
| $\theta$        | 0.019027         | 0.0067468           |
| $\theta$        | 0.067468         | 0.019027            |
| $\theta$        | 0.072220         | 0.018872            |

### Table 3.2.4

VALUES OF $U$, $\theta$ AND $Nu$ FOR STEADY HEATING AND STEADY COOLING

FOR $\sigma = 50$, $Pr = 7$, $N = 100$

<table>
<thead>
<tr>
<th>$Ra$</th>
<th>$U$</th>
<th>$\theta$</th>
<th>$Nu$</th>
<th>$U$</th>
<th>$\theta$</th>
<th>$Nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.04050</td>
<td>-0.2689</td>
<td>11.7692</td>
<td>0.039487</td>
<td>-0.256889</td>
<td>11.8572</td>
</tr>
<tr>
<td>50</td>
<td>0.04694</td>
<td>-0.3489</td>
<td>11.3644</td>
<td>0.035690</td>
<td>-0.215583</td>
<td>12.1932</td>
</tr>
<tr>
<td>100</td>
<td>0.06452</td>
<td>-0.6014</td>
<td>9.712</td>
<td>0.032690</td>
<td>-0.18463</td>
<td>12.5244</td>
</tr>
</tbody>
</table>
FIG. 3.2.1. VELOCITY PROFILE $U$ FOR $\sigma = 50$, $\text{Pr} = 7.0$, $N = 100$ AND $T_w > T_o$
FIG. 3.2.2. TEMPERATURE PROFILE $\Theta$ FOR $N=100$, $T_w > T_o$ AND $R_d=5$
FIG. 3.2.3. HEAT TRANSFER PARAMETER Nu FOR $T_w > T_o$, N=100, AND Pr=7.0
FIG. 3.2.4. VARIATION OF $M_{pd}/M_p$ WITH $\sigma$

$N = 100, Pr = 7.0, T_w > T_0$

$R_a = 100$

$R_a = 50$

$R_a = 5$
The influence of Prandtl number on temperature is brought out in Fig (3.2.2). The magnitude of temperature increases considerably with decrease in Prandtl number. The effect of Prandtl number on velocity, Nu, mass flow rate and friction factor is not considerable.

For steady cooling of ascending hot fluid the magnitude of velocity and temperature are smaller compared to steady heating, as seen in Table 3.2.4, which is opposite to the behaviour observed in the absence of dissipation. The velocity, temperature and mass flow rate decrease with increase in $R_a$ whereas Nu and friction factor exhibit an opposite trend.

The important result of the present investigation is that the effect of dissipations are significant when the dimensionless pressure gradient is high. The results of the present investigation emphasize the fact that dissipations are to be included in flow and stability problems to obtain physically relevant results.

3.3 UNSTEADY LAMINAR CONVECTION IN THE PRESENCE OF DISSIPATIONS AND A HEAT GENERATING SOURCE

Section 3.2 highlights the role of dissipations in combined natural and forced convection in a vertical channel filled with porous medium. The concept of internally energised porous medium is of interest in various applications. These include nuclear fuel shielding, solar
collectors, and novel methods of energy production. The solid particles forming the porous structure may be electrically heated. The enormous specific surface area of porous media enables high specific ratings even with small temperature driving forces, between the solid and fluid. Unsteady internal flows with unsteady heat transfer are encountered in a wide variety of heat transfer devices like starting of a rocket engine, shutting of a nuclear reactor and during changes in propulsion power of a power plant. The above engineering applications have stimulated our interest in unsteady laminar convection in a heat generating vertical channel filled with porous medium.

A survey of literature on unsteady flows in the absence of porous medium has been furnished by Varshney (1979). Yamamoto and Iwamure (1976) have investigated the flow with convective acceleration through a porous medium. Gulab Ram and Mishra (1977) applied these equations to study the MHD flow of a conducting fluid through porous media.

Effects of free convection and mass transfer flow through porous medium has been studied by Raptis, Tzivanidis and Kafousias (1981). Raptis and Perdikis (1985) have analysed unsteady flow through a porous medium bounded by an infinite vertical plate. Natural convection heat transfer in enclosures due to uniform heat generation in a porous media has been investigated for different geometries by Gasser &
Karimi (1976), Hardee and Nilson (1977). Saatdjian and Caltagirone (1980) presented a numerical study of convection in porous media between two horizontal impermeable planes heated from below. As the media is heated the porous matrix decomposed exothermically into gaseous products, simulating heat generation and mass transfer. The present section presents a numerical study of unsteady laminar convection in a vertical channel filled with porous medium in the presence of a uniform heat source and dissipations.

3.3.(a) **Mathematical formulation:**

The physical configuration is the same as in the previous sections and the following additional features are incorporated.

1. The flow is assumed to be steady initially and changes to unsteady for $t^* > 0$, when wall temperature varies with time as $T_w = T_0 + \left(\frac{\Gamma}{a}\right) \times \phi (t^*)$, where $\phi (t^*)$ is an arbitrary function of $t^*$ indicating unsteadiness in wall temperature.

2. A heat generating source $Q_g$ is enclosed in the channel in addition to the viscous and Darcy dissipations.

The governing equations of momentum and energy under the above assumptions are:

$$\rho \frac{\partial u}{\partial t} = - \frac{\partial p}{\partial x} + \frac{\mu}{\varepsilon} \frac{\partial^2 u}{\partial y^2} - \frac{\mu}{K} u + \rho g$$  \hspace{1cm} (3.3.1)

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2} + \left(\frac{\mu}{\rho_0 C_p}\right) \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\mu u^2}{\varepsilon \kappa p C_p}\right) + \frac{Q_g}{\rho_0 C_p}$$  \hspace{1cm} (3.3.2)
where $\alpha$, $u$, $\rho$, $\mu$, $\mu/\alpha$, $\rho_0$, $\alpha_\phi$ and $C_p$ are defined in section 3.1. and $Q_g$ is the heat generating term. The second and third terms on R.H.S of (3.3.2) are viscous and Darcy dissipation terms respectively. The fourth term denotes internal heat generation.

The boundary conditions on velocity and temperature fields are:

$$T = T_w, u = 0 \text{ at } y = a \text{ for } t^* = 0 \quad \ldots (3.3.3)$$

$$\frac{\partial T}{\partial y} = 0, \frac{\partial u}{\partial y} = 0 \text{ at } y = 0$$

Equations (3.3.1) and (3.3.2) are non-dimensionalised using the following quantities,

$$x = a x, y = a Y, u = \frac{\alpha_\phi U}{a}, t = \frac{a^2 t^*}{\alpha} \ldots (3.3.4)$$

$$T_w - T = \Gamma \Theta \phi (t^*), \frac{T_w - T_o}{T_w - T_o} = \phi (t^*)$$

Incorporating (3.3.4) in (3.3.1) and (3.3.2) the dimensionless form of momentum and energy equations are:

$$\frac{1}{Pr} \frac{\partial \theta}{\partial t^*} = N + \frac{\partial^2 \theta}{\partial Y^2} - \phi^2 \theta - R_a^* (\phi - 1) - R_a \phi \theta \ldots (3.3.5)$$

$$\frac{\partial \theta}{\partial t^*} + \left( \frac{\theta}{\phi} \right) \frac{\partial \phi}{\partial t^*} = \frac{U}{\phi} + \frac{\partial^2 \theta}{\partial Y^2} - \left( \frac{N'}{\phi} \right) \left( \frac{\theta U}{\phi} \right)^2 - \left( \frac{\theta p}{\phi} \right) U^2$$

$$\ldots (3.3.6)$$

In equations (3.3.5) and (3.3.6) $N$, $R_a$ and $\sigma$ are dimensionless pressure, Rayleigh number and porous parameter respectively as defined in section 3.1; $E$, $Pr$ and $R_a$ denote Eckert number,
Prandtl number and Reynolds number as defined in section 3.2;

\[ \frac{Q_d}{a^2} \] is the dimensionless heat source parameter;

\[ R_a = \frac{g \beta a^2 K}{\alpha} \] is the modified number; \( \phi(t') = 1 + \delta t'^2 \);

\[ N' = \frac{E}{R^2 \beta} \quad \text{and} \quad P = \frac{E \sigma^2}{R^2 \beta} \]

The initial conditions are:

\[ \begin{cases} \frac{\partial^2 U}{\partial Y^2} - \sigma^2 U = -N' + R_a \phi & t' = 0 \\ \frac{\partial^2 \theta}{\partial Y^2} = -U + N' \left( \frac{\partial U}{\partial Y} \right)^2 + P U^2 & \end{cases} \] \hspace{1cm} (3.3.7)

It may be remarked that for \( \frac{\partial \phi}{\partial t'} = 0 \& \phi = 1 \), \( N' = 0 \) and \( P = 0 \) equations (3.3.5) and (3.3.6) reduce to those equations analysed in section (3.1)

3.3 (b) Analysis:

The partial differential equations (3.3.5) and (3.3.6) are solved with the boundary conditions (3.3.3) and initial conditions (3.3.7) using the implicit finite difference scheme as described in section 2.4. A mesh size (400, 21) is chosen for the problem, and solutions are obtained by assuming \( E/R_0^2 = 1 \), \( R_a = R_0^2 \) and \( \phi = 1 + \delta t'^2 \), where \( \delta \) is a small constant very much less than one.

The quantities of physical interest mass flow rate,
heat transfer and skin friction are calculated for different values of \( \sigma, \Pr, t', R_a, Q' \) and \( R' \) as in the previous section.

For steady cooling of ascending hot fluid the governing equations are solved with a negative Rayleigh number for various values of \( t', \sigma, \Pr, R_a, R' \) and \( Q' \) and the effect of these parameters on mass flow rate, skin friction and Nusselt number are calculated.

3.3.(c) Results and discussion

The present analysis differs from that of section 3.2 by the presence of unsteadiness in wall temperature and heat generating term. Hence, only the effects of \( t' \) and \( Q' \) on the flow and heat transfer characteristics are highlighted in Figures (3.3.1 to 3.3.8) and Table 3.3.1.

The problem is governed by the parameters \( t', R_a, R'_a, N \) and \( Q' \). The behaviour of velocity, temperature and heat transfer rate with respect to \( R_a, \sigma \) and \( N \) are similar to that observed in section 3.2.

The effects of \( t' \) and \( \sigma \) on velocity and temperature fields are evident from Figs (3.3.1 - 3.3.2). For small values of the porous parameter the velocity and temperature increase with increase in \( t' \). For \( \sigma > 25 \) velocity increases till certain value of \( t' \) and after that it steadily decreases, whereas temperature increases with time \( t' \). From Figures (3.3.1 and 3.3.2) it is seen that both velocity and
### TABLE 3.3.1

**VARIATION OF \( U, \theta \) AND \( \text{Nu} \) WITH \( \Omega^* \)**

*For \( \sigma = 50, \ P_r = 7, \ N = 10, \ R_a = 5\)*

<table>
<thead>
<tr>
<th>( \Omega^* ) = 0.0</th>
<th>( \Omega^* ) = 2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y )</td>
<td>( U )</td>
</tr>
<tr>
<td>0.0</td>
<td>0.004001</td>
</tr>
<tr>
<td>0.2</td>
<td>0.004001</td>
</tr>
<tr>
<td>0.4</td>
<td>0.004001</td>
</tr>
<tr>
<td>0.6</td>
<td>0.004001</td>
</tr>
<tr>
<td>0.8</td>
<td>0.003999</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \Omega^* ) = 4</th>
<th>( \Omega^* ) = 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y )</td>
<td>( U )</td>
</tr>
<tr>
<td>0.0</td>
<td>0.007989</td>
</tr>
<tr>
<td>0.2</td>
<td>0.007830</td>
</tr>
<tr>
<td>0.4</td>
<td>0.007351</td>
</tr>
<tr>
<td>0.6</td>
<td>0.006552</td>
</tr>
<tr>
<td>0.8</td>
<td>0.005434</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
FIG. 3.3.1. EFFECT OF $t^*$, $\sigma$ AND $R_a^*$ ON VELOCITY PROFILE FOR $N=10, Q^*=0, R_a=5$ AND $\phi=1+0.1t^*$
FIG. 3.3.2. EFFECT OF $t^*$, $\sigma$ AND $R_a^*$ ON TEMPERATURE PROFILE FOR $N=10, Q^*=0, R_a=5, T_w > T_o$ AND $\phi = 1 + 0.1 t^*^2$
temperature decrease with increase in $\sigma$ and increase in $R^*_a$.

Fig (3.3.3) depicts the effect of $t^*$, $\sigma$ and $\phi (t^*)$ on Nusselt number. $Nu$ increases with increase in $t^*$ initially and later on decreases steadily. Nusselt number decreases with increase in $R^*_a$ and the magnitude of $Nu$ obtained for $\phi = 1 - \delta t^*$ (when wall temperature is decreased with $t^*$), is greater than that obtained for $\phi = 1 + \delta t^*$ (when the wall temperature is increased with $t^*$) as illustrated in Figure (3.3.3).

The ratio of $M_{pd}/M_p$ increases with increase in $t^*$ and increase in $R^*_a$ as shown in Fig 3.3.4. Here $M_{pd}$ is the mass flow rate in the presence of dissipations and heat generating source, and $M_p$ is the mass flow rate in the absence of the above.

Figure 3.3.5 illustrates that the heat generating term augments velocity in the absence and presence of dissipations. In the presence of dissipation and heat generating source, the velocity profile increases with increase in $t^*$ and exhibits a reverse trend in the absence of dissipation. The temperature profile in figure 3.3.6 indicates that the presence of heat generating source and dissipation reverses the direction of heat transfer. It is also seen that temperature increases with increase in $t^*$.

The heat generating parameter considerably increases the magnitude of temperature, velocity and $Nu$. 122
FIG. 3.3.3. EFFECT OF $t^*$, $\sigma$, $R_a$ AND $Q^*$ ON Nu FOR
which is evident from Table 3.3.1.

The skin friction decreases with increase in $t$ and $R^*$ but increases with increase in $R$ as illustrated in Fig 3.3.7.

For steady cooling of ascending fluid, the velocity profile and temperature profile decrease with increase in $t^*$ as shown in Fig 3.3.8.

The nett effect of dissipation, heat generation and wall temperature variation is to enhance heat transfer and mass flow rate.

Problems of unsteady motion are significant in flow through turbomachinery blades, dynamics of lifting surfaces, flutter of helicopter, rotor blades and accelerated and decelerated rocket. Another class of unsteady problems involve the study of unsteady forces on bluff bodies, buildings and structures in the atmosphere. Convection in porous media induced by internal heat generation arises in various physical problems such as heat removal from nuclear fuel debris in nuclear reactors, underground disposal of radioactive waste materials, and exothermic chemical reactions in packed bed reactors.

Moreover, this phenomena can be encountered during the storage of agricultural products where heat is generated as a result of metabolism of products. Hence the heat transfer rates, mass flow rates obtained in the present section will
FIG. 3.3.5. EFFECT OF $t^*$, $R_a$, DISSIPATION TERMS AND $Q^*$ ON VELOCITY PROFILE FOR $N=10, \phi=1+0.1t$
FIG. 3.3.6. EFFECT OF $t^*$, $R_a$, DISSIPATION TERMS AND $Q^*$ ON TEMPERATURE PROFILE $\Theta$ FOR $\sigma=50$, $N=10$, $R_a^*=5$, $T_w>T_o$ AND $\phi=1+0.1 \, t^{*2}$
FIG. 3.3.7. EFFECT OF $t^*$, $\sigma$, $R_d$ AND $R_a^*$ ON SKIN-FRICTION
$\gamma$ FOR $\phi=1+0.1t^*^2$, $Q^*=0$
FIG. 3.38. EFFECT OF $t^*$ AND $N$ ON VELOCITY AND TEMPERATURE PROFILES FOR $\sigma = 25, Q^* = 0$ AND $R_{\alpha} = 5$
help in the design of engineering devices.

3.4 LAMINAR CONVECTION IN A VERTICAL PIPE FILLED WITH POROUS MEDIUM

In sections (3.1 to 3.3) laminar convection in a vertical channel filled with porous medium is discussed. Recently convective heat transfer in porous media with different geometry has begun to attract a great deal of attention owing to the fact that cylindrical and spherical shapes of canisters have been proposed for nuclear waste disposal in subsea beds. Moreover, the design of heat exchangers for energy extraction in underground as well as for temperature control of a catalytic bed, demands better understanding of heat transfer process about a horizontal cylinder embedded in porous media.

Analytical studies of natural convection about a horizontal cylinder and sphere were performed by Merkin (1979) and Nilson, (1981) who used curvilinear coordinate system together with boundary layer approximation for simplifications of the mathematical problem. Using the same approach Cheng (1982) recently obtained similarity solution for mixed convection about an isothermal horizontal cylinder. In the present section the problem of laminar convection under a pressure gradient along a vertical pipe filled with saturated porous medium is analysed. The walls of the pipe are heated or cooled uniformly. The Brinkman model is used
and the governing equations are solved for small values of porous parameters using Runge-Kutta-Gill method. The present analysis is an extension of Morton's (1960) analysis. In the absence of porous medium for steady cooling of ascending hot fluid Morton noticed "thermal runaway" leading to higher velocities, as the Rayleigh number approached 33. One of the motives of the present analysis is to investigate whether a similar behaviour is observed in the presence of porous medium.

3.4(a) Mathematical formulation:

Steady flow of a fluid with density \( \rho \) under a constant pressure gradient along a vertical pipe of radius "a" filled with saturated porous medium is considered. The walls of the pipe are maintained at a uniform temperature gradient \( \frac{\Gamma}{a} \) in the direction of the axis. Cylindrical polar coordinates \((x, r, \phi)\) are used to describe the system with \( x \) axis vertically upwards. There is axial symmetry and convection will be independent of \( \phi \) and the velocity components are taken as \( (u(r), 0, 0) \) for the solution envisaged. From the assumptions described in section 3.1 the equations of motion and energy reduce to,

\[
\frac{\mu}{\varepsilon} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) - \frac{\mu}{k} \frac{u}{r} = \frac{\partial p}{\partial x} - \rho g \quad \quad \text{(3.4.1)}
\]

\[
u \frac{\partial T}{\partial x} = \alpha \left[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right] \quad \quad \text{(3.4.2)}
\]
where \( k, \alpha, \varepsilon, \lambda, \mu / \varepsilon \) refer to quantities defined in section 3.1.

Equations (3.4.1) and (3.4.2) are non-dimensionalised with the following non-dimensional variables namely:

\[
u = \frac{\alpha \cdot U}{a}, \quad r = \frac{a}{R}, \quad x = \frac{a}{X} \quad \ldots (3.4.3)
\]

\[T - T = \Gamma \Theta \]

The dimensionless governing equations with Boussinesq approximation are,

\[
\left( \frac{d^2}{dR^2} + \frac{1}{R} \frac{d}{dR} \right) U - \sigma^2 U = -N + R \cdot \Theta \Gamma \quad \ldots (3.4.4)
\]

\[
\left( \frac{d^2}{dR^2} + \frac{1}{R} \frac{d}{dR} \right) \Theta = -U \quad \ldots (3.4.5)
\]

In the above equations \( \sigma, N, R, \) and \( \nu \) are dimensionless parameters defined in section 3.1. \( \Gamma = \pm 1 \) refers to the two physical situations of steady heating and steady cooling discussed in previous sections.

The boundary conditions of the problem are:

\[ \Theta = 0, \quad U = 0 \text{ at } R = 1 \]

\[
\frac{d\Theta}{dR} = 0, \quad \frac{dU}{dR} = 0 \text{ at } R = 0 \quad \ldots (3.4.6)
\]

3.4(b) Mathematical Analysis:

The coupled differential equations (3.4.4) and (3.4.5)
are solved with the boundary conditions (3.4.6) using Runge-Kutta-Gill method. A step size of 0.1 is used. Results are obtained for arbitrary values of $\sigma$ up to 40 for two physical situations namely, upflow heated and upflow cooled. To assess the accuracy of the method, results of the present analysis are compared with that of Morton (1960), for $\sigma = 0$ and $R_a = 16$. The comparison Table 3.4.1 shows that the results obtained are in good agreement.

The two important quantities of physical interest namely $M_{pc}/M_p$ (where $M_{pc}$ is the mass flow rate in the pipe and $M_p$ mass flow rate in the channel) and Nusselt number are evaluated for various values of $\sigma$, $R_a$ and $N$ as described in the previous sections.

For steady cooling of ascending hot fluid the governing equations are solved using $\Gamma = -1$ (a negative Rayleigh number) for various values of $\sigma$, $R_a$ and $N$ and the influence of various parameters on $N_u$, and $M_{pc}$ are investigated in detail.

3.4(c) Results and Discussion:

The important parameters governing the problem are the porous parameter and the Rayleigh number. The influence of $\sigma$ and $R_a$ on $N_u$ is brought out in Fig 3.4.3. $N_u$ increases with increase in Rayleigh number and increase in $\sigma$ up to $R_a = 1000$. For $R_a = 1000$, $N_u$ decreases with increase in $\sigma$ which is similar to the behaviour observed in the case of the channel
### TABLE 3.4.1

**COMPARISON OF VELOCITY AND TEMPERATURE FOR σ = 0 Rₐ = 16, N=1**

<table>
<thead>
<tr>
<th>Y</th>
<th>Present Analysis</th>
<th>Morton (Graphical)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>U</td>
<td>θ</td>
</tr>
<tr>
<td>0.0</td>
<td>0.25000</td>
<td>0.0469</td>
</tr>
<tr>
<td>0.2</td>
<td>0.24000</td>
<td>0.0444</td>
</tr>
<tr>
<td>0.4</td>
<td>0.21000</td>
<td>0.0371</td>
</tr>
<tr>
<td>0.6</td>
<td>0.16000</td>
<td>0.0264</td>
</tr>
<tr>
<td>0.8</td>
<td>0.090</td>
<td>0.0132</td>
</tr>
<tr>
<td>1.0</td>
<td>0.000</td>
<td>0.0</td>
</tr>
</tbody>
</table>

### TABLE 3.4.2

**RATIO OF Mpc/Mp FOR N = 1**

<table>
<thead>
<tr>
<th>Rₐ</th>
<th>Mpc/Mp σ = 5</th>
<th>Mpc/Mp σ = 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.0589</td>
<td>1.0000</td>
</tr>
<tr>
<td>50</td>
<td>1.1932</td>
<td>1.0609</td>
</tr>
<tr>
<td>100</td>
<td>1.2857</td>
<td>1.0638</td>
</tr>
<tr>
<td>1000</td>
<td>1.3340</td>
<td>1.3560</td>
</tr>
<tr>
<td>5000</td>
<td>1.0600</td>
<td>1.3080</td>
</tr>
</tbody>
</table>
FIG. 3.4.1. COMPARISON OF VELOCITY PROFILE FOR $N=1$
FIG. 3.4.2. COMPARISON OF TEMPERATURE PROFILE FOR $N=1$
in section 3.1. The behaviour of velocity and temperature with respect to Rayleigh number is brought out in Figures (3.4.1 and 3.4.2). The velocity and temperature decrease with increase in $\sigma$ and $R_a$, which is similar to the behaviour observed in the channel.

Figures (3.4.1) and (3.4.2) also depict that the magnitude of velocity of the fluid in a pipe is more compared to that in a channel, whereas, the magnitude of the temperature in a pipe is less compared to that in the case of the channel. Table 3.4.2 indicates that the ratio of $M_p/M_p$ increases with increase in Rayleigh number, but decreases with increase in $\sigma$. Table 3.4.3 shows that heat transfer rate is lesser in a pipe compared to that in a channel.

The most important result observed in the case of steady cooling of ascending hot fluid is that there is no runaway behaviour in the presence of porous medium as observed by Morton (1960) in the absence of porous medium. Table 3.4.4 emphasizes the above result. Presence of porous medium controls the critical behaviour of temperature and velocity of the fluid. This fact can be made use of in controlling the working of nuclear reactors. The influence of $\sigma$, $R_a$ and $N$, on flow and heat transfer characteristics are similar to that observed in a channel.

The configuration and flow conditions investigated in the present section are those encountered in chemical reactor
### Table 3.4.3

**Comparison for Nu for Pipe and Channel for \( \sigma = 5 \)**

<table>
<thead>
<tr>
<th>( R_a )</th>
<th>Pipe Nu</th>
<th>Channel Nu</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>9.37552</td>
<td>10.69550</td>
</tr>
<tr>
<td>50</td>
<td>9.69920</td>
<td>11.14492</td>
</tr>
<tr>
<td>100</td>
<td>11.34768</td>
<td>11.60654</td>
</tr>
<tr>
<td>1000</td>
<td>12.20000</td>
<td>16.06921</td>
</tr>
<tr>
<td>5000</td>
<td>25.33000</td>
<td>24.03000</td>
</tr>
</tbody>
</table>

### Table 3.4.4

**Comparison of Velocity and Temperature for Upflow Cooled Situation for \( N = 10, R_s = 33 \)**

<table>
<thead>
<tr>
<th>( Y )</th>
<th>( U )</th>
<th>( \theta )</th>
<th>( U )</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.05003</td>
<td>0.01050</td>
<td>233.360</td>
<td>40.400</td>
</tr>
<tr>
<td>0.2</td>
<td>0.49060</td>
<td>0.01000</td>
<td>222.000</td>
<td>38.110</td>
</tr>
<tr>
<td>0.4</td>
<td>0.45800</td>
<td>0.00848</td>
<td>182.500</td>
<td>31.590</td>
</tr>
<tr>
<td>0.6</td>
<td>0.03960</td>
<td>0.00620</td>
<td>127.400</td>
<td>21.960</td>
</tr>
<tr>
<td>0.8</td>
<td>0.02672</td>
<td>0.00321</td>
<td>62.780</td>
<td>10.830</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
studies. The purpose of chemical reactor studies is to determine the heat transfer rate between the walls and the fluid-solid system and the local temperature within the reactor. The results from these non-reacting studies can be made use of to estimate the cooling rate needed for catalytic packed bed reactors.

3.5 EXISTENCE OF MULTIPLE SOLUTIONS FOR NATURAL CONVECTION.

Some insight into the conditions under which multiple solutions exist might be gained by extending the work of Joseph and Warner (1967), who established such conditions for the case of isothermal walls. Beckett et al. (1984) obtained multiple solutions for mixed convection in a vertical infinite channel in the presence of viscous dissipation with walls maintained at different temperatures. Criteria for existence and uniqueness was furnished by Amann (1976) in a comprehensive review paper of non-linear functional analysis. Beckett (1980) describes the existence of multiple solutions for combined natural and forced convection between parallel vertical walls. The present section analyses the existence of multiple solutions for natural convection in a vertical channel filled with porous medium for Darcy model.

3.5 (a) Mathematical formulation:

The incompressible viscous fluid is contained between two parallel vertical infinite walls \( y = 0 \) and \( d \) maintained at temperature \( T_0 + Ax \) and \( T_1 + Ax \) respectively, where \( x \) is
the distance measured vertically upwards. Fully developed steady two dimensional flow is assumed so that the velocity is everywhere parallel to the x axis and a function of y alone. With these assumptions and using the Boussinesq approximation the governing equations of motion reduces to,

\[
\frac{\mu u}{k} = - g \beta \rho_0 (T_r - T) \tag{3.5.1}
\]

where \( T_r = T_0 + A_x \)

The energy equation is,

\[
\alpha_x \frac{\partial^2 T}{\partial y^2} = u \frac{\partial T}{\partial x} - \frac{\mu u^2}{k \rho_0 C_p} \tag{3.5.2}
\]

The boundary conditions for these equations are:

\[
T = T_0 + A_x \text{ at } y = 0 \tag{3.5.3}
\]

\[
T = T_1 + A_x \text{ at } y = d \tag{3.5.4}
\]

The equations are non-dimensionalised with the following variables:

\[
u = \frac{\alpha_x U}{\alpha}, \quad x = aX, \quad y = aY \text{ and } \theta (y) = \frac{(T_r - T)/(T_0 - T_1)}{T_0 - T_1} \tag{3.5.4}
\]

From (3.5.4), the dimensionless form of (3.5.1) and (3.5.2) are:

\[
U = - R_a \theta \tag{3.5.5}
\]

\[
\frac{d^2 \theta}{dy^2} \frac{(T_0 - T_1)}{A} + U = \frac{\sigma^2}{Pr} \frac{E}{R_e^2} U^2 \tag{3.5.6}
\]

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where \( R_a = g \frac{\beta (T_0 - T_1)}{\alpha \nu} \) ka

Substituting (3.5.5) in (3.5.6) the resulting equation is,

\[
\frac{d^2 \theta}{dy^2} - R'_a \theta = \varepsilon^* \theta^2
\]  

..(3.5.7)

where \( R'_a = R_a \left( \frac{A}{T_1 - T_0} \right) \), \( \varepsilon^* = \left( \frac{\sigma^2}{P_r} \right) \left( \frac{E}{R_e^2} \right) \left( \frac{A}{T_1 - T_0} \right) \)

When \( T_0 < T_1 \), equation (3.5.7) is transformed to

\[
\frac{d^2 \theta}{dy^2} + R'_a \theta = \varepsilon^* \theta^2
\]  

..(3.5.8)

The dimensionless boundary conditions are:

\[ \theta (0) = 0 \text{ at } Y = 0 \]

\[ \theta (1) = 1 \text{ at } Y = 1 \]  

..(3.5.9)

3.5(b) **Mathematical Analysis:**

Equation (3.5.8) is solved with the boundary conditions (3.5.9) using Runge-Kutta-Gill method with Newton Raphson modification. Computations are carried out for small values of \( \varepsilon^* \) and for arbitrary values of \( R'_a \). Multiple solutions are obtained only for values of \( R'_a = -9.50, -88 \) and -100. For \( R'_a = 100, \varepsilon^* = 0.1 \) six solutions are obtained corresponding to six different values of \( \theta'(0) = -5716.1, \)
The numerical method yields the values of $\theta$ and $\theta'$ from which the magnitude of velocity can be calculated.

3.5(c) Results and discussion.

The results of the present investigation are presented in Figs (3.5.1 and 3.5.2). Multiple solutions exist only if the reference temperature $T_0$ is less than that of the other wall.

Equation (3.5.8) describes simple harmonic motion if $\sqrt{R_a'} = n \pi$ and $\epsilon' = 0$. There are two solutions for $R_a' = [0, 4\pi^2]$ and, for $R_a = 100$, $\epsilon' = 0.1$, six modes are possible with each mode growing from the basic mode.

From Fig 3.5.1 it is seen that for $R_a' = -88.9$, $\epsilon' = 0.1$ there are two types of curves for $\theta'(0) = 4826$ and 1420. It is seen that the curve corresponding to the former is parabolic whereas the latter is a sine curve. Similarly for $R_a' = -50$ and $\epsilon' = 0.1$ there are two solutions for $\theta'(0) = 1826$ and $\theta'(0) = 542$. For $R_a' = -9$ and $\epsilon' = 0.01$ there are two solutions corresponding to $\theta'(0) = 18$ and $\theta'(0) = 446$, both the curves are parabolic in shape. For $R_a' = -100$, $\epsilon' = 0.1$ Fig 3.5.2 depicts six solutions corresponding to $\theta'(0) = 5715.8$, 5769, -5716, -4744, -18.31, 2361, each mode growing from the basic mode.

Table 3.5.1 illustrates the behaviour of Nusselt number with respect to $\theta'(0)$. It is seen that for the same
FIG. 3.5.1. TEMPERATURE PROFILE FOR DIFFERENT VALUES OF $\theta'(o)$
FIG. 3.5.2. TEMPERATURE PROFILES FOR VARIOUS VALUES OF $\theta'(0)$
value of Rayleigh number corresponding to each value of $e^\prime(0)$ different Nusselt numbers are obtained.

The present section demonstrates that multiple solutions exist in particular physical situations. It is interesting to note that Darcy model admits multiple solutions for a particular physical situation.

**TABLE 3.5.1:**

<table>
<thead>
<tr>
<th>$R_a'$</th>
<th>$e^\prime$</th>
<th>$e^\prime(0)$</th>
<th>$Nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-60</td>
<td>0.1</td>
<td>-18.0</td>
<td>36.78</td>
</tr>
<tr>
<td>-100</td>
<td>0.1</td>
<td>5715.8</td>
<td>28.70</td>
</tr>
<tr>
<td>-100</td>
<td>0.1</td>
<td>5766.0</td>
<td>15.67</td>
</tr>
<tr>
<td>-100</td>
<td>0.1</td>
<td>23617</td>
<td>63.22</td>
</tr>
<tr>
<td>-100</td>
<td>0.1</td>
<td>-4744</td>
<td>28.69</td>
</tr>
<tr>
<td>-100</td>
<td>0.1</td>
<td>-18.37</td>
<td>121.16</td>
</tr>
<tr>
<td>-88.9</td>
<td>0.1</td>
<td>1429</td>
<td>90.11</td>
</tr>
<tr>
<td>-88.9</td>
<td>0.1</td>
<td>4826</td>
<td>15.07</td>
</tr>
<tr>
<td>-50</td>
<td>0.1</td>
<td>1826</td>
<td>48.65</td>
</tr>
<tr>
<td>-50</td>
<td>0.1</td>
<td>-18</td>
<td>242.39</td>
</tr>
</tbody>
</table>
3.6 **REFINED ANALYSIS OF MIXED CONVECTION.**

In a medium composed of a solid phase and a fluid phase, the averages over the separate phases are incorporated by assuming that conduction through the solid acts over a volume \((1-\varepsilon)\) and the conduction through the fluid in void spaces over a volume \(\varepsilon\). This approach has been used to determine the effective thermal conductivity.

In sections (3.1) and (3.2), though effective thermal diffusivity was incorporated in the energy equation, its explicit dependence on the porosity of the medium, and ratio of thermal conductivity of solid to fluid phase was not evident since the velocity was non-dimensionalised using effective thermal diffusivity. Moreover the analysis in sections (3.1) and (3.2) were confined to arbitrary values of the parameter "\(\sigma\)" \((\sigma = a\sqrt{\varepsilon}/nk, \text{where } k \text{ is the permeability})\). In the present section the dependence of effective thermal diffusivity on porosity and ratio of conductivities of solids to fluids is brought out, by incorporating the relation connecting porosity and thermal conductivity. The velocity is non-dimensionalised using thermal diffusivity of the fluid. The present analysis incorporates practical values of \(\sigma\) obtained using Kozney Blake expression connecting porosity and permeability for different values of bed to particle diameter ratios.
3.6 (a) **Mathematical analysis:**

The physical model is the same as in section 3.1. Under the basic assumptions discussed in section 3.1 the governing equations for laminar convection in a vertical channel filled with porous medium in the presence and absence of dissipation terms are:

\[
\frac{d^2 u}{dY^2} - \sigma^2 U = - N + R_s \theta \Gamma \quad \ldots (3.6.1)
\]

\[
(\varepsilon + (1-\varepsilon)b) \frac{d^2 \theta}{dY^2} = - U \quad \ldots (3.6.2)
\]

\[
(\varepsilon + (1-\varepsilon)b) \frac{d^2 \theta}{dY^2} = - U + \left( \frac{E}{R_s} \right) \left( \frac{1}{Pr} \left( \frac{dU}{dY} \right)^2 + \frac{\sigma^2}{Pr^2} U^2 \right) \quad \ldots (3.6.3)
\]

Equations (3.6.1) and (3.6.2) denote convection in the absence of dissipation terms while (3.6.1) and (3.6.3) denote convection in the presence of dissipation terms.

In equations (3.6.1) to (3.6.3), parameters \( N, R_s, E, Pr \) and \( \sigma \) are the same as defined in section 3.1 and 3.2;

\[ k = \varepsilon d^2 / 150 (1-\varepsilon)^2 \]

\( d \) - diameter of the particle forming the bed.

\( b = \lambda_s / \lambda_f \) (ratio of thermal conductivities of solid to fluid)

The governing equations (3.6.1) to (3.6.3) are solved using boundary conditions (3.1.8) and the influence of various parameters on the flow and heat transfer characteristics of the problem are analysed. The computations
are carried out for $b = 2$ (water/glass beads), $b = 7.35$ (oil/glass), $b = 38.46$ (air/glass) and $b = 100$ (water/steel). The value of porosity used in the analysis is 0.5 and results are obtained for bed to particle diameter ratios 10.6, 6.25, 2.4 and 11.76.

3.6 (c) Results and discussions:

The influence of Rayleigh number, pressure and porous parameter on the flow and heat transfer characteristics are similar to those observed in sections 3.1 and 3.2. The present section highlights the effects of using the expression connecting thermal diffusivity and porosity through Tables 3.6.1 to 3.6.2 and Figs (3.6.1 to 3.6.4).

The influence of $b$ on velocity and temperature profiles are evident from Fig 3.6.1 and 3.6.2. It is seen from Figures (3.6.1 and 3.6.2) that the velocity increases with increase in $b$ and the temperature decreases with increase in $b$. In the presence of dissipations the velocity decreases with increase in $b$ and the temperature exhibits the same behaviour as observed in the absence of dissipations as seen in Figs. (3.6.3 and 3.6.4). Table 3.6.1 shows that $b$ does not influence Nusselt number considerably. It is seen from Table 3.6.2 that the influence of ratio of thermal diffusivities ($b$), is predominant on temperature profiles rather than on velocity profiles.

The present analysis refines the results obtained in
### TABLE 3.6.1

**Comparison of Nusselt Number obtained in Section 3.1 and in the improved analysis for \( n = 1000, R_a = 5 \) and \( \sigma = 69. \)**

<table>
<thead>
<tr>
<th>( b )</th>
<th>( \text{Nu (present analysis)} )</th>
<th>( \text{Nu (previous analysis)} )</th>
<th>( \text{Nu (present analysis)} )</th>
<th>( \text{Nu (previous analysis)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>11.69127</td>
<td></td>
<td>11.65490</td>
<td></td>
</tr>
<tr>
<td>7.35</td>
<td>11.69072</td>
<td>11.69194</td>
<td>11.79320</td>
<td></td>
</tr>
<tr>
<td>38.46</td>
<td>11.68921</td>
<td></td>
<td>11.83076</td>
<td>11.61494</td>
</tr>
<tr>
<td>100</td>
<td>11.68954</td>
<td></td>
<td>11.84000</td>
<td></td>
</tr>
</tbody>
</table>
### TABLE 3.6.2

**VARIATION OF VELOCITY AND TEMPERATURE WITH \( b \) IN THE PRESENCE AND ABSENCE OF DISSIPATION FOR \( \sigma = 34, \ N = 1000, \ R = 5 \)**

#### PRESENCE OF DISSIPATION

\[ b = 7.35 \quad \text{or} \quad b = 38.46 \]

<table>
<thead>
<tr>
<th>( y )</th>
<th>( u )</th>
<th>( \theta )</th>
<th>( u )</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.900206</td>
<td>-16.07906</td>
<td>0.8461</td>
<td>-3.0655</td>
</tr>
<tr>
<td>0.2</td>
<td>0.897515</td>
<td>-15.70740</td>
<td>0.8456</td>
<td>-2.9420</td>
</tr>
<tr>
<td>0.4</td>
<td>0.889200</td>
<td>-13.44000</td>
<td>0.8440</td>
<td>-2.5700</td>
</tr>
<tr>
<td>0.6</td>
<td>0.875700</td>
<td>-10.18330</td>
<td>0.8415</td>
<td>-1.9550</td>
</tr>
<tr>
<td>0.8</td>
<td>0.856100</td>
<td>-5.67800</td>
<td>0.8317</td>
<td>-1.0950</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

#### ABSENCE OF DISSIPATION

\[ b = 7.35 \quad \text{or} \quad b = 38.46 \]

<table>
<thead>
<tr>
<th>( y )</th>
<th>( u )</th>
<th>( \theta )</th>
<th>( u )</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.83296</td>
<td>0.0995</td>
<td>0.83329</td>
<td>0.021082</td>
</tr>
<tr>
<td>0.2</td>
<td>0.83298</td>
<td>0.0995</td>
<td>0.83330</td>
<td>0.020238</td>
</tr>
<tr>
<td>0.4</td>
<td>0.83303</td>
<td>0.0836</td>
<td>0.83330</td>
<td>0.017704</td>
</tr>
<tr>
<td>0.6</td>
<td>0.83310</td>
<td>0.0636</td>
<td>0.83332</td>
<td>0.013480</td>
</tr>
<tr>
<td>0.8</td>
<td>0.83240</td>
<td>0.0354</td>
<td>0.83259</td>
<td>0.007567</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
FIG. 3.6.1. INFLUENCE OF $b$ ON VELOCITY PROFILE IN THE ABSENCE OF DISSIPATION TERMS
Fig. 3.6.2. Influence of $b$ on temperature profile in the absence of dissipation terms

$\sigma = 34.63, b=2$

$b=7.35$

$b=100$

$Ra = 5, N = 1000, T_w > T_o$
Fig. 3.6.3. Influence of $b$ on temperature profile in the presence of dissipation terms.
FIG. 3.6.4. INFLUENCE OF $b$ ON VELOCITY PROFILES IN THE PRESENCE OF DISSIPATION TERMS.
sections 3.1 and 3.4. Hence the results have relevance in industrial applications and enables an accurate estimate of flow and heat transfer rate in similar situations.

The analysis in the present chapter are based on the following manuscripts.
