Chapter 7

Maximal Semitotal- Point

Domination in Graphs
7.1 Introduction.

A set $D \subseteq V$ of a graph $G = (V, E)$, is a dominating set, if every vertex not in $D$ is adjacent to at least one vertex in $D$. The domination number $\gamma(G)$ of $G$, is the minimum cardinality of a dominating set. A dominating set $D$ of a graph $G$, is called a maximal dominating set, if $V - D$ is not a dominating set of $G$. The maximal domination number $\gamma_m(G)$ of $G$, is the minimum cardinality of a maximal dominating set. The maximal domination in graphs was introduced by Kulli and Janaki-ram [46].

A dominating set $D$ of a graph $H$, is a maximal semitotal-point dominating set of $H$, if $V(H) - D$ is not a dominating set of $H$. The maximal semitotal-point dominating domination number $\gamma_{mtp}(G)$ of $G$, is the minimum cardinality of a maximal semitotal-point dominating dominating set of $H$.

In this chapter, many bounds on $\gamma_{mtp}(G)$ are obtained in terms of elements of $G$. The aim in this chapter is to establish maximal domination on semitotal-point graph and express the results in terms of elements of $G$ but not in terms of elements of $H$.

The following figure shows the formation of $\gamma_m(G)$ and $\gamma_{mtp}(G)$. 

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In $G$, $|V(G)| = 5$, $|E(G)| = 5$ and $D = \{a, b, c\}$. Therefore, $\gamma_m(G) = |D| = 3$.

In $T_2(G)$, $|V(T_2(G))| = 10$, $|E(T_2(G))| = 15$ and $D' = \{a, b, c, e_2\}$. Therefore, $\gamma_{mtp}(G) = |D'| = 4 = |V(G) - 1|$.

The following theorems are used in the proof of further results.

**Theorem 7.A[6].** In a tree $T$, if every nonpendant vertex is adjacent to at least one pendant vertex then, $\gamma(T) = \gamma_{tp}(T)$.

**Theorem 7.B[46].** For any graph $G$ of order $p$, $\gamma_m(G) = p$ if and only if $G = K_p$.

**Theorem 7.C[45].** For any graph $G$ with a pendant vertex

$$\gamma_s(T) = \gamma(T).$$

**Remark 7.1.** $\gamma_{tp}(G)$ and $\gamma(T_2(G))$ both denote the domination number
7.2 Results

In the following theorem $\gamma_{mtp}(G)$ of some standard graphs are established.

**Theorem 7.1** (i) For any complete graph $K_p$ with $p \geq 2$

$$\gamma_{mtp}(K_p) = p + 1 \text{ and } \gamma_{mtp}(K_3) = p.$$  

(ii) For any cycle $C_p$ with $p \geq 4$

$$\gamma_{mtp}(C_p) = \left\lceil \frac{p}{2} \right\rceil + 2$$

(iii) For any path $P_p$ with $p \geq 3$

$$\gamma_{mtp}(P_p) = \left\lceil \frac{p}{2} \right\rceil + 1$$

(iv) For any star $K_{1,p-1}$ with $p \geq 2$

$$\gamma_{mtp}(K_{1,p-1}) = 3.$$ 

In the following theorems the bounds of $\gamma_{mtp}(G)$ are established.

**Theorem 7.2** For any graph $G$ of order $p$, $3 \leq \gamma_{mtp}(G) \leq p + 1$.

**Proof.** For the lower bound, the smallest possible connected graph that can be considered is $G = K_2$. Then $T_2(G) = C_3$ or $K_3$ and
\( \gamma_m(C_3) = \gamma_{mtp}(G) = 3 \). Clearly, \( \gamma_{mtp}(G) \geq 3 \).

For the sharpness of the lower bound, let \( D_m = D_1 \cup D_2 \) be the maximal dominating sets of \( G \), where \( D_1 = \{u\} \) such that \( \text{deg}(u) = p - 1 \) and \( D_2 = \{v_i\} \) such that \( \text{deg}(v_i) = 1 \) for \( 1 \leq i \leq p - 1 \). Let \( e_1, e_2, \ldots, e_q \) be the edge vertices in \( T_2(G) \). For every edge \( e_i; 1 \leq i \leq q \), there exists an edge vertex \( e_j \) in \( T_2(G) \) and \( N(e_j) = \{u\} \cup \{v_j\} \). Thus \( D_m \cup \{e_j\} \) forms a maximal dominating set of \( T_2(G) \). Hence
\[
\gamma_{mtp}(G) = |D_m \cup \{e_j\}| = |u| + |v_i| + |e_j| = 3.
\]

For the upper bound, let \( D_m = \{v_1, v_2, \ldots, v_n\} \) be a maximal dominating set of \( G \) and \( \{e_1, e_2, \ldots, e_q\} \) be the edge vertices of \( T_2(G) \). The following cases are discussed.

**Case 1.** Suppose \( \delta(G) = \Delta(G) = p - 1 \). Then \( G = K_p \). By the definition of \( T_2(G) \), \( K_p \subseteq T_2(G) \) and also in \( T_2(G) \) each edge vertex \( e_i; 1 \leq i \leq q \) will form a cycle of length 3. Since \( \gamma_m(K_p) = p \), therefore \( \gamma_{mtp}(G) = |D_m \cup \{e_i\}| = p + 1 \). Thus, \( \gamma_{mtp}(G) = p + 1 \).

**Case 2.** Suppose \( \delta(G) < \Delta(G) = p - 1 \) and \( \delta(G) \leq \Delta(G) < p - 1 \), then by Case 1, we get \( 3 \leq \gamma_{mtp}(G) \leq p \). Thus from these two cases, \( \gamma_{mtp}(G) \leq p + 1 \). The sharpness of the upper bound follows from Case 1.

The following theorem gives the relation between \( \gamma_m(T) \) and \( \gamma_{mtp}(T) \).
Theorem 7.3  For any nontrivial tree $T$, $\gamma_m(T) \leq \gamma_{mtp}(T)$. The equality holds if every nonpendant vertex is adjacent to at least one pendant vertex of a tree.

**Proof.** Suppose $D_m$ be the maximal dominating set of $T$ and $e_1, e_2, \ldots, e_q$ be the line vertices in $T_2(T)$.

Let $D_1 = D_m \cup D'_m$, where $D'_m \subset [V(T_2(T)) - V(T)]$. Suppose $v_i \in (D_m \cup D'_m)$ be a vertex not adjacent to any vertex of $[V(T_2(T)) - (D_m \cup D'_m)]$. Thus one can easily verify that $(D_m \cup D'_m)$ is a maximal semitotal-point dominating set $T_2(T)$.

Hence $\gamma_m(T) = |D_m|$

$$\leq |D_m \cup D'_m|$$

$$= |D_1|$$

$$= \gamma_{mtp}(T) .$$

Equality follows from Theorem 7.4.  

The following theorem establishes the lower bound on $\gamma_{mtp}(G)$ in terms of vertices and edges of $G$.

**Theorem 7.4**  For any $(p, q)$ graph $G$, $\frac{2q - p(p-3)}{2} \leq \gamma_{mtp}(G)$.

**Proof.** Let $D_m$ be a maximal dominating set of $G$ and $e_1, e_2, \ldots, e_q$ be the line vertices in $T_2(G)$. By a result in [34], we have

$q \leq pC_2 - [p - \gamma_{mtp}(G)] = \frac{p(p-1)}{2} - p + \gamma_{mtp}(G)$. Thus, the result.
**Theorem 7.5** For any graph $G$ of order $p \geq 2$ and maximum degree $\Delta$, $\gamma_{mtp}(G) - p \leq \left\lceil \frac{p}{1+\Delta(G)} \right\rceil$. The equality holds for $G = K_p (\neq K_3)$ for $p \geq 2$ vertices.

**Proof.** By Theorem 7.2, we have $\gamma_{mtp}(G) \leq p + 1$. Therefore $\gamma_{mtp}(G) - p \leq 1$. Also $\left\lceil \frac{p}{1+\Delta(G)} \right\rceil \leq \gamma(G)$. Since $\gamma(G) \leq \frac{p}{2}$, therefore $\left\lceil \frac{p}{1+\Delta(G)} \right\rceil \leq \frac{p}{2}$. Hence from these two results we get $1 \leq \frac{p}{2}$. Thus $\gamma_{mtp}(G) - p \leq \left\lceil \frac{p}{1+\Delta(G)} \right\rceil$. Equality follows from the result $(i)$ of Theorem 7.1. 

**Theorem 7.6** For any graph $G$ with $p \geq 2$ vertices, $\gamma_{mtp}(G) \leq p - e + 2$, where $e$ is the number of pendant vertices of $G$. The equality holds for $K_{1,p}$.

**Proof.** Let $S$ be the set of all pendant vertices of $G$ with $|S| = e$. For an end vertex $v_i \in S$, the set $D = (V - S) \cup \{v_i\}$ is a maximal dominating set of $G$. Let $\{e_1, e_2, \cdots, e_q\}$ be the set of line vertices in $T_2(G)$ and every vertex $v_i \in S$ is adjacent to corresponding line vertex $e_i \in T_2(G)$ which is incident in $G$. Thus $D \cup \{e_i\}$ is a maximal dominating set of $T_2(G)$. Hence

$$\gamma_{mtp}(G) \leq |D \cup \{e_i\}|$$

$$= |(V - S) \cup \{v_i\} \cup \{e_i\}|$$

$$= p - e + 2.$$
The equality follows from the result (iv) of Theorem 7.1.

The relation between $\gamma_m(G)$ and $\gamma_{mtp}(G)$ is obtained in the following theorem.

**Theorem 7.7** For any graph $G \neq T$, $\gamma_m(G) + 1 \leq \gamma_{mtp}(G)$. Further, the equality holds for $G = K_p (\neq K_3)$ with $p \geq 2$.

**Proof.** Since for any graph $G$, $\gamma_m(G) < \gamma_{mtp}(G)$, therefore $\gamma_m(G) + 1 \leq \gamma_{mtp}(G)$. Equality follows from Theorem 7.2 and result (i) of Theorem 7.1.

**Theorem 7.8** For any non trivial tree $T \neq K_{1,p-1}$, $\gamma_{mtp}(T) = s + 1$, where $s$ is the number of cutvertices of $G$. The bound is attained if and only if every cutvertex is adjacent to at least one pendant vertex.

**Proof.** Let $D_m$ be a maximal dominating set of $T_2(T)$ and $K = \{c_1, c_2, \cdots, c_s\} \subset V(T)$ be the set of all cutvertices of $T$ with $|K| = s$. Let $\{e_1, e_2, \cdots, e_q\}$ be the set of line vertices in $T_2(T)$. The set $K$ dominates the line vertices in $T_2(T)$. Now consider $X \subset \{e_1, e_2, \cdots, e_q\}$ such that one can find a vertex in $V(T_2(T))$ which is not adjacent to any vertex in $(V(T_2(T)) - D_m)$. Clearly $D_m^{'} = K \cup X$ is a maximal dominating set of $T_2(T)$. Thus
\[ \gamma_m(T) \leq |D_m| \]
\[ \leq |K \cup X| \]
\[ \leq |K| + |X| \]
\[ \leq s + 1 \]

Further, assume that there exists a cutvertex \( v_i \in D_m \) which is not adjacent to any pendant vertex in \( T \). Then the cardinality of maximal dominating set of \( T_2(T) \) is at most \( s \), a contradiction. Hence each cutvertex is adjacent to at least one pendant vertex.

The converse is obvious.

Now the upper bound of \( \gamma_{mtp}(G) \) in terms of vertex covering number of \( G \) is obtained.

**Theorem 7.9** For any graph \( G \), \( \gamma_{mtp}(G) \leq \alpha_0(G) + 2 \), where \( \alpha_0(G) \) is vertex covering number of \( G \). Further, the equality holds for \( G = K_p (\neq K_3) \) or \( K_{1,p-1} \) with \( p \geq 2 \).

**Proof.** Let \( X = \{v_1, v_2, \cdots, v_r\} \) be a vertex covering set of \( G \). Then for any vertex \( v_i \in X \) for \( 1 \leq i \leq r \), the set \( D_m = (V - X) \cup \{v_i\} \) is a maximal dominating set of \( G \). Let \( \{e_1, e_2, \cdots, e_q\} \) be the edge vertices in \( T_2(G) \) and for any edge set \( E_i \) in \( G \) for \( 1 \leq i \leq q \), there exist a vertex \( v_j \in D_m \) such that \( v_j \) is incident with an edge of \( E_i \). Thus one
can easily find at least one line vertex $e_i$ for $1 \leq i \leq q$ which is not adjacent to any vertex of $\{(V \subset T_2(G)) - (D_m \cup \{e_i\})\}$ in $T_2(G)$. Since $\gamma_m(G) \leq \alpha_0(G) + 1$, therefore

$$\gamma_{mtp}(G) \leq |D_m \cup \{e_i\}|$$

$$\leq \gamma_m(G) + |e_i|$$

$$\leq \alpha_0(G) + 1 + 1$$

$$= \alpha_0(G) + 2. \quad \blacksquare$$

The next result gives the relation between $\beta_0(T)$ and $\gamma_{mtp}(G)$.

**Theorem 7.10** For any nontrivial tree $T$, $\gamma_{mtp}(T) \leq \beta_0(T) + 1$, where $\beta_0(T)$ is the independence number of $T$. Equality holds for $P_p$ with $p \geq 2$ vertices.

**Proof.** Let $X$ be a maximum independent set of $G$ which is a dominating set of $T_2(T)$. Suppose $S \subseteq X$ be the set of all pendant vertices in $T_2(T)$. Then for a pendant vertex $v_k \in S$, the set $D_m = (V - S) \cup \{v_k\}$ is a maximal dominating set of $T$ and $|D_m| \leq |X|$ such that $v_j \in X_i$ and for a vertex $v_k \in S$ is adjacent to the corresponding line vertex $e_k \in T_2(T)$; $1 \leq k \leq q$. Therefore, $D_m \cup \{e_k\}$ is a maximal dominating set of $T_2(T)$. Hence

$$\gamma_{mtp}(T) \leq |D_m \cup \{e_k\}|$$
\[ \leq |X| + |e_k| \]
\[ \leq |X| + 1 \]
\[ = \beta_0(T) + 1. \]

Equality can be easily verified when \( G \) is a path of length at least two.

Now the relation between the vertex connectivity of \( G \) and \( \gamma_{mtp}(G) \) is established.

**Theorem 7.11** For any graph \( G \), \( \kappa(G) + 2 \leq \gamma_{mtp}(G) \), where \( \kappa(G) \) is the vertex connectivity of \( G \).

**Proof.** The proof follows from the lower bound of Theorem 7.2.

**Theorem 7.12** For any graph \( G \), \( \delta(G) + 2 \leq \gamma_{mtp}(G) \). Further, the equality holds for \( G = K_p (\neq K_3) \) with \( p \geq 2 \) vertices.

**Proof.** Let \( D_m \) be a maximal dominating set of \( G \). Let \( \{e_1, e_2, \cdots, e_q\} \) be a set of edge vertices in \( T_2(G) \) and \( \{v_1, v_2, \cdots, v_p\} \) be a set of point vertices in \( T_2(G) \). Then there exist at least one vertex \( v_i \) which is not in \( D_m \) and the degree of \( v_i' \) of the corresponding vertex of \( T_2(G) \) will be twice the degree of vertex \( v_i \) in \( G \). That is \( deg_{T_2(G)}(v_i') = 2deg_G(v_i) \). Therefore \( deg_{T_2(G)}(v_i') \leq \gamma_{mtp}(G) - 2 \). Since \( \delta(G) \leq deg_G(v_i) \), therefore \( \delta(G) + 2 \leq \gamma_{mtp}(G) \).
For equality, suppose \( G = K_p (\neq K_3) \), for \( p \geq 2 \) vertices. Then \( \gamma_{mtp}(K_p) = p + 1 \) and \( \delta(K_p) = p - 1 \). Hence from these two, the result follows.

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In the next result a relation between \( \gamma_c(G) \) and \( \gamma_{mtp}(G) \) is established.

**Theorem 7.13** For any graph \( G \), \( \gamma_{mtp}(G) \leq \gamma_c(G) + \delta(G) + 1 \). Equality holds for \( K_p (\neq K_3) \) and \( K_{1,p-1} \) with \( p \geq 2 \) vertices.

**Proof.** Let \( D_c \) be a connected dominating set of \( G \) and \( \{e_1, e_2, \ldots, e_q\} \) be the set of edge vertices in \( T_2(G) \). Let \( v \in V(G) \) be a vertex of minimum degree in \( G \) such that the set \( N(v) \) contains the elements of \( D_c \). Then \( D_m = D_c \cup \{v\} \) forms a maximal dominating set of \( G \). Since the \( \langle D_c \rangle \) is connected, therefore there exists a line vertex \( e_i \in T_2(G) \); \( 1 \leq i \leq q \) which is adjacent to the corresponding incident vertices in \( D_c \). Hence \( D_c \) must contain at least one vertex from each edge of \( G \) which are incident to \( e_i \). Clearly

\[
\gamma_{mtp}(G) \leq |D \cup \{e_i\}|
\]

\[
\leq |D \cup \{v\} \cup \{e_i\}|
\]

\[
\leq \gamma_c(G) + \delta(G) + 1.
\]

The equality follows form Theorem 7.7, and from the fact that \( \gamma_c(K_p) = \ldots \)
\[ \gamma_c(K_{1,p-1}) = 1 \text{ and } \delta(K_p) = p - 1, \quad \delta(K_{1,p-1}) = 1. \]

The next result gives the relation between \( \gamma_t(G) \) and \( \gamma_{mtp}(G) \).

**Theorem 7.14** For any graph \( G \) with maximum degree \( \Delta \), \( \gamma_{mtp}(G) \leq \gamma_t(G) + \Delta(G) \). The equality holds for \( G = K_p \neq K_3 \) for \( p \geq 2 \).

**Proof.** Let \( D_t \) be a total dominating set of \( G \) and \( v \in D_t \) be a vertex of maximum degree in \( G \). Let \( X \subseteq (V - T) \), where \( T \) is the set of all vertices not adjacent to \( v \in G \). Let \( \{e_1, e_2, \cdots, e_q\} \) be the set of all line vertices in \( T_2(G) \). Then each line vertex is incident with at least one vertex of \( D_t \). Therefore

\[
\gamma_{mtp}(G) \leq |D_t \cup X| = \gamma_t(G) + \deg(v)
\]

Since \( \deg(v) \leq \Delta(G) \), therefore \( \gamma_{mtp}(G) \leq \gamma_t(G) + \Delta(G) \).

Equality follows from Theorem 7.2.

The following theorem establishes the relation between \( \gamma_s(G) \) and \( \gamma_{mtp}(G) \).

**Theorem 7.15** For any non trivial tree \( T \), \( \gamma_s(T) \leq \gamma_{mtp}(T) \).

**Proof.** The result follows from Theorem 7.C, Theorem 7.3 and from the fact that \( \gamma(T) \leq \gamma_m(T) \).
Corollary 7.1. In a tree, if every nonpendant vertex is adjacent to at least one pendant vertex, then

\[(i) \quad \gamma_{mtp}(T) = \gamma_s(T) + 1\]

\[(ii) \quad \gamma_{mtp}(T) = \gamma(T) + 1.\]