Chapter 6

Domination in Semitotal-Point Graphs
6.1 Introduction

A set $S \subseteq V$ is a dominating set if each vertex in $V$ is dominated by at least one vertex of $S$. The domination number $\gamma(G)$ is the smallest cardinality of a dominating set. Each vertex $v \in V$ dominates every vertex in its closed neighborhood $N[v]$.

A dominating set $D$ of a graph $G$ is a split dominating set if the induced subgraph $\langle V - D \rangle$ is disconnected. The split domination number $\gamma_s(G)$ of $G$ is the minimum cardinality of a split dominating set.

A dominating set $D$ of a graph $G$ is a connected dominating set if the induced subgraph $\langle D \rangle$ is connected. The connected domination number $\gamma_c(G)$ of $G$ is the minimum cardinality of a connected dominating set.

A dominating set $D$ is a total dominating set if the induced subgraph $\langle D \rangle$ has no isolated vertex. The total domination number $\gamma_t(G)$ of a graph $G$ is the minimum cardinality of a total dominating set.

A set $D$ of vertices of a graph $G$ is a maximal dominating set of $G$ if $D$ is a dominating set and $V - D$ is not a dominating set. The maximal domination number $\gamma_m(G)$ of $G$ is the minimum cardinality of a maximal dominating set.

A dominating set $D$ of $G$ is a cototal dominating set if the induced sub-
The cototal domination number $\gamma_{\text{cot}}(G)$ of $G$ is the minimum cardinality of a cototal dominating set. A dominating set $D$ of vertices in a graph $G$ is a dominating clique if the induced subgraph $\langle D \rangle$ is a complete graph. The clique domination number $\gamma_{\text{cl}}(G)$ of $G$ is the minimum cardinality of a dominating clique. A dominating set $D$ of a connected graph $G$ is a path nonsplit dominating set if the induced subgraph $\langle V - D \rangle$ is a path in $G$. The path nonsplit domination number $\gamma_{\text{pns}}(G)$ is the minimum cardinality of a path nonsplit dominating set.

The maximum number of classes of a domatic partition of $G$ is called the domatic number of $G$ and is denoted by $d(G)$. The vertex independence number $\beta_0(G)$ is the maximum cardinality among the independent set of vertices of $G$.

E.Sampatkumar and S.B.Chikkodimath[56] introduced the new graph valued function, namely the semitotal-point graph of a graph.

The semitotal-point graph $T_2(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$, where two vertices are adjacent if and only if (i) they are adjacent vertices of $G$ or (ii) one is a vertex of $G$ and the other is an edge of $G$, incident with it.
In Figure 6.1, a graph $G$ and its semitotal-point graph $T_2(G)$ are shown.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.1.png}
\caption{Figure 6.1}
\end{figure}

In this chapter, some results on domination number of semitotal-point graphs are obtained.

The following Theorems are required for further results.

**Theorem 6.A[45].** For any graph $G$ with a pendant vertex, $\gamma(G) = \gamma_s(G)$.

**Theorem 6.B[19].** If $G$ is a connected graph with $p \geq 3$ vertices, then $\gamma_t(G) \leq \frac{2p}{3}$.

**Theorem 6.C[57].** If $T$ is a tree with $p \geq 3$ vertices, then $\gamma_c(T) = p - e$, where $e$ denotes the number of pendant vertices in $T$.

**Theorem 6.D[32].** For any nontrivial connected $(p, q)$ graph $G$,
\[\alpha_0 + \beta_0 = p = \alpha_1 + \beta_1.\]

**Theorem 6.E[67].** For any connected \((p, q)\) graph \(G\),
\[
\left\lfloor \frac{p}{p-\delta(G)} \right\rfloor \leq d(G).
\]

**Theorem 6.F[22].** If \(G\) is a graph without isolated vertices of order \(p\), then \(\gamma(G) \leq \frac{p}{2}\).

### 6.2 Results

The following theorem gives the domination number of semitotal-point graph of some standard class of graphs.

**Theorem 6.1** (i) For any path \(P_p\) with \(p \geq 2\), \(\gamma(T_2(P_p)) = \left\lfloor \frac{p}{2} \right\rfloor\).

(ii) For any cycle \(C_p\) with \(p \geq 3\), \(\gamma(T_2(C_p)) = \left\lfloor \frac{p}{2} \right\rfloor\).

(iii) For any wheel \(W_p\) with \(p \geq 4\), \(\gamma(T_2(W_p)) = \left\lfloor \frac{p}{2} \right\rfloor + 1\).

(iv) For any star \(K_{1,p-1}\) with \(p \geq 2\), \(\gamma(T_2(K_{1,p-1})) = 1\).

(v) For any \(K_p\) with \(p \geq 2\) vertices, \(\gamma(T_2(K_p)) = p - 1\).

**Theorem 6.2** For any \((p, q)\) graph \(G\), \(\gamma(G) \leq \gamma(T_2(G))\). The equality holds for \(G = K_2\) or \(\overline{K_p}\).
Proof. The following cases are considered.

**Case 1.** Let $G$ be a graph without isolated vertices. Then by Theorem 6.1, $\gamma(G) \leq \frac{p}{2}$. Therefore by definition of $T_2(G)$, $\gamma(T_2(G)) \leq \frac{p+q}{2}$. Hence $\gamma(G) \leq \gamma(T_2(G))$.

**Case 2.** Let $G$ be a graph with isolated vertices of the type $G = H \cup nK_1$, where $H = (p', q')$ is any connected graph. Then $T_2(G) = H' \cup nK_1$. By Case 1, $\gamma(H) \leq \gamma(H')$. Therefore $\gamma(G) \leq \frac{p'}{2} + n$ and $\gamma(T_2(G)) \leq \frac{p'+q}{2} + n$. Hence the result.

Equality can be easily verified for $G = K_2$ or $\overline{K}_p$.

**Theorem 6.3** Let $G = T$ be a tree then, $\gamma(T_2(G)) = \gamma(G)$ except $G \neq P_p$ for $p \geq 6$ vertices.

**Proof.** Suppose $G = P_p$ for all $p \geq 6$. Then $\gamma(P_p) = \left\lceil \frac{p}{3} \right\rceil$ and by Theorem 6.1, $\gamma(T_2(P_p)) = \left\lfloor \frac{p}{2} \right\rfloor$. Therefore, $\gamma(T_2(P_p)) \neq \gamma(P_p)$ for all $p \geq 6$.

Let $G = T$ be a tree which is not a path of order $p \geq 6$. Let $D_1 = \{v_1, v_2, \ldots v_r\}$ be a subset of vertex set such that $\deg(v_i) \geq 2$ for $1 \leq i \leq r$ and $D_2 = \{u_1, u_2, \ldots u_s\}$ where $\deg(u_i) = 1$ for $1 \leq i \leq s$ be the subsets of $V(T)$ where $V(T) = D_1 \cup D_2$. Clearly $D_1$ is a minimum dominating set of $T$. Therefore $\gamma(T) \leq |D_1| = r$. 

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Let $D'_1 = \{v'_1, v'_2, \cdots v'_t\}$ and $D'_2 = \{u'_1, u'_2, \cdots u'_s\}$ be the corresponding vertices of $D_1$ and $D_2$ in $T_2(G)$ respectively. Let $u_1, u_2, \cdots u_n \in E(T)$ and $u'_1, u'_2, \cdots u'_n$ be the corresponding edge vertices in $T_2(G)$. For each edge $u_i$ for $1 \leq i \leq n$ there exists a vertex $v_j \in D'_1$. Clearly $\gamma(T_2(G)) \geq |D'_1| = |D_1|$. Hence $\gamma(T_2(G)) = \gamma(G)$. \hfill \blacksquare

**Corollary 6.1.** In a tree if every nonpendant vertex is adjacent to at least one pendant vertex, then $\gamma(T_2(T)) = \gamma(T)$.

In the following theorem we establish an upper bound for $T_2(G)$ in terms of vertices of $G$.

**Theorem 6.4** For any $(p, q)$ graph $G$, $\gamma(T_2(G)) \leq p$. The equality holds for $G = \overline{K}_p$.

**Proof.** The following cases are considered.

**Case 1.** Suppose $G$ is a connected graph. Then $\Delta(G) \leq p - 1$. Therefore by Theorem 6.1, $\gamma(T_2(G)) \leq p - 1$.

**Case 2.** Let $G$ be a disconnected graph with $n$ components that is $G = H_1 \cup H_2 \cup \cdots \cup H_n$. Then $T_2(G) = H'_1 \cup H'_2 \cup \cdots \cup H'_n$. Consider the following subcases of case 2.

**Subcase 2.1.** Suppose each component $H_i$; $1 \leq i \leq n$ is a complete graph of order $\geq 2$, then by Case 1, $\gamma(H'_1) \leq \frac{P'_1}{2}$, $\gamma(H'_2) \leq \frac{P'_2}{2}$, $\cdots$. 
\(\gamma(H'_n) \leq \frac{P'_n}{2}\), where \(P'_1 = V(H'_1), P'_2 = V(H'_2), \ldots, P'_n = V(H'_n)\).

Then \(\gamma(T_2(G)) = p - n\), where \(n\) is the number of components in \(G\).

Hence \(\gamma(T_2(G)) < p - 1\).

**Subcase 2.2.** Suppose the components \(H_i; 1 \leq i \leq n\), are either complete or other than complete graphs then by subcase 2.1, \(\gamma(T_2(G)) < p - n < p - 1\).

The equality is easy to verify for \(G = \overline{K}_p\).

The following result shows the upper bound of \(\gamma(T_2(G))\) in terms of \(\alpha_0\) and \(\beta_0\).

**Corollary 6.2** For any connected graph \(G\), \(\gamma(T_2(G)) \leq \alpha_0 + \beta_0\).

**Proof.** Follows from Theorem 6.4 and Theorem 6.4.

**Proposition 6.1** For any connected \((p, q)\) graph \(G\), \(\gamma(T_2(G)) \leq p - 1\).

The following theorem gives the relation between number of vertices, \(\Delta(G)\), \(\alpha_0(G)\) and \(\beta_0(G)\) with \(\gamma(T_2(G))\).

**Theorem 6.5** For any \((p, q)\) graph \(G\), \(\frac{p}{1+\Delta(G)} \leq \gamma(T_2(G)) \leq \alpha_0 + \beta_0\).

Further the equality of lower bound is attained if \(G = \overline{K}_p\).

**Proof.** Firstly, the lower bound is considered.

Let \(D\) be a dominating set of \(G\). Each vertex dominates at most itself and other vertices of \(\Delta(G)\). Hence \(\frac{p}{1+\Delta(G)} \leq \gamma(T_2(G))\).
If $G = \overline{K}_p$, then by Theorem 6.4, $\gamma(T_2(\overline{K}_p)) = p$. Hence the lower bound is attained.

The upper bound follows from the Proposition 6.1. ■

Next theorem gives the relation between number of vertices and $\beta_0(G)$ with $\gamma(T_2(G))$.

**Theorem 6.6** For any $(p, q)$ graph $G$, $p - \beta_0(G) \leq \gamma(T_2(G)) \leq p$.

Further, the equality of lower bound is attained if $G = P_p$ or $C_p$.

**Proof.** In the beginning, the lower bound is considered.

It is obvious that for any graph $G$, $p - \beta_0(G) \leq \gamma(T_2(G))$.

For equality, the following cases arise:

**Case 1.** If $G = P_p$ for $p \geq 2$ vertices then $\beta_0(P_p) = \left\lfloor \frac{p}{2} \right\rfloor$. Therefore

$p - \beta_0(G) = p - \left\lfloor \frac{p}{2} \right\rfloor = \left\lceil \frac{p}{2} \right\rceil$

By Theorem 6.1, $\gamma(T_2(P_p)) = \left\lceil \frac{p}{2} \right\rceil$. Hence the equality.

**Case 2.** If $G = C_p$ for all $p \geq 3$ vertices then $\beta_0(C_p) = \left\lfloor \frac{p}{2} \right\rfloor$. Therefore

$p - \beta_0(G) = p - \left\lfloor \frac{p}{2} \right\rfloor = \left\lceil \frac{p}{2} \right\rceil$

By Theorem 6.2, $\gamma(T_2(C_p)) = \left\lceil \frac{p}{2} \right\rceil$

The upper bound follows from Theorem 6.4. ■

**Corollary 6.3.** For any graph $G$, $\alpha_0(G) \leq \gamma(T_2(G))$.

The next result gives the upper bound for $\gamma(T_2(G))$ in terms of
Theorem 6.7 For any graph $G$, $\gamma(T_2(G)) \leq \text{diam}(G)$.

**Proof.** Let $A = \{e_1, e_2, \ldots, e_t\}$ be the set of edges which constitutes the longest path between any two vertices of $G$ such that $|A| = \text{diam}(G)$.

Consider the following cases:

**Case 1.** If $G$ contains only one edge then $T_2(G) = C_3$. Therefore $\gamma(T_2(G)) = 1 = |A|$.

**Case 2.** Let $u_1, u_2, \ldots, u_m \in E(G)$ and $u'_1, u'_2, \ldots, u'_m$ be the corresponding edge vertices of $T_2(G)$. Let $v_1, v_2, \ldots, v_n \in V(G)$ and $v'_1, v'_2, \ldots, v'_n$ be the corresponding vertices of $T_2(G)$. Since all the edge vertices are incident with $V(T_2(G)) - u'_m$, therefore $\gamma(T_2(G)) \leq |V(T_2(G)) - u'_m| = |D| \leq |A|$. Hence $\gamma(T_2(G)) \leq \text{diam}(G)$.

The following theorems give the relation between domination number of $T_2(G)$ and other domination parameters.

**Theorem 6.8** For any connected graph $G$, $\gamma(T_2(G)) \leq \gamma_m(G)$, where $\gamma_m(G)$ is the maximal domination number of $G$.

**Proof.** Let $G$ be any connected graph. Suppose $\gamma(T_2(G)) > \gamma_m(G)$. Then each $u_i$, $1 \leq i \leq m$ of a corresponding edge vertex of $G$ in $T_2(G)$
will be adjacent to at least two vertices of $V(T_2(G) - v'_n)$ in $T_2(G)$, where $v'_n$ is the corresponding vertex set of $G$ in $T_2(G)$. Therefore by taking alternate vertices $v_j$, $1 \leq j \leq n$ of $T_2(G)$, the set $v_j, j = 1, 2, \ldots, n$ will be a dominating set of $T_2(G)$. Since each $u_i$, $1 \leq i \leq m$ is an independent vertex in $T_2(G)$, then the set $u_i, i = 1, 2, \ldots, n$ is maximal independent set. Since every maximal independent set is a minimal dominating set, therefore $D' = |u_m| = V(T_2(G)) - D$ is also a dominating set of $T_2(G)$. Thus $\gamma(T_2(G)) < \gamma_m(G)$, a contradiction.

In the next theorem, a relation between $\gamma_c(G)$ and $\gamma(T_2(G))$ is established.

**Theorem 6.9** For any tree $T$ of order $p \geq 2$ vertices, $\gamma_c(T) \geq \gamma(T_2(T))$.

**Proof.** Let $T$ be a tree of order $p \geq 2$. Let $D = \{v_1, v_2, \ldots, v_r\}$ be such that $\text{deg}(v_i) \geq 2$ for $1 \leq r$. Then in $T_2(G)$ degree of each $u_i$ and each $v_r$ will become twice. Therefore, each $v'_i$ in $T_2(G)$ will be a minimal dominating set. Therefore $\gamma(T_2(T)) = p - e$, where $e$ is the number of pendant vertices in $T$. Hence by Theorem 6.C, $\gamma_c(T) \geq \gamma(T_2(T))$. ■

**Theorem 6.10** For any graph $G$, $\gamma(T_2(G)) \leq \gamma_{cat}(G)$.
**Proof.** Since $\gamma_t(G) \leq \gamma_{cot}(G)$, therefore by Theorem 6.8, we get the required result.

**Theorem 6.11** For any $(p,q)$ graph $G$, $\gamma_{cot}(T_2(G)) = 1$ if and only if $G = K_{1,p-1}$.

**Proof.** If $G = K_{1,p-1}$, then $deg(v'_i) = 2deg(v_i)$ where $v'_i$ is the corresponding vertex $v_i$ of $G$ in $T_2(G)$. Each component of $T_2(G)$ will be a triangle and each triangle will have one common vertex which is adjacent to all other vertices of $T_2(G)$. Therefore $\gamma_{cot}(T_2(G)) = 1$.

Conversely, suppose $\gamma_{cot}(T_2(G)) = 1$ and $G \neq K_{1,p-1}$, then there exists a vertex of $\Delta(T_2(G)) = p - 1$. Let $u$ be a vertex of degree $p - 1$, then all other vertices of $T_2(G)$ are adjacent to $u$. But in $T_2(G)$ this is the only case when $G$ is a star. Otherwise $\Delta(T_2(G)) < p - 1$. Then $\gamma(T_2(G)) \geq 2$, a contradiction.

**Proposition 6.2.** For any path $P_p$ with $p \geq 2$ vertices,

$$
\gamma_{pms}(T_2(G)) = q,
$$

where $q$ is the number of edges in $P_p$.

**Proof.** Let $G$ be a path with $p \geq 2$ vertices. Let $u'_m$ be the corresponding edge vertices in $T_2(G)$ and $v'_n$ is the corresponding vertices of $G$ in $T_2(G)$, then

$$
\gamma_{pms}(T_2(G)) = |V(T_2(G) - u'_m)|
$$
\[
= |u_m'|
\]
\[
= q
\]

**Theorem 6.12** For any tree \( T \), \( \gamma(T_2(T)) \geq \gamma_s(T) \)

**Proof.** By Theorem 6.3 and Theorem 6.A, we get the required result. ■

**Theorem 6.13** For any connected graph \( G \), \( \gamma(T_2(G)) \leq \gamma_t(G) \).

**Proof.** By Corollary 6.2 and Theorem 6.B, the required result. ■

**Theorem 6.14** For any \((p,q)\) graph \( G \), \( \gamma_{cl}(T_2)(G) = 1 \) if and only if \( G = K_{1,p-1} \).

**Proof.** Proof is similar to the Theorem 6.11. ■

Next theorem gives the relation between domatic number of \( G \) and domination number of \( T_2(G) \).

**Theorem 6.15** For any graph \( G \), \( d(G) \leq \gamma(T_2(G)) \), where \( d(G) \) is the domatic number of \( G \).

**Proof.** For any graph \( G \), \( d(G) \leq \delta(G) + 1 \). Let \( u \) be a vertex of minimum degree in \( G \), then degree of the corresponding vertex \( u_i' \) in \( T_2(G) \) will be twice. Also by Theorem 6.4, \( \gamma(T_2(G)) \leq p \). Therefore
\[ d(G) \leq \gamma(T_2(G)). \]

If \( G = \overline{K}_p \), then \( d(\overline{K}_p) = p = \gamma(T_2(G)) \). Hence the equality holds. \[ \square \]

**Theorem 6.16** For any graph \( G \), \( \frac{p}{p - \delta(G)} \leq \gamma(T_2(G)). \)

**Proof.** Follows from Theorem 6.E and Theorem 6.15. \[ \square \]

**Nordhaus-Gaddum type results**

**Theorem 6.17** For any connected \((p, q)\) graph \( G \)

\[
(i) \quad \gamma(T_2(G)) + \gamma(T_2(\overline{G})) \leq 2p - 1
\]

\[
(ii) \quad \gamma(T_2(G)) \cdot \gamma(T_2(\overline{G})) \leq p(p - 1).
\]

**Proof.** By using Theorem 6.1, the required result. \[ \square \]