Chapter 4

On Chainos Total-ctree Graph of a Graph
4.1 Introduction

A block of a graph is maximal nonseparable subgraph. A block of a graph $G$ is called a pendant block if it contains exactly one cutvertex of $G$. A block of a graph $G$ is called a nonpendant block if it contains at least two cutvertices of $G$. If two distinct blocks $b_1$ and $b_2$ are incident with a common cutvertex then they are called adjacent blocks. The block degree of a block $b$ of a graph $G$, is the number of blocks adjacent to it. For a connected graph $G$ with blocks $\{b_i\}$ and cutvertices $\{w_j\}$, the block-cutvertex graph of a graph $G$, denoted by $bc(G)$, is defined as the graph having vertex set $\{b_i\} \cup \{w_j\}$, with two vertices adjacent if one corresponds to a block $\{b_i\}$ and the other to a cutvertex $\{w_j\}$ and $\{w_j\}$ is in $\{b_i\}$. A nontrivial connected graph $G$ is called a chain if every block is incident with at most two cutvertices of $G$ and every cutvertex is incident with exactly two blocks. In other words a graph is a chain if its block-cutvertex graph $bc(G)$ is a path.

The concept of pathos of a graph $G$ was introduced by Harary [33], as a collection of minimum number of edge-disjoint open paths whose union is $G$.

On a similar line, the chainos of a graph $G$ was introduced by Basa-
vanagoud et al[9], as a collection of minimum number of block disjoint open chains whose union is $G$. The chainoslength is the number of blocks which lie on a particular chain $C_i$ of chainos of $G$. A pendant chainos is a chain $C_i$ of chainos having single block which corresponds to a pendant block in $G$. The chain number of a graph $G$ is the number of chains in chainos. The chain number of a graph $G$ is equal to $k$, where $2k$ is the sum of pendantblocks of $G$ and the number of cutvertices incident with odd number of blocks of $G$.

The cutvertices and blocks of a graph are called its members. If a block is incident with cutvertices $w_1, w_2, \ldots, w_r, \ r \geq 2$, we say that $w_i$ and $w_j$ are coadjacent where $i \neq j, \ 1 \leq i, j \leq r$. The ctree $tc(G)$ of a graph $G$ is the graph whose vertices can be put in one-to-one correspondence with the set of blocks and cutvertices of $G$ in such a way that two vertices of $tc(G)$ are adjacent if and only if the corresponding blocks of $G$ are adjacent or the corresponding members of $G$ are incident[44]. The total-ctree $Tc(G)$ of a graph $G$ is the graph whose vertices can be put in one-to-one correspondence with the set of blocks and cutvertices of $G$ in such a way that two vertices of $Tc(G)$ are adjacent if and only if the corresponding members of $G$ are adjacent, coadjacent or incident[44].
A graph $G$ and its total-ctree are shown in Figure 4.1.

![Diagram of G and Tc(G)](image)

**Figure 4.1**

The chainos ctree graph $C_{tc}(G)$ of a graph $G$ is the graph whose vertex set is the union of blocks, set of chains of chaions and set of cutvertices of $G$, in which two vertices are adjacent if and only if the corresponding blocks of $G$ are adjacent and the block lies on the chain of chainos and the blocks are incident to the cutvertex. Here a new class of graphs known as chainos total-ctree graph of a graph is introduced.

The chainos total-ctree graph of a graph $G$ denoted as $C_{Tc}(G)$ is defined as the graph whose vertex set is the union of blocks, set of chains of chaions and set of cutvertices of $G$, in which two vertices are adjacent if and only if the corresponding members of $G$ are adjacent, coadjacent or incident and the block lies on the chain of chainos. Since the system
of chain of chainos for a graph is not unique, the corresponding chainos total-ctree graph is also not unique. This graph is defined for a graph having at least one cutvertex.

In Figure 4.2, a graph and its different chainos total-ctree graphs $C_{Te}(G)$ are shown.
Figure 4.2
In this chapter, some properties of chainos total-ctree graphs are obtained. Further eulerian and hamiltonian properties of these graphs are investigated. In addition, characterizations of those graphs whose chainos total-ctree graphs are planar, outerplanar and maximal outerplanar are presented. Also, the necessary and sufficient conditions for chainos total-ctree graphs to have crossing number one or two are established.

The following Theorems and observations are necessary in the proofs of further results.

**Theorem 4.A[9].** If $G$ is a connected graph with $p$ vertices and $b_i$ is the number of blocks to which the vertex $v_i$ belongs in $G$, cutvertex $w$ incident with $w_j$ blocks then the chainos ctree graph $C_{tc}(G)$ has $w + k + 1 + \sum (b_i - 1)$ vertices and $1 + \sum \frac{(b_i+2)(b_i-1)}{2} + \sum (w_j)$ edges, where $k$ is the chain number.

**Theorem 4.B[9].** The chainos ctree graph of a graph $G$ with $b \geq 1$ blocks is noneulerian.

**Theorem 4.C[9].** The chainos ctree graph of a graph $G$ is hamiltonian if and only if every cutvertex of $G$ lies on even number of blocks.

**Theorem 4.D[44].** The vertices of total-ctree graph $Tc(G)$ corresponding to a cutvertex and its $n$ incident blocks, generate a subgraph $K_{n+1}$
of $Tc(G)$. Also the vertices of $Tc(G)$ corresponding to a block and its $n$ incident cutvertices, generate a subgraph $K_{n+1}$ of $Tc(G)$.

**Theorem 4.E[44].** The total-ctree graph $Tc(G)$ of a graph $G$ is planar if and only if the following hold:

(i) Every cutvertex of $G$ is incident with at most three blocks.

(ii) Every block of $G$ is incident with at most three cutvertices.

**Theorem 4.F[44].** No vertex of $Tc(G)$ of a graph $G$ is a cutvertex.

**Theorem 4.G[9].** The chainos ctree $C_{tc}(G)$ of a graph $G$ is planar if and only if every vertex of $G$ is incident with at most three blocks.

**Remark 4.1.** For any graph $G$ with $b \geq 2$ blocks

(i) $tc(G) \subset Tc(G)$

(ii) $Tc(G) \subset C_{Tc}(G)$

(iii) $C_{tc}(G) \subset C_{Tc}(G)$

**Remark 4.2.** If the block degree of the nonpendant block $b$ in a graph $G$ is even(odd), then the corresponding vertex in $C_{Tc}(G)$ is of odd(even) degree.

**Remark 4.3.** If the degree of the pendant block in a graph $G$ is even(odd), then the corresponding vertex in $C_{Tc}(G)$ is even(odd) degree.
Remark 4.4. The degree of the chainos vertex in $C_{T_c}(G)$ is equal to the chainos length of the corresponding chain $C_i$ of chainos of $G$.

Remark 4.5. Every pendant chainos in a graph $G$ adds one cutvertex to $C_{T_c}(G)$.

Remark 4.6. If a vertex $v$ of $C_{T_c}(G)$ corresponds to a cutvertex $w$ of a graph $G$, then $deg_{C_{T_c}(G)}v = r + n$, where $r$ is the number of blocks of $G$ containing $w$ and $n$ is the number of cutvertices of $G$ adjacent to $v$.

4.2 Results

In the following theorem the number of vertices and edges in $C_{T_c}(G)$ are calculated.

Theorem 4.1 If $G$ is a connected graph with $p$ vertices and $b_i$ is the number of blocks to which the vertex $v_i$ belongs in $G$, cutvertices $w$ incident with $w_j$ blocks and $c$ be the number of coadjacent cutvertices of each block, then chainos total-ctree graph $C_{T_c}(G)$ has $w+k+1+\sum(b_i-1)$ vertices and $1 + \sum \frac{(b_i+2)(b_i-1)}{2} + \sum w_j + \sum \frac{c(c-1)}{2}$ edges, where $k$ is the chainnumber.
Proof. By the definition of $C_{Tc}(G)$, the number of vertices in $C_{Tc}(G)$ is $w + k + 1 + \sum (b_i - 1)$.

By Theorem 4.A, the number of edges in $C_{Tc}(G)$ is $1 + \sum \frac{(b_i+2)(b_i-1)}{2} + \sum w_j$. The number of edges in $C_{Tc}(G)$ is the sum of the edges in $C_{tc}(G)$ and the number of edges of the whole graph formed by coadjacent cutvertices of the each block. Hence the number of edges in $C_{Tc}(G)$ is $1 + \sum \frac{(b_i+2)(b_i-1)}{2} + \sum w_j + \sum \frac{e(e-1)}{2}$.  

Theorem 4.2 The number of cutvertices in chainos total-ctree graph $C_{Tc}(G)$ of a graph $G$ is equal to the number of pendant chains of $G$.

Proof. Consider the following cases:

Case 1. By Theorem 4.F, no vertex of total-ctree graph $Tc(G)$ of a graph $G$ is a cutvertex. Assume all the chains of chainos are of length $\geq 2$. Each of the chainosvertex is adjacent to two or more vertices corresponding to the blocks on which it lies. Thus resulting in $C_{Tc}(G)$ in which again no vertex is a cutvertex. Thus $C_{Tc}(G)$ is a block.

Case 2. Suppose $G$ has at least one pendant chain. By Theorem 4.F, no vertex of total-ctree graph $Tc(G)$ of a graph $G$ is a cutvertex. By Case 1, $C_{Tc}(G)$ remains a block for all the chains of length $\geq 2$. By Remark 4.5, every pendant chain adds a cutvertex to $C_{Tc}(G)$. Hence the
number of cutvertices in $C_{Tc}(G)$ of $G$ is equal to the number of pendant chains of $G$.

Theorem 4.3 The chainos total-ctree graph of a graph $G$ with $b \geq 1$ blocks is noneulerian if $G$ contains pendant chainos.

Proof. Let $C_i$ be a pendant chainos of a graph $G$. Then by the definition of $C_{Tc}(G)$ of a graph $G$, the degree of the corresponding chainosvertex $C_i$ will be one. Hence $C_{Tc}(G)$ is noneulerian.

Theorem 4.4 The chainos total-ctree $C_{Tc}(G)$ of a graph $G$ is hamiltonian if and only if every cutvertex of $G$ lies on even number of blocks.

Proof. By definition, the number of vertices in chainos total-ctree graph and chainos ctree graph is same. By Theorem 4.C, the chainos ctree graph $C_{tc}(G)$ of a graph $G$ is hamiltonian if and only if every cutvertex of $G$ lies on even number of blocks and by Remark 4.1, $C_{tc}(G)$ is a subgraph of $C_{Tc}(G)$. Hence $C_{Tc}(G)$ has hamiltonian cycle. Thus $C_{Tc}(G)$ is hamiltonian.
4.3 (Non-) Planar Chainos Total-ctree Graphs

In this section characterizations for planarity of chainos total-ctree graphs are obtained.

**Theorem 4.5** The chainos total-ctree graph $C_{Tc}(G)$ of a graph $G$ is planar if and only if the following hold:

(i) Every cutvertex of $G$ is incident with at most three blocks.

(ii) Every block of $G$ is incident with at most three cutvertices.

**Proof.** Suppose $C_{Tc}(G)$ is planar. By Remark 4.1 and Theorem 4.G, (i) holds.

To prove (ii), assume a block $B$ of the graph $G$ which is incident with cutvertices $w_1, w_2, \ldots, w_n$ and $n \geq 4$. By Theorem 4.D, the vertices of $Tc(G)$ corresponding to the block $b, w_1, w_2, \ldots, w_n$ generate $K_{n+1}$ as a subgraph of $Tc(G)$. Clearly $K_5$ is a subgraph of $Tc(G)$. By Remark 4.1, $K_5$ is a subgraph of $C_{Tc}(G)$. This contradicts the planarity of $C_{Tc}(G)$.

Thus (ii) holds.

Conversely, suppose $G$ satisfies the given conditions. From condition (i) and Theorem 4.G, $C_{tc}(G)$ is planar and (ii) implies by Theorem 4.D, that the blocks together with incident cutvertices do not generate
subgraphs homeomorphic to $K_{3,3}$ or $K_5$. This implies $C_{Tc}(G)$ is planar.

A characterization for graphs whose $C_{Tc}(G)$ graphs are outerplanar and maximal outerplanar are established.

**Theorem 4.6** The chainos total-ctree graph $C_{Tc}(G)$ of a graph $G$ is outerplanar if and only if $G$ is a chain of length at most two.

**Proof.** Suppose $C_{Tc}(G)$ is outerplanar. Then clearly $C_{Tc}(G)$ is planar. By Theorem 4.5, every cutvertex of $G$ is incident with at most three blocks and every block of $G$ is incident with at most three cutvertices. Assume that $G$ has a cutvertex incident with three blocks and a block incident with three cutvertices. Then by Theorem 4.4, $Tc(G)$ has two $K_4$ as subgraphs. By Remark 4.1, $C_{Tc}(G)$ has two $K_4$ subgraphs. Thus $C_{Tc}(G)$ is nonouterplanar, a contradiction. Hence $G$ must be a chain.

Now, suppose $G$ is a chain of length $\geq 3$. In the optimal drawing of $C_{Tc}(G)$, one can easily see that the vertices to the nonpendant block lie in the interior region of $C_{Tc}(G)$. Thus $C_{Tc}(G)$ is nonouterplanar, a contradiction. Thus $G$ must be a chain of length at most two.

Conversely, suppose $G$ is a chain of length at most two. Assume $G$ is a chain of length one. Clearly $C_{Tc}(G)$ is $K_2$ which is outerplanar.
Assume $G$ is a chain of length two. Clearly $Tc(G)$ is $K_3$. Since $G$ is a chain of length two, it has only one chain of chainos. The chainos vertex is adjacent to vertices corresponding to the blocks on which it lies, that is in $Tc(G)$, thus forming $C_{Tc}(G)$ which is $K_4 - x$. Hence $C_{Tc}(G)$ is outerplanar.

**Theorem 4.7** The chainos total-ctree graph $C_{Tc}(G)$ of a graph $G$ is maximal outerplanar if and only if $G$ is a chain of length two.

**Proof.** Suppose $C_{Tc}(G)$ of $G$ is maximal outerplanar. By Theorem 4.6, $C_{Tc}(G)$ is outerplanar and $G$ is a chain of length two.

Conversely, suppose $G$ is a chain of length two. The blocks, cutvertices and chains of chainos together form $K_4 - x$ in $C_{Tc}(G)$. Clearly $K_4 - x$ has two nonadjacent vertices whose join alters the outerplanarity of $C_{Tc}(G)$. Hence $C_{Tc}(G)$ is maximal outerplanar.

A criterion for $C_{Tc}(G)$ to be minimally nonouterplanar is established in the following theorem.

**Theorem 4.8** Chainos total-ctree graph $C_{Tc}(G)$ of a graph $G$ is minimally nonouterplanar if and only if $G$ is a chain of length three or $G$ has only one cutvertex which is incident with exactly three blocks.
**Proof.** Suppose $C_{T_c}(G)$ of $G$ is minimally nonouterplanar. Assume a cutvertex $v$, incident with $n \geq 4$ blocks. Then by Theorem 4.5, $C_{T_c}(G)$ is nonplanar, a contradiction. Hence every cutvertex of $G$ is incident with at most three blocks.

Assume every cutvertex of $G$ is incident with $n < 3$ blocks. Clearly $G$ is a chain. If $G$ is a chain of length $> 3$, then by Theorem 4.6, $C_{T_c}(G)$ is nonouterplanar, a contradiction. Hence $G$ is a chain of length $\leq 3$.

Suppose $G$ is a chain of length $\leq 2$. Then by Theorem 4.6, $C_{T_c}(G)$ is outerplanar, a contradiction. Assume $G$ is a chain of length three. In the optimal drawing of $C_{T_c}(G)$ the vertices corresponding to the cutvertices, blocks and the chainosvertex form a wheel in $C_{T_c}(G)$. Thus $C_{T_c}(G)$ is minimally nonouterplanar.

Assume two cutvertices $u$ and $v$ of $G$ such that $u$ is incident with exactly three blocks of $G$ and $v$ is incident with exactly two blocks of $G$. In the optimal drawing of $C_{T_c}(G)$, we observe that the vertices corresponding to the cutvertices, blocks and the chainosvertex form $K_4$ and a wheel as subgraphs of $C_{T_c}(G)$. Clearly $i[C_{T_c}(G)] \geq 2$, a contradiction.

Assume $G$ has two cutvertices $u_1$ and $v_1$ each of which is incident with three blocks. By Theorem 4.4, $T_c(G)$ has two $K_4$ as subgraphs.
By Remark 4.1, $C_{Tc}(G)$ has two $K_4$ subgraphs. Clearly $i[C_{Tc}(G)] \geq 2$, a contradiction. Hence $G$ must have only one cutvertex incident with exactly three blocks.

Conversely, suppose $G$ is a chain of length three or $G$ has only one cutvertex incident with exactly three blocks. If $G$ is a chain of length three, then $C_{Tc}(G)$ is a wheel. Clearly $i[C_{Tc}(G)] = 1$. If $G$ has only one cutvertex which is incident with exactly three cutvertices, then by Theorem 4.4, $Tc(G)$ has $K_4$ as a subgraph. By Remark 4.1, $C_{Tc}(G)$ has $K_4$ as a subgraph. Also by definition of chainos, $G$ has two chains $C_1$ and $C_2$ of length 2 and 1 respectively. By the definition of $C_{Tc}(G)$, the chainosvertex $C_1$ is adjacent with two vertices and the chainosvertex $C_2$ is adjacent to only one vertex of $Tc(G)$. Thus $K_4$ remains a subgraph of $C_{Tc}(G)$. Hence $i[C_{Tc}(G)] = 1$.

4.4 Characterization of $C_{Tc}(G)$ with Crossing Number One or Two

**Theorem 4.9** The chainos total-ctree graph $C_{Tc}(G)$ of a graph $G$ has crossing number one if and only if either of the following condition holds:

(i) Every cutvertex of $G$ is incident with at most three blocks and ev-
Every block is incident with four cutvertices with a unique block incident with four cutvertices.

(ii) Every block of \( G \) is incident with at most three cutvertices and every cutvertex is incident with at most four blocks with a unique cutvertex incident with four blocks.

**Proof.** Suppose \( C_{Tc}(G) \) of \( G \) has crossing number one. Assume that a cutvertex of \( G \) is incident with \( n \geq 3 \) blocks and a block incident with \( m > 4 \) cutvertices.

The following cases are discussed.

**Case 1.** Suppose \( G \) has a cutvertex \( u \) incident with four blocks and a block incident with five cutvertices. By Theorem 4.D, the cutvertex \( u \) incident with four blocks form \( \langle K_5 \rangle \) and the block incident with five cutvertices form \( \langle K_6 \rangle \) in \( Tc(G) \). By Remark 4.1, \( C_{Tc}(G) \) has \( \langle K_5 \rangle \) and \( \langle K_6 \rangle \) as subgraphs. Hence \( cr[C_{Tc}(G)] > 1 \), a contradiction.

**Case 2.** Suppose \( G \) has at least two blocks each of which is incident with four cutvertices. By Theorem 4.D, \( Tc(G) \) contains at least two subgraphs as \( \langle K_5 \rangle \). By Remark 4.1, \( C_{Tc}(G) \) has at least two \( \langle K_5 \rangle \) as subgraphs. Clearly \( cr[C_{Tc}(G)] \geq 2 \), a contradiction.

To prove (ii), assume a block of \( G \) which is incident with \( n > 3 \)
cutvertices and a cutvertex is incident with $m > 4$ blocks.

The following cases:

**Case 1.** Suppose $G$ has a block incident with four cutvertices and a cutvertex incident with five blocks. By Theorem 4.4 the block with its four incident cutvertices form $\langle K_5 \rangle$ and the cutvertex incident with five cutvertex form $\langle K_6 \rangle$ in $Tc(G)$. By Remark 4.1, $C_{Tc}(G)$ has $\langle K_5 \rangle$ and $\langle K_6 \rangle$ as subgraphs. Hence $cr[C_{Tc}(G)] > 1$, a contradiction.

**Case 2.** Suppose $G$ has at least two blocks each of which is incident with four cutvertices. By Theorem 4.4, $Tc(G)$ contains at least two subgraphs as $\langle K_5 \rangle$. By Remark 4.1, $C_{Tc}(G)$ has at least two $\langle K_5 \rangle$ as subgraphs. Clearly $cr[C_{Tc}(G)] > 1$, a contradiction.

Conversely, suppose $G$ holds both the conditions of the theorem. Then by necessity the result follows.

**Theorem 4.10** The chainos total-ctree graph of a graph $G$ has crossing number two if and only if either of the following conditions hold:

(i) $G$ has exactly two cutvertices, each of which is incident with exactly four blocks with any other cutvertex of $G$ is incident with at most three blocks and every block is incident with at most three cutvertices.

(ii) $G$ has exactly two blocks, each of which is incident with exactly
four cutvertices with any other block of \( G \) is incident with at most three cutvertices and every cutvertex of \( G \) is incident with at most three blocks.

**Proof.** Suppose the chainos total-ctree graph \( C_{Tc}(G) \) of a graph \( G \) has crossing number two. To prove \((i)\) we discuss the following cases:

**Case 1.** Assume \( G \) has a cutvertex \( v \) incident with five blocks and a block incident with four cutvertices. Then by Theorem 4.D, the points of \( Tc(G) \) corresponding to the cutvertex \( v \) with its \( n \) incident blocks and the block with its \( n \) incident cutvertices form a complete graph \( K_{n+1} \) of \( Tc(G) \). This implies \( Tc(G) \) has \((K_5 \text{ and } K_6)\) as subgraphs. Since \( cr(K_{n+1}) \geq 3\); for \( n \geq 5 \) and by Remark 4.1, \( C_{Tc}(G) \) has \( K_5 \) and \( K_6 \) as subgraphs. Clearly \( cr(C_{Tc}(G)) \geq 3 \), a contradiction.

**Case 2.** Assume \( G \) has at least three cutvertices each of which is incident with four blocks. Then by Theorem 4.D, \( Tc(G) \) has at least three \( K_5 \) as subgraphs. By Remark 4.1, \( C_{Tc}(G) \) has at least three \( K_5 \) as a subgraphs. Clearly \( cr(C_{Tc}(G)) \geq 3 \), a contradiction.

In each of the above cases we have a contradiction and from Case 1 and Case 2 we conclude that condition \((i)\) holds.

To prove \((ii)\) the following cases arise.

**Case 1.** Assume \( G \) has a block incident with five cutvertices and a
cutvertex incident with four blocks. Then by Theorem 4.D, the vertices of $Tc(G)$ corresponding to the block with its $n$ incident cutvertices and the cutvertex with its $n$ incident blocks form a complete subgraph $K_{n+1}$ as subgraphs of $Tc(G)$. This implies $Tc(G)$ has ($K_5$ and $K_6$) as subgraphs. Since $cr(K_{n+1}) > 2$; therefore $cr(C_{Tc}(G)) > 2$, a contradiction.

**Case 2.** Assume $G$ has at least three blocks each of which is incident with four cutvertices. Then by Theorem 4.D, $Tc(G)$ has at least three $K_5$ as subgraphs. By Remark 4.1, $C_{Tc}(G)$ has at least three $K_5$ as a subgraphs. Clearly $cr(C_{Tc}(G)) \geq 3$, a contradiction.

In each of the above cases we have a contradiction and from Case 1 and Case 2 we conclude that the condition $(ii)$ holds.

Conversely, $G$ satisfies either of the conditions $(i)$ or $(ii)$. Then by Theorem 4.D and Remark 4.1, $C_{Tc}(G)$ has exactly two subgraphs homeomorphic to $K_5$ and other remaining subgraphs are either $K_3$ or $K_4$. Clearly $C_{Tc}(G)$ has exactly two crossings.