CHAPTER 4

Publications based on this Chapter;


- *Unsteady Conducting Dusty Fluid Flow Through a Rectangular Channel in Frenet Frame Field System*, COMMUNICATED.

- *Unsteady Conducting Dusty Fluid Flow between two Circular Cylinders under Varying Pressure Gradient*, COMMUNICATED.
Chapter 4

Flow of an Unsteady Conducting Dusty Fluid in Different Channels

4.1 Introduction

P.G.Saffman [77] has discussed the stability of the laminar flow of a dusty gas in which the dust particles are uniformly distributed. He assumed that the dust particles are uniform in size and shape and the bulk concentration of the dust is very small to be neglected. On the other hand density of the dust material is large compared with the gas density so that the mass concentration of dust is an appreciable fraction of unity. Liu [43] has studied the flow induced by an oscillating infinite flat plate in a dusty gas. Rossow [72] has studied the flow of a viscous, incompressible and electrically conducting fluid in presence of an external magnetic field due to the impulsive motion of an infinite flat plate. Ong and Nicholls [55] have extended the study to cover the case of flow near an infinite wall which executes simple harmonic motion parallel to itself. Michael and Norey [47] investigated the motion of dusty gas with uniform distribution of the dust particles between two rotating cylinders. Ghosh [32] has obtained the analytical solutions for the dusty visco-elastic fluid between two infinite parallel plates under the influence of time
dependent pressure gradient, using appropriate boundary conditions. Authors like Amos [1] and Datta [15] have studied the fluid flow due to pulsatile pressure gradient, further, Datta and Dalal [14] have studied the Pulsatile flow and heat transfer of a dusty fluid through an infinitely long annular pipe.

The objective of this chapter is to study the geometry of laminar flow of an electrically conducting viscous incompressible fluid with embedded non-conducting identical spherical particles through different channels like rectangular, cylindrical and between two cylinders. The flow is due to the influence of uniform magnetic field and time dependent pressure gradient. Initially the fluid and dust particles are assumed to be at rest. This chapter is divided into three sections. In section A, we consider the flow in rectangular region and obtain the analytical expressions for velocities of fluid and dust particles by solving partial differential equations using variable separable method and Laplace transform technique. The skin friction at the boundary plates is also calculated. Finally, the changes in velocity profiles for fluid and dust particles have been determined and the effect of strength of magnetic field on velocity profiles at fixed time has been depicted graphically. In section B the flow is considered in a circular cylinder under the same assumptions as in the above case. The section C deals with the study of flow between two circular cylinders. The flow is due to the influence of time dependent pressure gradient and differential rotations of the cylinders. It is assumed that the inner and outer cylinders rotate with the different angular velocities. The flow analysis is carried out as same in above two sections.
4.2 Section A:

4.2.1 Formulation of the Problem

Consider a flow of viscous incompressible, conducting dusty fluid through a rectangular channel of nonconducting walls. The flow is due to the influence of time dependent pressure gradient and magnetic field of uniform strength $B_0$. Both the fluid and the dust particle clouds are supposed to be static at the beginning. The dust particles are assumed to be spherical in shape and uniform in size. The number density of the dust particles is taken as a constant throughout the flow. It is assumed that the dust particles are electrically nonconducting and neutral. As Figure-4.2.1 shows, the axis of the channel is along binormal direction and the velocity components of both fluid and dust particles are respectively given by:

$$\vec{u} = u_0 \vec{b}, \quad \vec{v} = v_0 \vec{b}$$

(4.2.1)

i.e., $u_s = u_n = 0$ and $v_s = v_n = 0$, where $(u_s, u_n, u_b)$ and $(v_s, v_n, v_b)$ denote the velocity components of fluid and dust respectively.

![Figure 4.2.1: Schematic diagram of dusty fluid flow in a rectangular channel.](image)
4.2.2 Solution of the Problem

By virtue of system of equations (1.3.1) the intrinsic decomposition of equations (2.2.2) and (2.2.4) give the following forms;

\[
0 = -\frac{1}{\rho} \frac{\partial p}{\partial s} - 2\sigma_n \frac{\partial u_b}{\partial n} + \tau_s k_s u_b \tag{4.2.2}
\]

\[
0 = -\frac{1}{\rho} \frac{\partial p}{\partial n} - 2\tau_s \frac{\partial u_b}{\partial s} + \sigma'_n k'_n u_b \tag{4.2.3}
\]

\[
\frac{\partial u_b}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial b} + \nu \left[ \frac{\partial^2 u_b}{\partial s^2} + \frac{\partial^2 u_b}{\partial n^2} \right] - C_r u_b + \frac{kN}{\rho} (v_b - u_b) - D u_b \tag{4.2.4}
\]

\[
\frac{\partial v_b}{\partial t} = \frac{k}{m} (u_b - v_b) \tag{4.2.5}
\]

\[
v_b^2 k''_b = 0 \tag{4.2.6}
\]

where \( C_r = (\tau'^2_s + \sigma'^2_n) \) is called curvature number [9].

From equation (4.2.6) we see that \( v_b^2 k''_b = 0 \) which implies either \( v_b = 0 \) or \( k''_b = 0 \). The choice \( v_b = 0 \) is impossible, since if it happens then \( u_b = 0 \), which shows that the flow doesn’t exist. Hence \( k''_b = 0 \), it suggests that the curvature of the streamline along binormal direction is zero. Thus no radial flow exists.

Since we have assumed that the exponential pressure gradient to be impressed on the system for \( t > 0 \), we can write

\[
-\frac{1}{\rho} \frac{\partial p}{\partial b} = \alpha_0 + \sum_{\beta=1}^{\infty} \alpha_\beta e^{-\beta t} \tag{4.2.7}
\]

where \( \alpha \) and \( \beta \) are reals.
Consider the initial and boundary conditions as:

\[
\begin{cases}
\text{Initial condition; at } t = 0; \quad u_b = 0, v_b = 0 \\
\text{Boundary condition; for } t > 0; \quad u_b(0, n) = 0, u_b(h_1, n) = 0, \\
\phantom{B} \quad u_b(s, 0) = 0, u_b(s, h_2) = 0
\end{cases}
\]  

(4.2.8)

We define Laplace transforms of \( u_b \) and \( v_b \) as

\[
U = \int_0^\infty e^{-st}u_b dt \quad \text{and} \quad V = \int_0^\infty e^{-st}v_b dt
\]  

(4.2.9)

Applying the Laplace transform to equations (4.2.4), (4.2.5) and to boundary conditions, then by using initial conditions one obtain the following differential equations:

\[
xU = \sum_{\beta=0}^{\infty} \frac{\alpha_\beta}{x + \beta} + \nu \left[ \frac{\partial^2 U}{\partial s^2} + \frac{\partial^2 U}{\partial n^2} + \frac{C_s U}{l} \right] + \frac{1}{\tau} (V - U) - DU = 0
\]  

(4.2.10)

\[
xV = \frac{1}{\tau} (U - V)
\]  

(4.2.11)

\[
U(0, n) = 0, \quad U(h_1, n) = 0,
\]  

(4.2.12)

\[
U(s, 0) = 0, \quad U(s, h_2) = 0
\]  

(4.2.13)

where \( l = \frac{mN}{\rho} \) and \( \tau = \frac{m}{k} \). Equation (4.2.11) implies

\[
V = \frac{U}{(1 - x\tau)}
\]  

(4.2.14)

Eliminating \( V \) from (4.2.10) and (4.2.14) we obtain the following equation

\[
\frac{\partial^2 U}{\partial s^2} + \frac{\partial^2 U}{\partial n^2} - Q^2 U + R = 0
\]  

(4.2.15)
where

$$Q^2 = \left( C_r + \frac{D}{\nu} + \frac{x}{\nu} + \frac{xL}{\nu(1 + xT)} \right) \quad \text{and} \quad R = \sum_{\beta=0}^{\infty} \frac{\alpha_\beta}{x + \beta}$$

To solve equation (4.2.15) we assume the solution in the following form \( [90] \)

$$U(s, n) = X(s, n) + Y(s) \quad (4.2.16)$$

Substitution of \( U(s, n) \) in equation (4.2.15) yields

$$\frac{\partial^2 X}{\partial s^2} + \frac{\partial^2 Y}{\partial s^2} + \frac{\partial^2 X}{\partial n^2} - Q^2(X + Y) + R = 0$$

so that if \( Y \) satisfies

$$\frac{\partial^2 Y}{\partial s^2} - Q^2Y + R = 0$$

then

$$\frac{\partial^2 X}{\partial s^2} + \frac{\partial^2 X}{\partial n^2} + Q^2X = 0 \quad (4.2.17)$$

In similar manner if \( U(s, n) \) is inserted in no slip boundary conditions, one can obtain

\[
\begin{cases}
U(0, n) = X(0, n) + Y(0) = 0, U(h_1, n) = X(h_1, n) + Y(h_1) = 0, \\
U(s, 0) = X(s, 0) + Y(s) = 0, U(s, h_2) = X(s, h_2) + Y(s) = 0
\end{cases}
\]

By solving the problem

$$\frac{\partial^2 Y}{\partial s^2} + Q^2Y + R = 0$$

$$Y(0) = 0, \quad Y(h_1) = 0$$

one can obtain the solution in the form

$$Y(s) = \frac{R}{Q^2} \left( \frac{\sinh(Q(s - h_1)) - \sinh(Qs)}{\sinh(Qh_1)} + 1 \right) \quad (4.2.18)$$
Using variable separable method, the solution of the problem (4.2.17) with the conditions

\[ X(0, n) = 0, \quad X(h_1, n)) = 0, \]
\[ X(s, 0) = -Y(s), \quad X(s, h_2) = -Y(s) \]
is obtained in the form

\[
X(s, n) = \frac{2h_1^2R}{\pi} \sum_{r_1=0}^{\infty} \frac{(1 - (-1)^{r_1})}{r_1^2\pi^2 + h_1^2Q^2}
\times \left( \frac{\sinh(En) - \sinh(E(n - h_2))}{\sinh(Eh_2)} \right) \sin \left( \frac{r_1\pi}{h_1} s \right)
\]

where \( E^2 = Q^2 + \frac{r_1^2\pi^2}{h_1^2} \).

Now by substituting (4.2.18) and (4.2.19) in (4.2.16) we have

\[
U(s, n) = \frac{2h_1^2R}{\pi} \sum_{r_1=0}^{\infty} \frac{(1 - (-1)^{r_1})}{r_1^2\pi^2 + h_1^2Q^2}
\times \left( \frac{\sinh(En) - \sinh(E(n - h_2))}{\sinh(Eh_2)} \right) \sin \left( \frac{r_1\pi}{h_1} s \right)
+ \frac{R}{Q^2} \left( \frac{\sinh(Q(s - h_1)) - \sinh(Qs)}{\sinh(Qh_1)} + 1 \right)
\]

Using \( U \) in equation (4.2.14) one can see that

\[
V(s, n) = \frac{2h_1^2R}{\pi(1 + x\tau)} \sum_{r_1=0}^{\infty} \frac{(1 - (-1)^{r_1})}{r_1^2\pi^2 + h_1^2Q^2}
\times \left( \frac{\sinh(En) - \sinh(E(n - h_2))}{\sinh(Eh_2)} \right) \sin \left( \frac{r_1\pi}{h_1} s \right)
+ \frac{R}{Q^2(1 + x\tau)} \left( \frac{\sinh(Q(s - h_1)) - \sinh(Qs)}{\sinh(Qh_1)} + 1 \right)
\]
By applying Inverse Laplace transform to $U$ and $V$ we obtain the following relations:

$$u_b(s,n,t) = \frac{2h_1^2}{\pi} \sum_{\beta=0}^{\infty} \sum_{r_1=0}^{\infty} \frac{\alpha_{\beta} e^{-\beta t}}{r_1^2 \pi^2 + h_1^2 A^2} \left(1 - (-1)^{r_1}\right)$$

$$\times \left(\frac{\sinh(Fn) - \sinh(F(n - h_2))}{\sinh(Fh_2)}\right) \sin \left(\frac{r_1 \pi}{h_1} s\right)$$

$$\sum_{\beta=0}^{\infty} \alpha_{\beta} e^{-\beta t} \left[\frac{\sinh(A(s - h_1)) - \sinh(As)}{\sinh(Ah_1)} + 1\right]$$

$$+ \frac{2h_1^2}{\pi} \sum_{\beta=0}^{\infty} \sum_{r_1=0}^{\infty} \frac{\alpha_{\beta} (1 - (-1)^{r_1})}{\sinh(Fn)} \sin \left(\frac{r_1 \pi}{h_1} s\right) \sin \left(\frac{r_2 \pi}{h_2} n\right)$$

$$\times \left[\frac{e^{\nu t} (1 + x_1 \tau)^2}{(x_1 + \beta)[l + (1 + x_1 \tau)^2]} + \frac{e^{\nu t} (1 + x_2 \tau)^2}{(x_2 + \beta)[l + (1 + x_2 \tau)^2]}\right]$$

$$+ \frac{2\nu h_2}{\pi} \sum_{\beta=0}^{\infty} \sum_{r_1=0}^{\infty} \alpha_{\beta} (1 - (-1)^{r_1}) \sin \left(\frac{r_1 \pi}{h_1} s\right)$$

$$\times \left[\frac{e^{\nu t} (1 + y_1 \tau)^2}{(y_1 + \beta)[l + (1 + y_1 \tau)^2]} + \frac{e^{\nu t} (1 + y_2 \tau)^2}{(y_2 + \beta)[l + (1 + y_2 \tau)^2]}\right]$$

$$v_b(s,n,t) = \frac{2h_1^2}{\pi} \sum_{\beta=0}^{\infty} \sum_{r_1=0}^{\infty} \frac{\alpha_{\beta} e^{-\beta t}}{r^2 \pi^2 + h_1^2 A^2} \left(1 - (-1)^{r_1}\right)$$

$$\times \left(\frac{\sinh(Fn) - \sinh(F(n - h_2))}{\sinh(Fh_2)}\right) \sin \left(\frac{r_1 \pi}{h_1} s\right)$$
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\[ + \sum_{\beta=0}^{\beta} \frac{e^{-\beta t}}{1 - \beta \tau} \left[ \frac{\sinh(A(s - h_1)) - \sinh(As)}{\sinh(Ah_1)} + 1 \right] \]

\[ + \frac{2h_1^2}{\pi} \sum_{\beta=0}^{\beta} \sum_{r_1=0}^{r_1} \frac{\alpha_\beta (1 - (-1)^{r_1})}{y_1 - y_2} \sin \left( \frac{r_1 \pi}{h_1} s \right) \]

\[ \times \left[ \frac{e^{y_1t}}{(y_1 + \beta)(1 - y_1 \tau)} - \frac{e^{y_2t}}{(y_2 + \beta)(1 - y_2 \tau)} \right] \]

\[ + \frac{4\nu h_2}{\pi} \sum_{\beta=0}^{\beta} \sum_{r_1=0}^{r_1} \sum_{r_2=0}^{r_2} \frac{\alpha_\beta (1 - (-1)^{r_2})}{r_2} \sin \left( \frac{r_1 \pi}{h_1} s \right) \sin \left( \frac{r_2 \pi}{h_2} s \right) \]

\[ \times \left[ \frac{e^{\gamma_1t}(1 + x_1 \tau)}{(x_1 + \beta)[l + (1 + x_1 \tau)^2]} + \frac{e^{\gamma_2t}(1 + x_2 \tau)}{(x_2 + \beta)[l + (1 + x_2 \tau)^2]} \right] \]

\[ + \frac{2\nu}{\pi} \sum_{\beta=0}^{\beta} \sum_{r_1=0}^{r_1} \frac{\alpha_\beta}{r_1} (1 - (-1)^{r_1}) \sin \left( \frac{r_1 \pi}{h_1} s \right) \]

\[ \times \left[ \frac{e^{\gamma_1t}(1 + y_1 \tau)}{(y_1 + \beta)[l + (1 + y_1 \tau)^2]} + \frac{e^{\gamma_2t}(1 + y_2 \tau)}{(y_2 + \beta)[l + (1 + y_2 \tau)^2]} \right] \]

4.2.3 Shearing Stress (Skin Friction):

The Shear stress at the boundaries \( s = 0, \ s = h_1 \) and \( n = 0, \ n = h_2 \) are given by

\[ D_{0 \ n} = 2h_1 \sum_{\beta=0}^{\beta} \sum_{r_1=0}^{r_1} \frac{r_1 \alpha_\beta e^{-\beta t}}{\tau^2 \pi^2 + h_1^2 A^2} (1 - (-1)^{r_1}) \]

\[ \times \left( \frac{\sinh(Fn) - \sinh(F(n - h_2))}{\sinh(Fh_2)} \right) + \sum_{\beta=0}^{\beta} A \alpha_\beta e^{-\beta t} \left[ \frac{\cosh(Ah_1) - 1}{\sinh(Ah_1)} \right] \]
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\[ + 2h_1 \sum_{\beta=0}^{\infty} \sum_{r_1=0}^{\infty} \alpha_\beta r_1 (1 - (-1)^{r_1}) \left[ \frac{e^{2 \nu t}}{y_1 + \beta} - \frac{e^{2 \nu t}}{y_2 + \beta} \right] \]

\[ + \frac{4 \nu h_2}{h_1} \sum_{\beta=0}^{\infty} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \alpha_\beta r_1 (1 - (-1)^{r_1})(1 - (-1)^{r_2}) \sin \left( \frac{r_2 \pi}{h_2} \right) \sinh \left( \frac{r_1 \pi}{h_1} \right) \]

\[ \times \left[ \frac{e^{2 \nu t}(1 + x_1 \tau)^2}{(x_1 + \beta)[l + (1 + x_1 \tau)^2]} + \frac{e^{2 \nu t}(1 + x_2 \tau)^2}{(x_2 + \beta)[l + (1 + x_2 \tau)^2]} \right] \]

\[ + \frac{2 \nu}{h_1} \sum_{\beta=0}^{\infty} \alpha_\beta (1 - (-1)^{r_1}) \]

\[ \times \left[ \frac{e^{2 \nu t}(1 + y_1 \tau)^2}{(y_1 + \beta)[l + (1 + y_1 \tau)^2]} + \frac{e^{2 \nu t}(1 + y_2 \tau)^2}{(y_2 + \beta)[l + (1 + y_2 \tau)^2]} \right] \]

\[ D_{h_1 \ n} = -D_{0 \ n} \]

\[ D_{s \ 0} = 2h_1 \sum_{\beta=0}^{\infty} \sum_{r_1=0}^{\infty} \frac{r_1 \alpha_\beta e^{-\beta t}}{r_1^2 \pi^2 + h_1^2 A^2} (1 - (-1)^{r_1}) \cos \left( \frac{r_1 \pi}{h_1} s \right) \]

\[ + \sum_{\beta=0}^{\infty} A \left( \frac{\cosh(A(s - h_2)) - \cosh(As)}{\sinh(Ah_1)} \right) \]

\[ + 2h_1 \sum_{\beta=0}^{\infty} \sum_{r_1=0}^{\infty} \alpha_\beta r_1 (1 - (-1)^{r_1}) \cos \left( \frac{r_1 \pi}{h_1} s \right) \left[ \frac{e^{2 \nu t}}{y_1 + \beta} - \frac{e^{2 \nu t}}{y_2 + \beta} \right] \]

\[ + 4 \nu \sum_{\beta=0}^{\infty} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \alpha_\beta (1 - (-1)^{r_1})(1 - (-1)^{r_2}) \sin \left( \frac{r_1 \pi}{h_1} s \right) \]

\[ \times \left[ \frac{e^{2 \nu t}(1 + x_1 \tau)^2}{(x_1 + \beta)[l + (1 + x_1 \tau)^2]} + \frac{e^{2 \nu t}(1 + x_2 \tau)^2}{(x_2 + \beta)[l + (1 + x_2 \tau)^2]} \right] \]
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\[ + \frac{2\nu}{h_1} \sum_{\beta=0}^{\infty} \sum_{r_1=0}^{\infty} \alpha_{\beta}(1 - (-1)^{r_1}) \cos \left( \frac{r_1\pi}{h_1} \right) \]

\[ \times \left[ \frac{e^{\nu t}(1 + y_1\tau)^2}{(y_1 + \beta)(1 + (1 + y_1\tau)^2)} + \frac{e^{\nu t}(1 + y_2\tau)^2}{(y_2 + \beta)(1 + (1 + y_2\tau)^2)} \right] \]

\[ D_s \Delta t = D_s h_2 \]

where

\[ y_1 = -\frac{1}{2\tau} \left( 1 + l + D\tau + \nu C_r\tau + \nu \tau \frac{r_1^2\pi^2}{h_1^2} \right) \]

\[ + \sqrt{(1 + l + D\tau + \nu C_r\tau + \nu \tau \frac{r_1^2\pi^2}{h_1^2})^2 - 4\nu \tau (D + C_r + \nu \tau \frac{r_1^2\pi^2}{h_1^2})} \]

\[ y_2 = -\frac{1}{2\tau} \left( 1 + l + D\tau + \nu C_r\tau + \nu \tau \frac{r_1^2\pi^2}{h_1^2} \right) \]

\[ - \sqrt{(1 + l + D\tau + \nu C_r\tau + \nu \tau \frac{r_1^2\pi^2}{h_1^2})^2 - 4\nu \tau (D + C_r + \nu \tau \frac{r_1^2\pi^2}{h_1^2})} \]

\[ x_1 = -\frac{1}{2\tau} (1 + l + D\tau + \nu C_r\tau + G) \]

\[ + \frac{1}{2\tau} \sqrt{(1 + l + D\tau + \nu C_r\tau + G)^2 - 4\nu \tau (C_r + D + G)} \]

\[ x_2 = -\frac{1}{2\tau} (1 + l + D\tau + \nu C_r\tau + G) \]

\[ - \frac{1}{2\tau} \sqrt{(1 + l + D\tau + \nu C_r\tau + G)^2 - 4\nu \tau (C_r + D + G)} \]

\[ A = \left( C_r + \frac{D}{\nu} - \frac{\beta}{\nu} - \frac{\beta l}{\nu(1 - \beta\tau)} \right), \quad F = A^2 + \frac{r_1^2\pi}{h_1}, \quad G = \nu \tau \pi^2 \left( \frac{r_1^2}{h_1^2} + \frac{r_2^2}{h_2^2} \right) \]
4.3 Section B

4.3.1 Formulation of the Problem

In this section we consider the flow through a circular cylinder of radius $a$ and the remaining assumptions are same as in section A. As Figure-4.3.1 shows, the axis of the channel is along $z-$axis and the velocity components of both fluid and dust particles are respectively given by:

$$\begin{align*}
u_r &= 0; \quad \nu_\theta = 0; \quad \nu_z = \nu_z(r,t); \\
v_r &= 0; \quad \nu_\theta = 0; \quad \nu_z = \nu_z(r,t)
\end{align*}$$

where $(\nu_r, \nu_\theta, \nu_z)$ and $(v_r, v_\theta, v_z)$ are velocity components of fluid and dust particles respectively.

![Figure 4.3.1: Schematic diagram of dusty fluid flow in a Circular Cylinder.](image)

By virtue of equation (4.3.1) the intrinsic decomposition of equations (2.2.2) and (2.2.4) in cylindrical polar coordinates give the following forms:

$$\begin{align*}
-\frac{1}{\rho} \frac{\partial p}{\partial r} &= 0, \quad (4.3.2) \\
\frac{\partial \nu_z}{\partial t} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left[ \frac{\partial^2 \nu_z}{\partial r^2} + \frac{1}{r} \frac{\partial \nu_z}{\partial r} \right] + \frac{kN}{\rho} (v_z - \nu_z) - \frac{\sigma B_0^2}{\rho} \nu_z, \quad (4.3.3) \\
\frac{\partial v_z}{\partial t} &= -\frac{k}{m} (u_z - v_z), \quad (4.3.4)
\end{align*}$$
Let us introduce the following non-dimensional quantities:

\[ R = \frac{r}{a}, \quad z = \frac{z}{a}, \quad \bar{p} = \frac{pa^2}{\rho a^2}, \quad T = \frac{t}{a^2}, \quad u = \frac{u}{a}, \quad v = \frac{v}{a} \] (4.3.5)

\[ \beta = \frac{l}{\gamma} = \frac{Nka^2}{\rho a}, \quad l = \frac{Nm}{\rho}, \quad \gamma = \frac{m}{ka^2}. \]

Transform the equations (4.3.3) to (4.3.4) to the non-dimensional forms as

\[ -\frac{\nu^2}{\rho} \frac{\partial p}{\partial R} = 0, \] (4.3.6)

\[ \frac{\partial u}{\partial T} = -\frac{\partial p}{\partial z} + \nu \left[ \frac{\partial^2 u}{\partial R^2} + \frac{1}{R} \frac{\partial u}{\partial R} \right] + \beta (v - u) - M^2 u, \] (4.3.7)

\[ \gamma \frac{\partial v}{\partial T} = (u - v). \] (4.3.8)

where \( M = B_0 a \sqrt{\sigma/\mu} \) = the Hartmann number.

Since we have assumed that the time dependent pressure gradient to be impressed on the system for \( t > 0 \), so we can write

\[ -\frac{1}{\rho} \frac{\partial p}{\partial z} = c + de^{\alpha t} \]

where \( c, \ d \) and \( \alpha \) are reals.

Eliminating \( v \) from (4.3.7) and (4.3.8) and then substituting the expression for pressure gradient, one can get

\[ \gamma \frac{\partial^2 u}{\partial T^2} + (l + 1) \frac{\partial u}{\partial T} - \gamma \frac{\partial}{\partial T} \left[ \frac{\partial^2 u}{\partial R^2} + \frac{1}{R} \frac{\partial u}{\partial R} \right] \]

\[ = c + de^{\alpha t} - \left[ \frac{\partial^2 u}{\partial R^2} + \frac{1}{R} \frac{\partial u}{\partial R} \right] - M^2 u, \] (4.3.9)
4.3.2 Solution of the Problem

Let the solution of the equation (4.3.9) in the form [90]

\[ u = U(R) + V(R, T) \]  \hspace{1cm} (4.3.10)

where \( U \) is the steady part and \( V \) is the unsteady part of the fluid velocity.

Separating the equation (4.3.9) into a steady part and an unsteady part as

\[ \frac{\partial^2 U}{\partial R^2} + \frac{1}{R} \frac{\partial U}{\partial R} + M^2 U = c, \]  \hspace{1cm} (4.3.11)

\[ \gamma \frac{\partial^2 V}{\partial T^2} + (l + 1) \frac{\partial V}{\partial T} - \gamma \frac{\partial}{\partial T} \left[ \frac{\partial^2 V}{\partial R^2} + \frac{1}{R} \frac{\partial V}{\partial R} \right] \]

\[ = d e^{at} + \left[ \frac{\partial^2 V}{\partial R^2} + \frac{1}{R} \frac{\partial V}{\partial R} \right] + M^2 V, \]  \hspace{1cm} (4.3.12)

Consider the equation (4.3.11) with the following boundary conditions

\[ U = 0, \hspace{0.5cm} \text{at} \hspace{0.5cm} R = 1, \]

\[ U = \text{finite, at} \hspace{0.5cm} R = 0. \]

Now, by solving equation (4.3.11) with above boundary conditions, one can get

\[ U = \frac{c}{M^2} \left( \frac{J_0(MR)}{J_0(M)} - 1 \right) \]  \hspace{1cm} (4.3.13)

where \( J_0 \) is Bessel's function of zeroth order.
Now consider the equation (4.3.12) with the following boundary conditions

**Initial condition:** \( U = V \) at \( T = 0 \),

**Boundary condition:** \( V = 0 \) at \( R = 1 \),
\[ V = 0, \quad \text{at} \quad R = 0. \]  
(4.3.14)

Assume the solution of the equation (4.3.12) is in the form

\[ V = g(R)e^{\alpha T}, \]  
(4.3.15)

where \( g(R) \) is an unknown function to be determined.

Using equation (4.3.15) in (4.3.12) one can obtain

\[ \frac{\partial^2 g}{\partial R^2} + \frac{1}{R} \frac{\partial g}{\partial R} + \lambda_1^2 g = \lambda_2, \]  
(4.3.16)

where \( \lambda_1 = \frac{(M^2 - \gamma^2) - \alpha(1 + \gamma)}{(1 + \alpha \gamma)} \) and \( \lambda_2 = \frac{\alpha}{(1 + \alpha \gamma)} \).

One can obtain the solution of (4.3.16)

\[ g(R) = A J_0(\lambda_1 R) + B K_0(\lambda_1 R) - \frac{\lambda_2}{\lambda_1^2} \]  
(4.3.17)

where \( A, B \) are constants. \( J_0 \) and \( K_0 \) are Bessel's function of first and second kind order zero respectively.

Since the fluid velocity is finite at the center of the circular tube, we have \( B = 0 \).

Hence using the boundary conditions (4.3.14) we get

\[ g(R) = \frac{\lambda_2}{\lambda_1^2} \left[ \frac{J_0(\lambda_1 R)}{J_0(\lambda_1)} - 1 \right] \]  
(4.3.18)

Using this in (4.3.15) we get \( V \) as

\[ V = e^{\alpha T} \frac{\lambda_2}{\lambda_1^2} \left[ \frac{J_0(\lambda_1 R)}{J_0(\lambda_1)} - 1 \right] \]  
(4.3.19)
Now, using equations (4.3.19) and (4.3.13) in (4.3.10) one can obtain the fluid velocity \( u \) as

\[
    u = \frac{c}{M^2} \left[ \frac{J_0(MR)}{J_0(M)} - 1 \right] + e^{\alpha \tau} \frac{\lambda_2}{\lambda_1^2} \left[ \frac{J_0(\lambda_1 R)}{J_0(\lambda_1)} - 1 \right] \quad (4.3.20)
\]

Also, the dust phase velocity is obtained from equation (4.3.8) as

\[
    v = \frac{c}{M^2} \left[ \frac{J_0(MR)}{J_0(M)} - 1 \right] \left[ 1 - e^{-\frac{1}{\lambda^2}} \right] + e^{\alpha \tau} \frac{\lambda_2}{\lambda_1^2} \left[ \frac{J_0(\lambda_1 R)}{J_0(\lambda_1)} - 1 \right] \left[ e^{\alpha \tau} - e^{-\frac{1}{\lambda^2}} \right] \quad (4.3.21)
\]

### 4.3.3 Shearing Stress (Skin Friction): 

The Shear stress at the boundary \( R = 1 \) is given by

\[
    D_1 = \frac{\mu c}{M} \left[ \frac{J'_0(M)}{J_0(M)} \right] + e^{\alpha \tau} \frac{\mu \lambda_2}{\lambda_1} \frac{J'_0(\lambda_1)}{J_0(\lambda_1)}
\]

From property of Bessel's function we know that \( J'_0 = -J_1 \). Hence The above equation becomes

\[
    D_1 = -\mu \left[ \frac{c}{M} \frac{J_1(M)}{J_0(M)} + e^{\alpha \tau} \frac{\lambda_2}{\lambda_1} \frac{J_1(\lambda_1)}{J_0(\lambda_1)} \right]
\]

### 4.4 Section C 

#### 4.4.1 Formulation of the Problem 

This section includes the study of flow in between two circular cylinders. The inner cylinder is of unit radius (i.e., 1) and outer cylinder is of radius \( b \) under the assumptions
same as in the above sections. In addition to these it is assumed that the flow is due to the differential rotations of the cylinders. Figure-4.4.1 describes the flow pattern.

![Diagram of dusty fluid flow between two Circular Cylinder](image)

Figure 4.4.1: Schematic diagram of dusty fluid flow between two Circular Cylinder.

### 4.4.2 Solution of the Problem

Solution in this section is obtained in three cases

**Case-4.4.1: Periodic Motion:**

For this case, we consider the boundary conditions as

\[
  u = u_1 \sin(\alpha T), \quad \text{at} \quad R = 1,
\]

\[
  u = u_2 \sin(\alpha T), \quad \text{at} \quad R = b.
\]

where \( u_1 \) and \( u_2 \) are uniform angular velocities.

Now, one can get the solutions for both fluid and dust phase velocities follows using the steps as in section B
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\[
\begin{align*}
    u &= \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) - T_2 K_0(MR)}{T_0} + 1 \right] \\
    &\quad + \frac{\lambda_2 e^{\alpha T}}{\lambda_1^2} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_2 K_0(\lambda_1 R)}{Q_0} - 1 \right] \\
    &\quad + e^{\alpha T} \sin(\alpha T) \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_2 K_0(\lambda_1 R)}{Q_0} \right] \\
    v &= \frac{c\gamma}{M^2} \left[ \frac{T_1 J_0(MR) - T_2 K_0(MR)}{T_0} + 1 \right] \\
    &\quad + \frac{\gamma \lambda_2 e^{\alpha T}}{\lambda_1^2 (1 + \alpha \gamma)} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] \\
    &\quad + \frac{e^{\alpha T}}{(\alpha \gamma)^2 + (1 + \alpha \gamma)^2} \left[ (1 + \alpha \gamma) \sin(\alpha T) - \alpha \gamma \cos(\alpha T) \right] \\
    &\quad \times \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right] + Ae^{-\frac{1}{4}T} \\
    \text{where} \\
    A &= \left[ \frac{\alpha \gamma}{(\alpha \gamma)^2 + (1 + \alpha \gamma)^2} \right] \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right] \\
    &\quad - \frac{\gamma \lambda_2}{\lambda_1^2 (1 + \alpha \gamma)} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] \\
    &\quad - \frac{c\gamma}{M^2} \left[ \frac{T_1 J_0(MR) - T_2 K_0(MR)}{T_0} + 1 \right]
\end{align*}
\]
4.4.3 Shearing Stress (Skin Friction):

The Shear stress at the boundary $R = 1$ and $R = b$ are respectively given by

$$D_1 = \frac{\mu c}{M} \left[ \frac{T_1 J'_0(M) - T_2 K'_0(M)}{T_0} \right]$$

$$+ e^{\alpha T} \frac{\mu \lambda_2}{\lambda_1} \left[ \frac{Q_1 J'_0(\lambda_1) + Q_3 K'_0(\lambda_1)}{Q_0} \right]$$

$$+ \mu \lambda_1 \sin(\alpha T) e^{\alpha T} \left[ \frac{Q_2 J'_0(\lambda_1) + Q_4 K'_0(\lambda_1)}{Q_0} \right]$$

$$D_b = \frac{\mu c}{M} \left[ \frac{T_1 J'_0(Mb) - T_2 K'_0(Mb)}{T_0} \right]$$

$$+ e^{\alpha T} \frac{\mu \lambda_2}{\lambda_1} \left[ \frac{Q_1 J'_0(\lambda_1 b) + Q_3 K'_0(\lambda_1 b)}{Q_0} \right]$$

$$+ \mu \lambda_1 \sin(\alpha T) e^{\alpha T} \left[ \frac{Q_2 J'_0(\lambda_1 b) + Q_4 K'_0(\lambda_1 b)}{Q_0} \right]$$

Case-4.4.2: Transition Motion:

For transition motion, we consider the boundary conditions as

$$u = u_1 H(T) e^{\alpha T}, \text{ at } R = 1,$$

$$u = u_2 H(T) e^{\alpha T}, \text{ at } R = b.$$

where $H(T)$ is the Heaviside’s unit step function.

Using these boundary conditions one can get the solutions for velocities of fluid and dust phase as
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\[ u = \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) - T_2 K_0(MR)}{T_0} + 1 \right] \]

\[ + \frac{\lambda_2 e^{\alpha T}}{\lambda_1^2} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] \]

\[ + H(T) e^{2\alpha T} \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right] \]

\[ v = \frac{c\gamma}{M^2} \left[ \frac{T_1 J_0(MR) - T_2 K_0(MR)}{T_0} + 1 \right] \]

\[ + \frac{\gamma \lambda_2 e^{\alpha T}}{\lambda_1^2(1 + \alpha \gamma)} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] \]

\[ + \left[ \frac{H(T) \gamma}{(1 + 2\alpha \gamma)} (e^{2\alpha T} - 1) \right] \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right] + A_1 e^{-\frac{1}{2}T} \]

where

\[ A_1 = - \frac{\gamma \lambda_2}{\lambda_1^2(1 + \alpha \gamma)} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] \]

\[ - \frac{c\gamma}{M^2} \left[ \frac{T_1 J_0(MR) - T_2 K_0(MR)}{T_0} + 1 \right] \]

4.4.4 Shearing Stress (Skin Friction):

The Shear stress i.e., the skin friction at the boundary \( R = 1 \) and \( R = b \) are respectively given by
\[
D_1 = \frac{\mu c}{M} \left[ \frac{T_1 J_0'(M) - T_2 K_0'(M)}{T_0} \right] \\
+ e^{\alpha T} \frac{\mu \lambda_2}{\lambda_1} \left[ \frac{Q_1 J_0'(\lambda_1) + Q_3 K_0'(\lambda_1)}{Q_0} \right] \\
+ \mu \lambda_1 H(T) e^{2 \alpha T} \left[ \frac{Q_2 J_0'(\lambda_1) + Q_4 K_0'(\lambda_1)}{Q_0} \right] \\
\]

\[
D_6 = \frac{\mu c}{M} \left[ \frac{T_6 J_0'(Mb) - T_2 K_0'(Mb)}{T_0} \right] \\
+ e^{\alpha T} \frac{\mu \lambda_2}{\lambda_1} \left[ \frac{Q_1 J_0'(\lambda_1b) + Q_3 K_0'(\lambda_1b)}{Q_0} \right] \\
+ \mu \lambda_1 H(T) e^{2 \alpha T} \left[ \frac{Q_2 J_0'(\lambda_1b) + Q_4 K_0'(\lambda_1b)}{Q_0} \right] \\
\]

**Case-4.4.3: Particular Case:**

For this case consider the boundary conditions as

\[ u = 0, \text{ at } R = 1, \]
\[ u = 0, \text{ at } R = b. \]

Now, with these boundary conditions we get the expression for both fluid and dust velocities as
\[ u = \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) - T_2 K_0(MR)}{T_0} + 1 \right] \]

\[ + \frac{\lambda_2 e^{\alpha T}}{\lambda_1^2} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] \]

\[ v = \frac{c \gamma}{M^2} \left[ \frac{T_1 J_0(MR) - T_2 K_0(MR)}{T_0} + 1 \right] \]

\[ + \frac{\gamma \lambda_2 e^{\alpha T}}{\lambda_1^2 (1 + \alpha \gamma)} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] + A_2 e^{-\frac{1}{\gamma} T} \]

where

\[ A_2 = -\frac{\gamma \lambda_2}{\lambda_1^2 (1 + \alpha \gamma)} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] \]

\[ - \frac{c \gamma}{M^2} \left[ \frac{T_1 J_0(MR) - T_2 K_0(MR)}{T_0} + 1 \right] \]

### 4.4.5 Shearing Stress (Skin Friction):

The Shear stress i.e., the skin friction at the boundary \( R = 1 \) and \( R = b \) are respectively given by

\[ D_1 = \frac{\mu c}{M} \left[ \frac{T_1 J_0'(M) - T_2 K_0'(M)}{T_0} \right] \]

\[ + \frac{e^{\alpha T} \mu \lambda_2}{\lambda_1} \left[ \frac{Q_1 J_0'(\lambda_1) + Q_3 K_0'(\lambda_1)}{Q_0} \right] \]
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\[ D_b = \frac{\mu c}{M} \left[ \frac{T_1 J'_0(Mb) - T_2 K'_0(Mb)}{T_0} \right] \]

\[ + e^{\alpha T} \frac{\mu \lambda_2}{\lambda_1} \left[ \frac{Q_1 J'_0(\lambda_1 b) + Q_3 K'_0(\lambda_1 b)}{Q_0} \right] \]

where

\[ T_0 = J_0(M) K_0(Mb) - J_0(Mb) K_0(M), \quad T_1 = K_0(M) - K_0(Mb), \]

\[ T_2 = J_0(M) - J_0(Mb), \quad Q_0 = J_0(\lambda_1) K_0(\lambda_1 b) - J_0(\lambda_1 b) K_0(\lambda_1), \]

\[ Q_1 = K_0(\lambda_1 b) - K_0(\lambda_1), \quad Q_2 = u_1 K_0(\lambda_1 b) - u_2 K_0(\lambda_1) \]

\[ Q_3 = J_0(\lambda_1) - J_0(\lambda_1 b), \quad Q_4 = u_2 J_0(\lambda_1) - u_1 J_0(\lambda_1 b) \]

4.5 Conclusion

From the graphs of all the sections one can see the parabolic nature of velocity profiles for fluid and dust particles. From these it is observed that the velocity of fluid particles is parallel to velocity of dust particles. Also it is evident from the graphs that, as we increase the strength of the magnetic field it reduces the velocities of fluid and dust particles. It means that it has an appreciable effect on the velocities of the both the phases. Further one can see that if the magnetic field is zero then results are in agreement with the Couette flow. The velocity is symmetrical with the center of the channel, also, if the dust is very fine i.e., mass of the dust particles is negligibly small then the relaxation
time of dust particles decreases and ultimately as $\tau \to 0$ the velocities of fluid and dust particles will be the same. Finally, we see that the fluid particles will reach the steady state earlier than the dust particles.

As a particular case as we observed if $\beta = 0$ and magnetic field is zero then the results coincides with the those of [29] i.e., the flow with constant pressure gradient.

![Figure 4.2.2: Variation of fluid velocity with $s \& n$ for $D = 0.2$. (Section A)](image-url)
Figure 4.2.3: Variation of fluid velocity with \( s \& n \) for \( D = 0.4 \). (Section A)

Figure 4.2.4: Variation of dust velocity with \( s \& n \) for \( D = 0.2 \). (Section A)
Figure 4.2.5: Variation of dust velocity with \( s \) & \( n \) for \( D = 0.4 \). (Section A)

Figure 4.3.2: Variation of fluid velocity with \( R \) (Section B)
Figure 4.3.3: Variation of dust phase velocity with $R$ (Section B)

Figure 4.4.2: Variation of fluid velocity with $R$ for Case-4.4.1 (Section C)
Figure 4.4.3: Variation of dust phase velocity with $R$ for Case-4.4.1 (Section C)

Figure 4.4.4: Variation of fluid velocity with $R$ for Case-4.4.2 (Section C)
Figure 4.4.5: Variation of dust phase velocity with $R$ for Case-4.4.2 (Section C)

Figure 4.4.6: Variation of fluid velocity with $R$ for Case-4.4.3 (Section C)
Figure 4.4.7: Variation of dust phase velocity with $R$ for Case-4.4.3 (Section C)