CHAPTER 3

Publications based on this Chapter


• *Special Finsler hypersurfaces admitting concurrent vector field*, Communicated.

• *C-conformal special Finsler hypersurface $F^{n-1}$ admitting concurrent vector field*, Communicated.
Chapter 3

Finsler Spaces admitting a Conformal Vector field

3.1 Introduction

In 1985, M. Matsumoto [27] studied the theory of Finslerian hypersurfaces. The theory of special Finsler spaces like quasi-C-reducible, C-reducible, Semi-C-reducible, P2-like, P-reducible, S3-like, C2-like and T-conditions of Finsler spaces and their properties were studied by M. Matsumoto [28] and C. Shibata [41].

The author M. Kitayama ([15], [16], [18]) have studied Finsler spaces admitting a parallel vector field and also studied Finslerian hypersurface and metric transformations. The concept of concurrent vector field on a Riemannian manifold had been studied by S. Sasaki and K. Yano ([39], [45]). The author T. Tachibana [44] has studied the concurrent vector field in Finsler spaces. Further the authors M. Matsumoto and K. Eguchi [33] have discussed the concurrent vector field in Finsler spaces.

The conformal theory and its related concepts of Finsler spaces was initiated by M.S. Knebelman in 1929. M. Hashiguchi [5] has introduced a special change named C-conformal change, which is a non-homothetic conformal change. The authors C. Shibata and H. Azuma [41] have studied C-conformal invariant tensors of Finsler metric.
In this chapter, we envisage to study the properties of special conformal Finsler hypersurfaces which are admitting a conformal parallel vector field $X^\alpha$ is defined on $F^{n-1}$. Also we have studied the properties of special Finsler hypersurface admitting concurrent vector field and C-conformal special Finsler hypersurface admitting concurrent vector field.

### 3.2 Special Finsler subspace admitting a conformal parallel vector field

Conformal Finsler subspace $F^m = (M^m, L(u, v))$ of a Finsler subspace $F^m = (M^m, L(u, v))$ may be parametrically represented by the equation $\bar{x}^i = \bar{x}^i(u^\alpha)$ where $i = 1, 2, ..., n$ and $\alpha = 1, 2, ..., m$. We use the following notions on conformal Finsler subspace:

\begin{align*}
a) \quad & \bar{g}_{\alpha\beta} = e^{2\sigma} g_{\alpha\beta}, \quad \bar{g}^{\alpha\beta} = e^{-2\sigma} g^{\alpha\beta}, \\
b) \quad & \bar{C}^\alpha_{\beta\gamma} = C^\alpha_{\beta\gamma}, \quad \bar{C}_{\alpha\beta\gamma} = e^{2\sigma} C_{\alpha\beta\gamma}, \quad \bar{C}_\alpha = C_\alpha, \\
c) \quad & \bar{l}^\alpha = e^{-\sigma} l^\alpha, \quad \bar{l}_\alpha = e^\sigma l_\alpha, \\
d) \quad & \bar{g}^\alpha = e^{2\sigma} g^\alpha, \\
e) \quad & \bar{h}_{\alpha\beta} = e^{2\sigma} h_{\alpha\beta}, \\
f) \quad & \bar{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + g_{\beta\gamma} \sigma^\alpha + \sigma_\gamma \delta^\alpha_\beta + \sigma_\beta \delta^\alpha_\gamma, \\
g) \quad & \bar{X}^\alpha = e^{-\sigma} X^\alpha, \quad \bar{X}_\alpha = e^\sigma X_\alpha, \\
h) \quad & \bar{N}^\alpha_\beta = N^\alpha_\beta - y_\beta \sigma^\alpha + \sigma_0 \delta^\alpha_\beta + \sigma_\beta y^\alpha.
\end{align*}

**Definition 1.** A vector field $X^\alpha = X^i B^\alpha_i$ is called parallel, if it satisfies the partial differential equations [15]

\begin{align*}
X^\alpha_{/\beta} &= \partial_\beta X^\alpha - N^\rho_\beta \partial_\rho X^\alpha + X^\rho \Gamma^\alpha_\rho = \partial_\beta X^\alpha + X^\rho \Gamma^\alpha_\rho = 0, \quad (3.2.2) \\
X^\alpha_{/\beta} &= \partial_\beta X^\alpha + X^\rho C^\alpha_\rho = X^\rho C^\alpha_\rho = 0, \quad (3.2.3)
\end{align*}
where $\partial_\beta$ and $\partial^\beta$ denote partial differentiations by $u^\beta$ and $\dot{u}^\beta$ respectively.

Under the conformal change the equation (3.2.2), can be written as

$$X^\alpha_{\beta} = \partial_\beta X^\alpha - N^\rho\partial_\rho X^\alpha + X^\rho \Gamma^\alpha_{\rho\beta}. \quad (3.2.4)$$

Using equations (3.2.1)(b,f,g,h), the equation (3.2.4), can be written as

$$X^\alpha_{\beta} = \partial_\beta (e^{-s} X^\alpha) + e^{-s} X^\rho (\Gamma^\alpha_{\rho\beta} + g_{\beta\rho} \sigma^\alpha + \delta^\alpha_{\beta} \sigma_\rho + \delta^\rho_{\beta} \sigma_\alpha),$$

$$X^\alpha_{\beta} = e^{-s} \partial_\beta X^\alpha + e^{-s} X^\rho \Gamma^\alpha_{\rho\beta} + e^{-s} X^\rho g_{\beta\rho} \sigma^\alpha + e^{-s} X^\rho \sigma_\rho \delta^\alpha_{\beta} + e^{-s} X^\rho \delta^\alpha_{\rho} \sigma_\beta, $$

$$X^\alpha_{\beta} = e^{-s} (\partial_\beta X^\alpha + X^\rho \Gamma^\alpha_{\rho\beta}) + e^{-s} (X^\rho \delta^\alpha_{\beta} \sigma_\rho + X^\rho \delta^\rho_{\beta} \sigma_\alpha + X^\rho g_{\beta\rho} \sigma^\alpha).$$

By virtue of equation (3.2.2), we get

$$X^\alpha_{\beta} = e^{-s} X^\rho (\partial^\rho_{\beta} \sigma_\rho + \delta^\rho_{\beta} \sigma_\alpha + g_{\beta\rho} \sigma^\alpha).$$

Therefore

$$X^\alpha_{\beta} = \partial_\beta X^\alpha + X^\rho \Gamma^\alpha_{\rho\beta} = 0, \quad (3.2.5)$$

if $X^\rho (\partial^\rho_{\beta} \sigma_\rho + \delta^\rho_{\beta} \sigma_\alpha + g_{\beta\rho} \sigma^\alpha) = 0$.

Now we consider equation (3.2.3), under conformal change it can be written as

$$X^\alpha_{\beta} = \partial_\beta X^\alpha + X^\rho C^\alpha_{\rho\beta},$$

$$X^\alpha_{\beta} = \dot{\partial}_\beta (e^{-s} X^\alpha) + X^\rho C^\alpha_{\rho\beta},$$

$$X^\alpha_{\beta} = X^\rho C^\alpha_{\rho\beta} = 0. \quad (3.2.6)$$

Thus from equation (3.2.5) and (3.2.6), we have:

$$X^\alpha_{\beta} = \partial_\beta X^\alpha + X^\rho \Gamma^\alpha_{\rho\beta} = 0, \text{ if } X^\rho (\partial^\rho_{\beta} \sigma_\rho + \delta^\rho_{\beta} \sigma_\alpha + g_{\beta\rho} \sigma^\alpha) = 0. \quad (3.2.7)$$

$$X^\alpha_{\beta} = X^\rho C^\alpha_{\rho\beta} = 0. \quad (3.2.8)$$

Thus we state
Lemma 3.2.1. A conformal Finsler subspace $\mathcal{F}_m$ admitting a conformal parallel vector field $\mathcal{X}^\alpha$, then it satisfies the equations (3.2.7) and (3.2.8).

From equation (3.2.6), we have

$$\mathcal{X}^\alpha C^\alpha_{\rho\beta} = 0, \quad (3.2.9)$$

contracting equation (3.2.9) by $g_{\alpha\gamma}$, we get

$$\mathcal{X}^\alpha C^\alpha_{\rho\gamma} = 0. \quad (3.2.10)$$

Again by symmetry of lower indices, we have

$$\mathcal{X}^\rho C^\rho_{\gamma\rho\gamma} = \mathcal{X}^\rho C^\rho_{\gamma\rho\gamma} = \mathcal{X}^\rho C^\rho_{\gamma\rho\gamma} = 0.$$

From the Bianchi identities, we know that

$$P_{hijk} = u_{(hi)}(C_{ijk}/h + C_{hjr}C^r_{ik}).$$

Contracting above equation by $B_{\gamma\alpha\beta\delta}^{hijk}$ and if we replace $r$ and $\tau$, we get

$$P_{hijk}B_{\gamma\alpha\beta\delta}^{hijk} = u_{(hi)}(C_{ijk}/h + C_{hjr}C^r_{ik})B_{\gamma\alpha\beta\delta}^{hijk},$$

$$P_{\gamma\alpha\beta\delta} = u_{(\gamma\alpha)}(C_{\alpha\beta\delta}/\gamma + C_{\gamma\delta\rho}C^\rho_{\alpha\delta}).$$

Under conformal change and contracting above equation by $\mathcal{X}^\beta$, we get

$$\mathcal{X}^\beta P_{\gamma\alpha\beta\delta} = u_{(\gamma\alpha)}(C_{\alpha\beta\delta}/\gamma + C_{\gamma\delta\rho}C^\rho_{\alpha\delta})\mathcal{X}^\beta,$$

using equation (3.2.10), we get

$$\mathcal{X}^\alpha P_{\gamma\alpha\beta\delta} = 0. \quad (3.2.11)$$

Consider the curvature tensor

$$S_{hijk} = C_{hkr}C^r_{ij} - C_{hjr}C^r_{ik},$$
contracting above equation by $B_{\gamma\alpha\beta\delta}^{hijk}$ and if we replace $r$ and $\tau$, we get

$$S_{hijk}B_{\gamma\alpha\beta\delta}^{hijk} = (C_{hkr}C_{ij})B_{\gamma\alpha\beta\delta}^{hijk} - (C_{hjr}C_{ik})B_{\gamma\alpha\beta\delta}^{hijk},$$

$$S_{\gamma\alpha\beta\delta} = C_{\gamma\delta\tau}C_{\alpha\beta} - C_{\gamma\delta\tau}C_{\alpha\delta}.$$  

Under conformal change, above equation can be written as

$$\bar{S}_{\gamma\alpha\beta\delta} = \bar{C}_{\gamma\delta\tau}C_{\alpha\beta} - \bar{C}_{\gamma\delta\tau}C_{\alpha\delta}.$$  

Contracting above equation by $\bar{X}^\tau$ and by equation (3.2.9), we get

$$\bar{X}^\tau\bar{S}_{\gamma\alpha\beta\delta} = 0. \quad (3.2.12)$$

Thus we state that:

**Lemma 3.2.2.** The Ricci identities $P_{\gamma\alpha\beta\delta}$ and $S_{\gamma\alpha\beta\delta}$ in the Finsler subspace $F^m$ are invariant under conformal change and satisfies the conditions (3.2.11) and (3.2.12).

Now we consider the special Finsler subspace like quasi-C-reducible, $P_2$-like, $S_3$-like, C-reducible, Semi-C-reducible, $C_2$-like and $T$-condition for Finsler subspace $F^m$ and then we prove all these special Finsler subspace are well defined in conformal Finsler subspace $F_m$ under some conditions.

A Finsler subspace $F^m$ is called a quasi-C-reducible, if the torsion tensor satisfies the equation

$$C_{\alpha\beta\gamma} = A_{\alpha\beta}C_{\gamma} + A_{\beta\gamma}C_{\alpha} + A_{\gamma\alpha}C_{\beta}.$$

Under conformal change the torsion tensor $C_{\alpha\beta\gamma}$ becomes

$$\bar{C}_{\alpha\beta\gamma} = \bar{A}_{\alpha\beta}\bar{C}_{\gamma} + \bar{A}_{\beta\gamma}\bar{C}_{\alpha} + \bar{A}_{\gamma\alpha}\bar{C}_{\beta}. \quad (3.2.13)$$
Transvecting equation (3.2.13) by $\bar{X}^\alpha \bar{X}^\beta$, we get

$$\bar{X}^\alpha \bar{X}^\beta C_{\alpha \beta \gamma} = (\bar{A}_{\alpha \beta} \bar{C}_{\gamma} + \bar{A}_{\beta \gamma} \bar{C}_{\alpha} + \bar{A}_{\alpha \gamma} \bar{C}_{\beta}) \bar{X}^\alpha \bar{X}^\beta,$$

$$\bar{X}^\alpha \bar{X}^\beta \bar{A}_{\alpha \beta} \bar{C}_{\gamma} = 0.$$ 

Therefore $\bar{C}_{\gamma} = 0$, provided $\bar{X}^\alpha \bar{X}^\beta \bar{A}_{\alpha \beta} \neq 0$. According to Deicke’s theorem and by virtue of (3.2.1), we state that:

**Theorem 3.2.3.** If a conformal quasi-C-reducible Finsler subspace $F^m$ admitting a conformal parallel vector field $X^\alpha$, then $F^m$ is Riemannian, provided $\bar{X}^\alpha \bar{X}^\beta \bar{A}_{\alpha \beta} \neq 0$.

A Finsler subspace $F^m$ is P2-like, if it is characterized by

$$P_{\delta \alpha \beta \gamma} = K_{\delta} C_{\alpha \beta \gamma} - K_{\alpha} C_{\delta \beta \gamma},$$  \hspace{1cm} (3.2.14) $$

where $K_{\delta} = K_{h} B_{\delta}^h$ is a covariant vector field.

Under conformal change the equation (3.2.14) can be written as

$$\bar{P}_{\delta \alpha \beta \gamma} = K_{\delta} \bar{C}_{\alpha \beta \gamma} - K_{\alpha} \bar{C}_{\delta \beta \gamma},$$  \hspace{1cm} (3.2.15) $$

where we set $K_{\alpha} = K_{\alpha}$ is a covariant vector field on $F^m$.

Contracting equation (3.2.15) by $\bar{X}^\delta$, we get

$$\bar{X}^\delta \bar{P}_{\delta \alpha \beta \gamma} = \bar{X}^\delta \bar{K}_{\delta} \bar{C}_{\alpha \beta \gamma} - \bar{X}^\delta \bar{K}_{\alpha} \bar{C}_{\delta \beta \gamma},$$

By virtue of (3.2.10) and (3.2.11), we get

$$\bar{X}^\delta \bar{K}_{\delta} \bar{C}_{\alpha \beta \gamma} = 0,$$

that implies $\bar{C}_{\alpha \beta \gamma} = 0$ provided $\bar{X}^\delta \bar{K}_{\delta} \neq 0$, hence by virtue of (3.2.1), we state:
Theorem 3.2.4. The $P^2$-like conformal Finsler subspace $F^n$ admitting a conformal parallel vector field $X^\alpha$, then $F^n$ is Riemannian provided $X^\alpha K^\alpha \neq 0$.

A Finsler subspace $F^n$ is called $S^3$-like, if the curvature tensor $S_{\delta\alpha\beta\gamma}$ is satisfies the equation

$$ L^2 S_{\delta\alpha\beta\gamma} = S(h_{\delta\beta}h_{\alpha\gamma} - h_{\delta\gamma}h_{\alpha\beta}), \quad (3.2.16) $$

where the scalar curvature $S = S_{\delta\alpha\beta\gamma}g^{\delta\beta}g^{\alpha\gamma}$ and $g^{\alpha\beta} = g^{ij}B_{ij}^{\alpha\beta}$.

Under conformal change the equation (3.2.16) can be written as

$$ L^2 \bar{S}_{\delta\alpha\beta\gamma} = \bar{S}(\bar{h}_{\delta\beta}\bar{h}_{\alpha\gamma} - \bar{h}_{\delta\gamma}\bar{h}_{\alpha\beta}). \quad (3.2.17) $$

Transvecting (3.2.17) by $g^{\alpha\gamma}X^\delta$, we obtain

$$ g^{\alpha\gamma}X^\delta L^2 \bar{S}_{\delta\alpha\beta\gamma} = \bar{S}(\bar{h}_{\delta\beta}\bar{h}_{\alpha\gamma} - \bar{h}_{\delta\gamma}\bar{h}_{\alpha\beta})g^{\alpha\gamma}X^\delta $$

using equation (3.2.12) and after simplification, we get $\bar{S} = 0$. Thus we state:

Theorem 3.2.5. If the conformal $S^3$-like Finsler subspace admitting a conformal parallel vector field $X^\alpha$, then the curvature tensor $\bar{S}_{\delta\alpha\beta\gamma}$ vanishes.

Now we shall consider the Finsler subspace satisfying T-condition characterized by

$$ T_{\delta\alpha\beta\gamma} = LC_{\delta\alpha\beta\gamma} + l_\delta C_{\alpha\beta\gamma} + l_\alpha C_{\delta\beta\gamma} + l_\beta C_{\delta\alpha\gamma} + l_\gamma C_{\delta\alpha\beta} = 0, $$

under conformal change above equation can be written as

$$ \bar{T}_{\delta\alpha\beta\gamma} = L\bar{C}_{\delta\alpha\beta\gamma} + \bar{l}_\delta \bar{C}_{\alpha\beta\gamma} + \bar{l}_\alpha \bar{C}_{\delta\beta\gamma} + \bar{l}_\beta \bar{C}_{\delta\alpha\gamma} + \bar{l}_\gamma \bar{C}_{\delta\alpha\beta} = 0. $$

Now contracting above equation by $X^\delta$ and by virtue of equation (3.2.10), we get

$$ \implies \bar{l}_4 \bar{X}^\delta \bar{C}_{\alpha\beta\gamma} = 0. $$

Thus by virtue of (3.2.1) we state:

Theorem 3.2.6. If $F^n$ is satisfying T-condition admitting a conformal parallel vector field $X^\alpha$, then it is Riemannian, provided $\bar{l}_4 \bar{X}^\delta \neq 0$. 

Finally the Finsler subspace satisfying generalized T-condition is defined by

$$T_{\alpha\beta} = L_{\alpha\beta} + l_{\alpha}C_{\beta} + l_{\beta}C_{\alpha} = 0.$$  

Under conformal change above equation can be written as

$$\bar{T}_{\alpha\beta} = \bar{L}_{\alpha\beta} + \bar{l}_{\alpha}\bar{C}_{\beta} + \bar{l}_{\beta}\bar{C}_{\alpha} = 0,$$

contracting above equation by $\bar{X}^\alpha$, we get

$$\bar{X}^\alpha T_{\alpha\beta} = \bar{X}^\alpha \bar{L}_{\alpha\beta} + \bar{X}^\alpha \bar{l}_{\alpha}\bar{C}_{\beta} + \bar{X}^\alpha \bar{l}_{\beta}\bar{C}_{\alpha} = 0. \tag{3.2.18}$$

that implies, $\bar{X}^\alpha \bar{l}_{\alpha}\bar{C}_{\beta} = 0$. Again by Deicke's theorem we state:

**Theorem 3.2.7.** If a conformal Finsler subspace $\bar{F}^m$ satisfies (3.2.18) admitting a conformal parallel vector field $\bar{X}^\alpha$, then it is Riemannian provided $\bar{l}_{\alpha}\bar{X}^\alpha \neq 0$.

A Finsler subspace $F^m$ is said to be C-reducible, if it satisfies the equation

$$mC_{\alpha\gamma\beta} = h_{\alpha\beta}C_{\gamma} + h_{\gamma\alpha}C_{\beta} + h_{\alpha\gamma}C_{\beta}, \tag{3.2.19}$$

where $m = (1, 2, ..., n + 1)$, $C_{\alpha} = C_{\gamma}B_{\alpha}^\gamma = g_{\beta\gamma}C_{\alpha\beta}$.  

Under conformal change (3.2.19) can be written as

$$m\bar{C}_{\alpha\beta\gamma} = \bar{h}_{\alpha\beta}\bar{C}_{\gamma} + \bar{h}_{\beta\gamma}\bar{C}_{\alpha} + \bar{h}_{\alpha\gamma}\bar{C}_{\beta},$$

Contracting above equation by $\bar{X}^{\alpha}\bar{X}^\beta$. Thus we have

$$m\bar{X}^{\alpha}\bar{X}^\beta \bar{C}_{\alpha\beta\gamma} = \bar{X}^{\alpha}\bar{X}^\beta \bar{h}_{\alpha\beta}\bar{C}_{\gamma} + \bar{X}^{\alpha}\bar{X}^\beta \bar{h}_{\beta\gamma}\bar{C}_{\alpha} + \bar{X}^{\alpha}\bar{X}^\beta \bar{h}_{\alpha\gamma}\bar{C}_{\beta},$$

that implies $\bar{X}^{\alpha}\bar{X}^\beta \bar{h}_{\alpha\beta}\bar{C}_{\gamma} = 0$. Thus $\bar{C}_{\gamma} = 0$, provided $\bar{X}^{\alpha}\bar{X}^\beta \bar{h}_{\beta\gamma} \neq 0$. we put $\bar{h}_{\alpha\beta}\bar{X}^{\alpha} = j_\beta \neq 0$ and $\bar{X}^\beta = j^\beta \neq 0$.

Therefore $j_{\beta}j^\beta = j^2 \neq 0$. we obtain:
**Theorem 3.2.8.** Let $F^m$ be a conformal $C$-reducible Finsler subspace admitting a conformal parallel vector field $\overline{X}^\alpha$, then it is Riemannian provided $j^2 \neq 0$.

A Finsler subspace $F^m$ with non-zero length $C$ of the torsion vector $C_\alpha$ is said to be semi-$C$-reducible, if the torsion tensor $C_{\alpha\beta\gamma}$ is of the form

$$C_{\alpha\beta\gamma} = p(h_{\alpha\beta}C_\gamma + h_{\beta\gamma}C_\alpha + h_{\gamma\alpha}C_\beta)/m + qC_\alpha C_\beta C_\gamma/C^2,$$  \hspace{1cm} (3.2.20)

where $m = (1, 2, \ldots, n+1)$, $C^2 = C_\alpha C^\alpha$, $C_\alpha = C_i B_i^\alpha$, $C^\alpha = C^i B_i^\alpha$.

Under conformal change (3.2.20) can be written as

$$\overline{C}_{\alpha\beta\gamma} = p(\overline{h}_{\alpha\beta}\overline{C}_\gamma + \overline{h}_{\beta\gamma}\overline{C}_\alpha + \overline{h}_{\gamma\alpha}\overline{C}_\beta)/n + q\overline{C}_\alpha \overline{C}_\beta \overline{C}_\gamma/\overline{C}^2,$$

Contracting above equation by $\overline{X}^\alpha \overline{X}^\beta$, we get $p\overline{h}_{\alpha\beta} \overline{X}^\alpha \overline{X}^\beta \overline{C}_\gamma = 0$

that implies $\overline{C}_\gamma = 0$, provided $pj^2 \neq 0$, thus we state:

**Theorem 3.2.9.** If $F^m$ is semi-$C$-reducible conformal Finsler subspace admitting a conformal parallel vector field $\overline{X}^\alpha$, then it is Riemannian provided $pj^2 \neq 0$. 
3.3 Special Finsler Hypersurfaces Admitting Concurrent Vector Field

A tangent vector field $X_i$ of a Finsler space $F^n$ is concurrent with respect to the Cartan connection $\mathcal{C}_i$, if [33]:

$$X^i_{\mid j} = \partial_j X^i - N^h_j \partial_h X^i + X^h F^i_{hj} = \partial_j X^i + X^h F^i_{hj} = -\delta^i_j,$$
(3.3.1)

$$X^i_{\mid j} = \partial_j X^i + X^h C^i_{hj} = X^h C^i_{hj} = 0,$$
(3.3.2)

are satisfied.

Transvecting (3.3.1) and (3.3.2) by $B^\alpha_i B^\beta_j$, we obtain

$$X^\alpha_{\mid \beta} = \partial_\beta X^\alpha - N^\delta_\beta \partial_\delta X^\alpha + X^\delta F^\alpha_\delta = \partial_\beta X^\alpha + X^\delta F^\alpha_\delta = -\delta^\alpha_\beta,$$
(3.3.3)

$$X^\alpha_{\mid \beta} = \partial_\beta X^\alpha + X^\delta C^\alpha_{\delta\beta} = X^\delta C^\alpha_{\delta\beta} = 0,$$
(3.3.4)

where $\partial_\beta$ and $\partial_\delta$ denote partial differentiation by $X^\beta$ and $Y^\beta$ respectively.

So under these conditions, we get the following Ricci identities [33]:

$$X^\delta R_{\delta\alpha\beta\gamma} = 0,$$
(3.3.5)

$$X^\delta P_{\delta\alpha\beta\gamma} + C_{\alpha\beta\gamma} = 0,$$
(3.3.6)

$$X^\delta S_{\delta\alpha\beta\gamma} = 0,$$
(3.3.7)

where $R_{\delta\alpha\beta\gamma}$, $P_{\delta\alpha\beta\gamma}$ and $S_{\delta\alpha\beta\gamma}$ are the components of the curvature tensors of $\mathcal{C}_i$ and $C_{\alpha\beta\gamma}$ are the components of the torsion tensors.

Since $P_{\delta\alpha\beta\gamma}$ are skew symmetric in $\delta$ and $\alpha$, we have from (3.3.6)

$$X^i C_{ijk} = 0.$$
(3.3.8)

We know the well known identity

$$P_{hijk} = C_{ijk}h - C_{hjki} + C_{hjr}C_{iklj} - C_{ijr}C_{hklj}.$$
Contracting above equation with $B^{\alpha\beta\gamma\delta}_{ijk}$, we get

$$P_{\delta\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - C_{\delta\beta\gamma\delta} + C_{\delta\beta\gamma\delta}C_{\alpha\gamma\delta} - C_{\alpha\beta\gamma\delta}C_{\delta\gamma\delta}. \quad (3.3.9)$$

Substitute (3.3.9) in (3.3.6), we get

$$P_{\delta\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - C_{\delta\beta\gamma\delta} + C_{\delta\beta\gamma\delta}C_{\alpha\gamma\delta} - C_{\alpha\beta\gamma\delta}C_{\delta\gamma\delta}. \quad (3.3.10)$$

We know that

$$C_{\alpha\beta\gamma\delta} = C_{\beta\gamma\delta}C_{\alpha\delta} + C_{\gamma\alpha\delta}C_{\beta\delta} + C_{\beta\alpha\delta}C_{\gamma\delta} - P_{\alpha\beta\gamma\delta}. \quad (3.3.11)$$

Contracting above equation by $X^\alpha$ and using (3.3.6), (3.3.1) and (3.3.8), we have

$$X^\alpha C_{\alpha\beta\gamma\delta} = C_{\beta\gamma\delta}. \quad (3.3.12)$$

Using (3.3.8) and (3.3.11) in (3.3.10), we get

$$X^\delta C_{\alpha\beta\gamma\delta} = 0. \quad (3.3.13)$$

For the later use, we shall state the following theorem and Lemma.

**Theorem 3.3.1 ([44]).** The covariant components $X_\alpha$ and the contravariant components $X^\alpha$ of a concurrent vector field of a Finsler space are functions of position only.

**Lemma 3.3.2 ([15]).** For the angular metric tensor $h_{\alpha\beta}$ and the covariant vector $m_\alpha$, we have

$$h_{\alpha\beta}X^\beta = m_\alpha \ (\neq 0), \quad (3.3.13)$$

$$m_\alpha X^\alpha = m^2 \ (\neq 0), \quad (3.3.14)$$

where $m^2 = g_{\alpha\beta}m^\alpha m^\beta$ and $m^\alpha = g^{\alpha\beta}m_\beta$. 
First we consider quasi-C-reducible Finsler hypersurface $F^{n-1}(n > 2)$, which is characterized by

$$C_{\alpha\beta\gamma} = A_{\alpha\beta}C_\gamma + A_{\beta\gamma}C_\alpha + A_{\gamma\alpha}C_\beta.$$ 

Contracting above equation by $X^\alpha X^\beta$ and using (3.3.8), we get

$$X^\alpha X^\beta A_{\alpha\beta\gamma} C_\gamma = 0.$$ 

Therefore by Deicke's theorem we state the following:

**Theorem 3.3.3.** A quasi-C-reducible Finsler hypersurface $F^{n-1}(n > 2)$ admits a concurrent vector field $X^\alpha$ then the space is Riemannian, provided that $X^\alpha X^\beta A_{\alpha\beta} \neq 0$.

Let $F^{n-1}(n > 2)$ be a C-reducible Finsler hypersurface and it is characterized by

$$(n + 1)C_{\alpha\beta\gamma} = h_{\alpha\beta}C_\gamma + h_{\beta\gamma}C_\alpha + h_{\gamma\alpha}C_\beta,$$ \hspace{1cm} (3.3.15)

where $C_\alpha = g^{\beta\gamma}C_{\alpha\beta\gamma}$.

Contracting (3.3.15) by $X^\alpha X^\beta$ and using (3.3.8), we obtain

$$X^\alpha X^\beta h_{\alpha\beta} C_\gamma = 0.$$ 

By virtue of Lemma (3.3.2), we state the following:

**Theorem 3.3.4.** A C-reducible Finsler hypersurface $F^{n-1}(n > 2)$ admits a concurrent vector field $X^\alpha$ then the Finsler hypersurface $F^{n-1}$ is Riemannian.

Now consider semi-C-reducible Finsler hypersurface $F^{n-1}(n > 2)$, which is characterized by

$$C_{\alpha\beta\gamma} = p(h_{\alpha\beta}C_\gamma + h_{\beta\gamma}C_\alpha + h_{\gamma\alpha}C_\beta)/(n) + qC_\alpha C_\beta C_\gamma/C^2,$$ \hspace{1cm} (3.3.16)

where $C_2 = C_\alpha C_\alpha$ and $p + q = 1$.

Contracting (3.3.16) by $X^\alpha X^\beta$ and using (3.3.8) and Lemma (3.3.2), we get $pC_\gamma = 0$. 

Then we obtain $p = 0$ because of $C^2 \neq 0$. In virtue of $p + q = 1$, we have $q = 1$. Thus we state the following theorem:

**Theorem 3.3.5.** If a semi-$C$-reducible Finsler hypersurface $F^{n-1}(n > 2)$ admits a concurrent vector field $X^\alpha$, then the Finsler hypersurface $F^{n-1}$ is $C_2$-like.

Let $F^{n-1}(n > 2)$ be a $C^h$-recurrent Finsler hypersurface, which satisfies

$$C_{\alpha\beta\gamma} = K_\delta C_{\alpha\beta\gamma}, \quad (3.3.17)$$

where $K_\delta$ is a covariant vector field.

Contracting (3.3.17) by $X^\delta$ and using (3.3.12), we have

$$C_{\alpha\beta\gamma} X^\delta K_\delta = 0.$$

We obtain the following theorem:

**Theorem 3.3.6.** If a $C^h$-recurrent Finsler hypersurface $F^{n-1}$ admits a concurrent vector field $X^\delta$, then the Finsler hypersurface is Riemannian, provided that $X^\delta K_\delta \neq 0$.

Let $F^{n-1}(n > 2)$ be a $P_2$-like Finsler hypersurface and it is characterized by

$$P_{\delta\alpha\beta\gamma} = K_\delta C_{\alpha\beta\gamma} - K_\alpha C_{\delta\beta\gamma}, \quad (3.3.18)$$

where $K_\delta$ is a covariant vector field.

Contracting (3.3.18) by $X^\delta$ and using (3.3.6) and (3.3.8), we obtain

$$(X^\delta K_\delta + 1) C_{\alpha\beta\gamma} = 0.$$

Hence we state following theorem:

**Theorem 3.3.7.** If a $P_2$-like Finsler hypersurface $F^n$ admits a concurrent vector field $X^\delta$, then the space is Riemannian, provided that $(X^\delta K_\delta + 1) \neq 0$. 
The following expression is well known

\[ P_{\alpha\beta\gamma} = C_{\alpha\beta\gamma0}. \]  

(3.3.19)

Let \( F^{n-1} \) be a \( P \)-reducible Finsler hypersurface, which is characterized by

\[ P_{\alpha\beta\gamma} = (h_{\alpha\beta} P_\gamma + h_{\beta\gamma} P_\alpha + h_{\gamma\alpha} P_\beta)/(n), \]

(3.3.20)

where \( P_\alpha = C_{\alpha\gamma0} \).

Contracting (3.3.20) by \( X^\alpha X^\beta \) and using (3.3.19) and (3.3.8), we obtain

\[ X^\alpha X^\beta h_{\alpha\beta} P_\gamma = 0. \]

Using Lemma (3.3.2), we get \( P_\gamma = 0 \). Hence we have:

**Theorem 3.3.8.** If a \( P \)-reducible Finsler hypersurface \( F^{n-1} \) admits a concurrent vector field \( X^\alpha \), then the space \( F^{n-1} \) is Landsberg space.

Let \( F^n \) be a S3-like Finsler hypersurface \( F^{n-1} \) and it is characterized by

\[ L^2 S_{\alpha\beta\gamma} = S(\delta_{\delta\gamma} h_{\alpha\gamma} - h_{\delta\gamma} h_{\alpha\beta}), \]

(3.3.21)

where the scalar curvature \( S = S_{\delta\alpha\beta\gamma} g^{\delta\beta} g^{\alpha\gamma} \) is a function of position alone.

Contracting (3.3.21) by \( X^\delta g^{\alpha\gamma} \) and using (3.3.7), (3.3.13) we get \( S = 0 \).

Therefore we have:

**Theorem 3.3.9.** If a S3-like Finsler hypersurface \( F^{n-1} \) admits a concurrent vector field, then the curvature tensor \( S_{\delta\alpha\beta\gamma} \) vanishes.

Let \( F^{n-1}(n > 2) \) be a h-isotropic Finsler space, which satisfies

\[ R_{\delta\alpha\beta\gamma} = K(g_{\delta\beta} g_{\alpha\gamma} - g_{\delta\gamma} g_{\alpha\beta}), \]

(3.3.22)
where $K$ is a Finsler scalar.

Contracting (3.3.22) by $X^\alpha m^\beta$ and using (3.3.5) and (3.3.14), we obtain

$$K(m^2g_{\alpha\gamma} - m_\alpha X_\gamma) = 0.$$ 

Further contracting above by $g^{\alpha\gamma}$, we have $K = 0$, hence we have:

**Theorem 3.3.10.** If a $h$-isotropic Finsler hypersurface $F^{n-1}(n > 2)$ admits a concurrent vector field $X^\alpha$, then the curvature tensor $R_{\delta\alpha\beta\gamma}$ vanishes.

Now we shall consider the T-condition

$$T_{\delta\alpha\beta\gamma} = L_C_{\delta\alpha\beta\gamma} + l_\delta C_{\alpha\beta\gamma} + l_\alpha C_{\delta\beta\gamma} + l_\beta C_{\delta\alpha\gamma} + l_\gamma C_{\delta\alpha\beta} = 0,$$

where the T-tensor is completely symmetric.

Contracting above equation by $X^\delta$ and using (3.3.8), we have $X^\delta l_\delta C_{\alpha\beta\gamma} = 0$. But $X^\delta l_\delta \neq 0$ that implies $C_{\alpha\beta\gamma} = 0$.

Therefore we have:

**Theorem 3.3.11.** If a Finsler hypersurface satisfying T-condition admits a concurrent vector field $X^\delta$, then the space is Riemannian provided $X^\delta l_\delta \neq 0$.

Finally we consider generalized T-condition and it is characterized by

$$T_{\alpha\beta} = T_{\alpha\beta\gamma} g^{\delta\gamma} = L_C_{\alpha\beta} + l_\alpha C_{\beta} + l_\beta C_{\alpha} = 0.$$ 

Contracting above equation by $X^\alpha$ and using (3.3.8), we get $l_\alpha C_{\beta} = 0$. But $l_\alpha \neq 0$, hence by Deicke's theorem we state the following theorem:

**Theorem 3.3.12.** If a Finsler hypersurface satisfying generalized T-condition admits a concurrent vector field $X^\alpha$, then the space is Riemannian provided $l_\alpha \neq 0$. 
We are concerned with a space of scalar curvature in Berwald's sense. It is characterized by the equation

\[ R_{\alpha\beta\gamma} = R_{\alpha\beta\gamma} y^\alpha = L^2 K h_{\alpha\gamma}, \tag{3.3.23} \]

or \[ R_{\alpha\beta\gamma} = h_{\alpha\gamma} K_\beta - h_{\alpha\beta} K_\gamma, \quad K_\beta = L^2 \partial_\beta K/3 + L K \partial_\beta, \]

where \( h_{\alpha\gamma} \) is the angular metric tensor and \( K \) is a Finsler scalar field.

Contracting (3.3.23) by \( x^\alpha \) and using (3.3.5) and (3.3.13), we obtain the following:

**Lemma 3.3.13.** If a Finsler hypersurface \( F^{n-1} \) of scalar curvature \( K \) admits a concurrent vector field, then the scalar curvature \( K \) vanishes.

From above Lemma and (3.3), we immediately get \( R_{\alpha\beta\gamma} = 0 \), then Berwald's curvature tensor

\[ H_{\delta\alpha\beta\gamma} = \partial_\delta R_{\alpha\beta\gamma} - 2 C_{\alpha\delta\beta} R_{\gamma}^\alpha = 0. \]

Therefore we state the following theorem:

**Theorem 3.3.14.** If a Finsler hypersurface \( F^{n-1} \) of scalar curvature admits a concurrent vector field \( x^\alpha \), then the Berwald's curvature tensor \( H_{\delta\alpha\beta\gamma} \) vanishes.

### 3.4 C-conformal special Finsler hypersurface \( \overline{F}^{n-1} \)

admitting concurrent vector field

If \( F^{n-1} \) is a Finsler hypersurface admitting a concurrent vector field \( x^\alpha \) and \( \overline{x}^\alpha = e^{-\sigma(x)} x^\alpha \) is a vector field on \( \overline{F}^{n-1} \) then from (3.3.3) and (3.3.4), we have

\[ \overline{x}^\alpha_{\beta} = \partial_\beta \overline{x}^\alpha - N^h_\beta \overline{x}^h + \overline{x}^h F^\alpha_{\delta\beta} = -\sigma_\beta \overline{x}^\alpha + \delta_\beta^\alpha (\sigma_\delta \overline{x}^\delta + 1) = 0, \tag{3.4.1} \]

\[ \overline{x}^\alpha_{\beta} = \sigma_\beta \overline{x}^\alpha + \overline{x}^\delta C^\alpha_{\delta\beta} = 0. \tag{3.4.2} \]

Thus we state:
Lemma 3.4.1. Let $F^{n-1}$ be $C$-conformal Finsler hypersurface of $F^n$. If $X^\alpha$ is a concurrent vector field on $F^{n-1}$, then $C$-conformal vector field $\overline{X}^\alpha$ on $F^{n-1}$ is concurrent if (3.4.1) and (3.4.2) are satisfied.

Under $C$-conformal change and from Ricci identities, we have the following integrability conditions

\[
\overline{X}^\delta \overline{P}_{\delta \alpha \beta \gamma} + \overline{C}_{\alpha \beta \gamma} = 0, \tag{3.4.3}
\]

\[
\overline{X}^\delta \overline{S}_{\delta \alpha \beta \gamma} = 0. \tag{3.4.4}
\]

Since $\overline{P}_{\delta \alpha \beta \gamma}$ are skew symmetric in $\delta$ and $\alpha$, we have from (3.4.3)

\[
\overline{X}^\alpha \overline{C}_{\alpha \beta \gamma} = 0. \tag{3.4.5}
\]

Consider the well known identity

\[
\overline{P}_{\delta \alpha \beta \gamma} = \overline{C}_{\alpha \beta \gamma \delta} - \overline{C}_{\delta \beta \gamma \alpha} + \overline{C}_{\delta \gamma \alpha \beta} - \overline{C}_{\alpha \beta \gamma \delta}. \tag{3.4.6}
\]

We know that

\[
\overline{C}_{\alpha \beta \gamma \delta} = \overline{C}_{\beta \gamma \delta \alpha} + \overline{C}_{\gamma \alpha \delta \beta} - \overline{C}_{\alpha \gamma \delta \beta}. \tag{3.4.7}
\]

Contracting above equation by $\overline{X}^\alpha$ and using (3.4.3), (3.4.2) and (3.4.5), we have

\[
\overline{X}^\alpha \overline{C}_{\alpha \beta \gamma \delta} = \overline{C}_{\beta \gamma \delta}. \tag{3.4.8}
\]

Using (3.4.5) and (3.4.7) in (3.4.6), we get

\[
\overline{X}^\delta \overline{C}_{\alpha \beta \gamma \delta} = 0. \tag{3.4.8}
\]

Thus we state:
Lemma 3.4.2. Under C-conformal change the torsion tensor $\overline{C}_{\alpha \beta \gamma}$ and a concurrent vector field $\overline{X}^\delta$, we have $\overline{X}^\delta \overline{C}_{\alpha \beta \gamma \delta} = 0$.

Under C-conformal change the torsion tensor $\overline{C}_{\alpha \beta \gamma}$ of $\overline{F}^{m-1}$ is given by

$$\overline{C}_{\alpha \beta \gamma} = \overline{A}_{\alpha \beta} \overline{C}_\gamma + \overline{A}_\beta \overline{C}_\alpha + \overline{A}_\gamma \overline{C}_\beta.$$  \hspace{1cm} (3.4.9)

By virtue of (1.4.3(c)) and where we assume that $\overline{A}_{\alpha \beta} = A_{\alpha \beta}$. From (3.4.9) we obtain

$$C_{\alpha \beta \gamma} = e^{-2\sigma} (A_{\alpha \beta} C_\gamma + A_\beta C_\alpha + A_\gamma C_\beta).$$  \hspace{1cm} (3.4.10)

Contracting (3.4.10) by $X^\alpha X^\beta$ and by virtue of (3.3.2), we have

$$e^{-2\sigma} X^\alpha X^\beta A_{\alpha \beta} C_\gamma = 0.$$

Therefore we obtain

$C_\gamma = 0$ provided $X^\alpha X^\beta A_{\alpha \beta} \neq 0$. According to Deicke's theorem and by Lemma (3.4.1), we state the following:

Theorem 3.4.3. If $F^{n-1}(n > 2)$ and $\overline{F}^{n-1}(n > 2)$ are quasi-C-reducible Finsler hypersurfaces admits concurrent vector fields $X^\alpha$ and $\overline{X}^\alpha$ respectively, then both the Finsler hypersurfaces $F^{n-1}(n > 2)$ and $\overline{F}^{n-1}(n > 2)$ are Riemannian provided that $A_{\alpha \beta} X^\alpha X^\beta \neq 0$.

Under C-conformal change the C-reducible condition becomes

$$(n)\overline{C}_{\alpha \beta \gamma} = \overline{h}_{\alpha \beta} \overline{C}_\gamma + \overline{h}_\beta \overline{C}_\alpha + \overline{h}_\gamma \overline{C}_\beta.$$  \hspace{1cm} (3.4.11)

Contracting (3.4.11) by $X^\alpha X^\beta$ and by virtue of (3.2.1(b),(c)), we obtain

$$X^\alpha X^\beta h_{\alpha \beta} C_\gamma = 0.$$

We set $h_{\alpha \beta} X^\alpha = j_\beta$ and $j_\beta X^\beta = j^\alpha$, where $j^2 = g_{\alpha \beta} m^\alpha m^\beta$ and $j^\alpha = g^{\alpha \beta} j_\beta$.

Consequently, taking into account of Lemma (3.4.1), we obtain:
Theorem 3.4.4. If $F^{n-1}(n > 2)$ and $\overline{F}^{n-1}(n > 2)$ are $C$-reducible Finsler hypersurface admits concurrent vector fields $X^\alpha$ and $\overline{X}^\alpha$ respectively, then both the Finsler hypersurfaces $F^{n-1}(n > 2)$ and $\overline{F}^{n-1}(n > 2)$ are Riemannian provided $f^2 \neq 0$.

Under C-conformal change the semi-C-reducible condition becomes

$$\overline{C}_{\alpha\beta\gamma} = p(h_{\alpha\beta}\overline{C}_\gamma + \overline{h}_{\beta\gamma}\overline{C}_\alpha + \overline{h}_{\alpha\gamma}\overline{C}_\beta)/(n) + q\overline{C}_\alpha\overline{C}_\beta\overline{C}_\gamma/\overline{C}^2.$$  \hspace{1cm} (3.4.12)

Contracting (3.4.12) by $X^\alpha X^\beta$ and using (3.3.2), (3.2.1(b),(c)), we obtain

$$p h_{\alpha\beta} X^\alpha X^\beta C_\gamma = 0.$$

Therefore we get

$$C_\gamma = 0$$ provided $pm^2 \neq 0$. Thus we state the following:

Theorem 3.4.5. If $F^{n-1}(n > 2)$ and $\overline{F}^{n-1}(n > 2)$ are semi-C-reducible Finsler hypersurface admits concurrent vector fields $X^\alpha$ and $\overline{X}^\alpha$ respectively, then both the Finsler hypersurface $F^{n-1}(n > 2)$ and $\overline{F}^{n-1}(n > 2)$ are Riemannian provided $p\overline{m}^2 \neq 0$.

Under C-conformal change the $C^h$-recurrent Finsler hypersurface becomes

$$\overline{C}_{\alpha\beta\gamma} = \overline{C}_{\alpha\beta\gamma} K_\delta.$$  \hspace{1cm} (3.4.13)

Contracting (3.4.13) by $X^\delta$ and using (1.4.3(c)), we have

$$e^{-\sigma} X^\delta K_\delta C_{\alpha\beta\gamma} = 0.$$

Therefore we obtain $C_{\alpha\beta\gamma} = 0$ provided $X^\delta K_\delta \neq 0$.

Hence we state:

Theorem 3.4.6. If $F^{n-1}$ and $\overline{F}^{n-1}(n > 2)$ are $C^h$-recurrent Finsler hypersurface admits concurrent vector fields $X^\delta$ and $\overline{X}^\delta$ respectively, then both the Finsler hypersurface $F^{n-1}(n > 2)$ and $\overline{F}^{n-1}(n > 2)$ are Riemannian provided $X^\delta K_\delta \neq 0$. 
Under C-conformal change, contracting $P2$-like by $\overline{X}^\delta$ and using (3.4.1) and (3.4.3),
we get
\[(e^{-\sigma}X^\delta K_\delta + 1)C_{\alpha\beta\gamma} = 0.\]
Therefore we obtain $C_{\alpha\beta\gamma} = 0$ provided $e^{-\sigma}X^\delta K_\delta + 1 \neq 0$.

Hence we state the following:

**Theorem 3.4.7.** If $F^{n-1}$ and $F^{n-1}$ $(n > 2)$ are $P2$-like Finsler hypersurface admits concurrent vector fields $X^\delta$ and $\overline{X}^\delta$ respectively, then both the Finsler hypersurface $F^{n-1}$ and $\overline{F}^{n-1}$ are Riemannian provided $e^{-\sigma}X^\delta K_\delta + 1 \neq 0$.

Under C-conformal change, $S3$-like condition can be written as
\[
L^2S_{\delta\alpha\beta\gamma} = e^{4\sigma}S(\delta_{\delta\beta}h_{\alpha\gamma} - h_{\delta\gamma}h_{\alpha\beta}).
\] (3.4.14)
Contracting (3.4.14) by $X^\delta$, then by $g^{\alpha\gamma}$ and by virtue of (3.4.4), we have $S = 0$. Thus we state the following:

**Theorem 3.4.8.** If $F^{n-1}(n > 3)$ and $\overline{F}^{n-1}(n > 3)$ are $S3$-like Finsler hypersurface admits concurrent vector fields $X^\alpha$ and $\overline{X}^\alpha$ respectively, then both the curvature tensors of Finsler hypersurface $F^{n-1}(n > 3)$ and $\overline{F}^{n-1}(n > 3)$ vanishes.

Under C-conformal change, $T$-condition can be written as
\[
\overline{T}_{\delta\alpha\beta\gamma} = L\overline{C}_{\delta\alpha\beta\gamma} + \overline{l}_\delta \overline{C}_{\alpha\beta\gamma} + \overline{l}_\alpha \overline{C}_{\delta\beta\gamma} + \overline{l}_\beta \overline{C}_{\delta\alpha\gamma} + \overline{l}_\gamma \overline{C}_{\delta\alpha\beta} = 0.
\] (3.4.15)
contracting (3.4.15) by $X^\delta$, we get
\[
\overline{T}_{\delta\alpha\beta\gamma}X^\delta = L\overline{C}_{\delta\alpha\beta\gamma} + \overline{l}_\delta \overline{C}_{\alpha\beta\gamma} + \overline{l}_\alpha \overline{C}_{\delta\beta\gamma} + \overline{l}_\beta \overline{C}_{\delta\alpha\gamma} + \overline{l}_\gamma \overline{C}_{\delta\alpha\beta} = 0.
\]
By virtue of (3.2.1(b),(c)), we have
\[X^\delta l_\delta C_{\alpha\beta\gamma} = 0.
\]
Thus we state the following:
Theorem 3.4.9. If $F^{n-1}$ and $\overline{F}^{n-1}$ are Finsler spaces satisfying T-condition, admits concurrent vector fields $X^\alpha$ and $\overline{X}^\alpha$ respectively, then both the Finsler hypersurfaces $F^{n-1}$ and $\overline{F}^{n-1}$ are Riemannian provided $X^\alpha l_\alpha \neq 0$.

Under C-conformal change, contracting generalized T-condition by $\overline{X}^\alpha$, we have

$$T_{\alpha\beta} = T_{\alpha\beta\rho\sigma}g^{\rho\sigma} = LC_{\alpha}|_{\beta} + l_\alpha C_{\beta} + l_\beta C_{\alpha} = 0. \quad (3.4.16)$$

By virtue of (3.2.1(b),(c)) we get

$$e^{-2\alpha}X^\alpha l_\alpha C_{\beta} = 0.$$

Thus we obtain:

Theorem 3.4.10. If $F^{n-1}$ and $\overline{F}^{n-1}$ are Finsler hypersurfaces satisfying generalized T-condition, admits concurrent vector fields $X^\alpha$ and $\overline{X}^\alpha$ respectively, then both the Finsler hypersurfaces $F^{n-1}$ and $\overline{F}^{n-1}$ are Riemannian provided $X^\alpha l_\alpha \neq 0$.

### 3.5 Conclusion

The important findings of this chapter are as follows:

- If a conformal quasi-C-reducible Finsler subspace $\overline{F}^m$ admitting a conformal parallel vector field $\overline{X}^\alpha$, then $\overline{F}^m$ is Riemannian, provided $\overline{X}^\alpha \overline{X}^\beta \overline{A}_{\alpha\beta} \neq 0$.

- If a conformal quasi-C-reducible Finsler subspace $\overline{F}^m$ admitting a conformal parallel vector field $\overline{X}^\alpha$, then $\overline{F}^m$ is Riemannian, provided $\overline{X}^\alpha \overline{X}^\beta \overline{A}_{\alpha\beta} \neq 0$.

- The $P2-like$ conformal Finsler subspace $\overline{F}^m$ admitting a conformal parallel vector field $\overline{X}^\alpha$, then $\overline{F}^m$ is Riemannian provided $\overline{X}^\delta K_\delta \neq 0$. 
• If the conformal \( S^3 \)-like Finsler subspace admitting a conformal parallel vector field \( \overline{X}^\alpha \), then the curvature tensor \( \overline{\mathfrak{g}}_{\delta\alpha\beta\gamma} \) vanishes.

• If \( \overline{F}^m \) is satisfying T-condition admitting a conformal parallel vector field \( \overline{X}^\alpha \), then it is Riemannian, provided \( \overline{I}_\delta \overline{X}^\delta \neq 0 \).

• If a conformal Finsler subspace \( \overline{F}^m \) satisfies (3.2.18) admitting a conformal parallel vector field \( \overline{X}^\alpha \), then it is Riemannian provided \( \overline{I}_\alpha \overline{X}^\alpha \neq 0 \).

• Let \( \overline{F}^m \) is a conformal C-reducible Finsler subspace admitting a conformal parallel vector field \( \overline{X}^\alpha \), then it is Riemannian provided \( j^2 \neq 0 \).

• If \( \overline{F}^m \) is semi-C-reducible conformal Finsler subspace admitting a conformal parallel vector field \( \overline{X}^\alpha \), then it is Riemannian provided \( pj^2 \neq 0 \).

• A quasi-C-reducible Finsler hypersurface \( F^{n-1}(n > 2) \) admits a concurrent vector field \( X^\alpha \) then the space is Riemannian, provided that \( X^\alpha X^\beta A_{\alpha\beta} \neq 0 \).

• A C-reducible Finsler hypersurface \( F^{n-1}(n > 2) \) admits a concurrent vector field \( X^\alpha \) then the Finsler hypersurface \( F^{n-1} \) is Riemannian.

• If a semi-C-reducible Finsler hypersurface \( F^n-1(n > 2) \) admits a concurrent vector field \( X^\alpha \), then the Finsler hypersurface \( F^{n-1} \) is \( C_2 \)-like.

• If a \( C^h \)-recurrent Finsler space \( F^n \) \((n \geq 2) \) admits a concurrent vector field \( X^\delta \), then the Finsler hypersurface is Riemannian, provided that \( X^\delta K_\delta \neq 0 \).

• If a \( P^2 \)-like Finsler space \( F^n \) \((n > 2) \) admits a concurrent vector field \( X^\delta \), then the space is Riemannian, provided that \( (X^\delta K_\delta + 1) \neq 0 \).

• If a \( P \)-reducible Finsler hypersurface \( F^{n-1} \) admits a concurrent vector field \( X^\alpha \), then the space \( F^{n-1} \) is Landsberg space.
• If a $S^3$-like Finsler hypersurface $F^{n-1}$ admits a concurrent vector field, then the curvature tensor $S_{\alpha\beta\gamma}$ vanishes.

• If a h-isotropic Finsler hypersurface $F^{n-1}(n > 2)$ admits a concurrent vector field $X^\alpha$, then the curvature tensor $R_{\alpha\beta\gamma}$ vanishes.

• If a Finsler hypersurface satisfying T-condition admits a concurrent vector field $X^\delta$, then the space is Riemannian, provided $X^\delta l_\delta \neq 0$.

• If a Finsler hypersurface $F^{n-1}$ of scalar curvature admits a concurrent vector field $X^\alpha$, then the Berwald's curvature tensor $H_{\alpha\beta\gamma}$ vanishes.

• Let $F^n$ be C-conformal Finsler space of $F^n$. If $X^i$ is a concurrent vector field on $F^n$, then C-conformal vector field $\bar{X}^i$ on $F^n$ is concurrent if (3.4.1) and (3.4.2) are satisfied.

• If $F^{n-1}(n > 2)$ and $\bar{F}^{n-1}(n > 2)$ are quasi-C-reducible Finsler hypersurfaces admits concurrent vector fields $X^\alpha$ and $\bar{X}^\alpha$ respectively, then both the Finsler hypersurfaces $F^{n-1}(n > 2)$ and $\bar{F}^{n-1}(n > 2)$ are Riemannian provided that $A_{\alpha\beta} X^\alpha X^\beta \neq 0$.

• If $F^{n-1}(n > 2)$ and $\bar{F}^{n-1}(n > 2)$ are C-reducible Finsler hypersurface admits concurrent vector fields $X^\alpha$ and $\bar{X}^\alpha$ respectively, then both the Finsler hypersurfaces $F^{n-1}(n > 2)$ and $\bar{F}^{n-1}(n > 2)$ are Riemannian provided $j^2 \neq 0$.

• If $F^{n-1}(n > 2)$ and $\bar{F}^{n-1}(n > 2)$ are semi-C-reducible Finsler hypersurface admits concurrent vector fields $X^\alpha$ and $\bar{X}^\alpha$ respectively, then both the Finsler hypersurface $F^{n-1}(n > 2)$ and $\bar{F}^{n-1}(n > 2)$ are Riemannian provided $p j^2 \neq 0$.

• If $F^{n-1}(n > 2)$ and $\bar{F}^{n-1}(n > 2)$ are $C^h$-recurrent Finsler hypersurface admits concurrent vector fields $X^\delta$ and $\bar{X}^\delta$ respectively, then both the Finsler hypersurface $\bar{F}^{n-1}(n > 2)$ and $\bar{F}^{n-1}(n > 2)$ are Riemannian provided $p j^2 \neq 0$. 
\( F^{n-1}(n > 2) \) and \( \overline{F}^{n-1}(n > 2) \) are Riemannian provided \( X^\delta K_\delta \neq 0 \).

- If \( F^{n-1}(n > 2) \) and \( \overline{F}^{n-1}(n > 2) \) are P2-like Finsler hypersurface admits concurrent vector fields \( X^\delta \) and \( \overline{X}^\delta \) respectively, then both the Finsler hypersurface \( F^{n-1}(n > 2) \) and \( \overline{F}^{n-1}(n > 2) \) are Riemannian provided \( e^{-\sigma}X^\delta K_\delta + 1 \neq 0 \).

- If \( F^{n-1}(n > 3) \) and \( \overline{F}^{n-1}(n > 3) \) are S3-like Finsler hypersurface admits concurrent vector fields \( X^\alpha \) and \( \overline{X}^\alpha \) respectively, then both the curvature tensors of Finsler hypersurface \( F^{n-1}(n > 3) \) and \( \overline{F}^{n-1}(n > 3) \) vanishes.

- If \( F^{n-1} \) and \( \overline{F}^{n-1} \) are Finsler spaces satisfying T-condition, admits concurrent vector fields \( X^\alpha \) and \( \overline{X}^\alpha \) respectively, then both the Finsler hypersurfaces \( F^{n-1} \) and \( \overline{F}^{n-1} \) are Riemannian provided \( X^\delta l_\delta \neq 0 \).

- If \( F^{n-1} \) and \( \overline{F}^{n-1} \) are Finsler hypersurfaces satisfying generalized T-condition, admits concurrent vector fields \( X^\alpha \) and \( \overline{X}^\alpha \) respectively, then both the Finsler hypersurfaces \( F^{n-1} \) and \( \overline{F}^{n-1} \) are Riemannian provided \( X^\alpha l_\alpha \neq 0 \).