CHAPTER 1
Chapter 1
Basic Definitions and Notions

The fundamental idea of a Finsler space may be traced back to the famous lecture of Riemannian: "Uber die Hypothesen, Welche der Geometrie zugrunde liegen". In this memoir of 1854 Riemann discussed various possibilities by means of which an n-dimensional manifold may be endowed with a metric, and pays particular attention to a metric defined by the positive square root of a positive definite quadratic differential form. Thus the foundation of Riemann geometry are laid; nevertheless, it is also suggested that the positive fourth root of a fourth order differential form might serve as a metric function. Those functions have three properties in common: they are positive, homogeneous of the first degree in the differentials, and are also convex in the later. It would seem natural, therefore to introduce a further generalization to the effect that the distance $ds$ between two neighboring points represented by the coordinates $x^i$ and $x^i + dx^i$ be defined by some function $F(x^i, dx^i) : ds = F(x^i, dx^i)$ where this function satisfies these three properties.

This chapter includes preliminaries and it covers known definitions, tensor notations and some results which will be used in the coming chapters.
1.1 Finsler Spaces: Definition and Examples

Finsler spaces are the most natural generalization of Riemannian spaces. As a generalization of Riemannian space, Finsler space is considered to be a space in which the line element is a function \( L(x^i, dx^i) \) which is positive homogeneous of degree one with respect to \((dx^1, \ldots, dx^m)\). A formal definition of a Finsler space is as follows:

**Definition 1.1.1.** By a Finsler space, we mean a triple \( F^n = (M, D, L) \), where \( M \) denotes \( n \)-dimensional differential manifold, \( D \) is an open subset of a tangent vector bundle \( TM \) endowed with the differentiable structure induced by the differentiable structure of the manifold \( TM \) and \( L : D \rightarrow \mathbb{R} \) is a differentiable mapping having the following properties:

1. \( L(x, y) > 0 \), for \((x, y) \in D\),
2. \( L(x, \lambda y) = |\lambda|L(x, y) \), for any \((x, y) \in D\) and \( \lambda \in \mathbb{R} \), such that \((x, \lambda y) \in D\),
3. The \( d - \) tensor field \( g_{ij}(x, y) = \frac{1}{2} \partial_i \partial_j L^2 \), \((x, y) \in D\),

where \( \partial_i = \frac{\partial}{\partial x^i} \), is non-degenerate on \( D \).

**Example 1.1.1:** Let \( M \) be a real \( n \)-dimensional differentiable manifold endowed with a Riemannian metric \( g \) and a differentiable 1-form \( \omega \). Let \( H \) be a closed subset of \( \phi(U) \times \mathbb{R}^n \) consisting of all points \((x^i, y^i)\) such that \( \omega_i(x) = 0 \).

Let \( L(x^i, y^i) = \frac{\partial \phi(x)}{\omega(z)y^i} \) be the real valued function defined on the open set \( U^* = \phi(U) \times \mathbb{R}^n - \{H\} \). Let \( B \) denote the union of all open sets \( \phi^{-1}(U^*) \). It is clear that \( L \) satisfies the homogeneity property on \( B \) and satisfies \( \text{Rank}(\partial_i \partial_j L^2/2) = n \) on an open sub manifold \( A \) of \( B \). Then the pair \( F^n = (TM, L) \) is a Finsler space called \( Kropina space \).

**Example 1.1.2:** Let \( M \) be a real \( n \)-dimensional differentiable manifold endowed with a Riemannian metric \( g \) and a differentiable 1-form \( \omega \). \( g_{ij}(x) \) and \( \omega_j(x) \) be the components
of $g$ and $\omega$ with respect to the local chart $(U, \phi, R^n)$ and let $L$ be a real function defined on $\phi(U) \times R^n$ by

$$L(x^i, y^i) = \omega_i(x)y^i + (g_{ij}(x)y^i y^j)^{1/2}.$$ 

clearly $L$ is a global function on $TM$ given locally by the above expression. Moreover $L$ satisfies the homogeneity property and on an open submanifold $A$ of $TM$ and satisfies

$$\text{Rank}(\partial_i \partial_j L^2/2) = n.$$ 

Thus $L$ is the fundamental function of Finsler space $F^n = (TM, L)$ and this space is called Randers space.

### 1.2 Finsler Connections

The metric tensor and angular metric tensors of Finsler space are given by

$$g_{ij} = \partial_i \partial_j L^2/2, \quad g^{ij} = (g_{ij})^{-1},$$

$$h_{ij} = g_{ij} - l_i l_j.$$  \hspace{1cm} (1.2.1)

where $\partial_i = \partial/\partial y^i$, $l_i = \partial_i L$ and $l^i = y^i/L$.

Cartan's C-tensors constructed from the metric tensor $g_{ij}$ are

$$C_{ijk} = \frac{1}{2} \partial_k g_{ij},$$  \hspace{1cm} (1.2.2)

$$C^k_{ij} = \frac{1}{2} g^{kr} C_{rj}.$$  \hspace{1cm} (1.2.3)

The Christoffel's symbol $\gamma^i_{jk}$ constructed from $g_{ij}$ with respect to $\partial_i = \partial/\partial x^i$ is given by

$$\gamma^i_{jk} = \frac{1}{2} g^{ih} \{ \partial_k g_{jh} + \partial_j g_{kh} - \partial_h g_{jk} \}.$$
The curvature tensors derived from Christoffel’s functions are

\[ G^i = \frac{1}{2} \gamma^i_{jk} y^j y^k, \quad G^i_j = \partial_j G^i, \quad G^i_{jk} = \partial_k G^i_j, \quad G^i_{jkl} = \partial_l G^i_{jk}. \]

A non-linear connection \( N^i_j \) is given by

\[ N^i_j = G^i_j - P^i_j, \]

where \( P^i_j \) is an arbitrary tensor given by

\[ P^i_j = \frac{1}{2} (y^k \partial_k N^i_j - N^i_j). \]

The Cartan’s connection coefficient are given by

\[ F^i_{jk} = \frac{1}{2} g^{ir} (\delta_r g_{rk} + \delta_k g_{rj} - \delta_r g_{jk}), \]

where \( \delta_j = \partial_j - G^i_j \partial_r. \)

The Cartan’s and Berwald’s connection of \( F^m \) are given by \( C_G = (F^i_{jk}, N^i_j, C^i_{jk}) \) and \( B_G = (G^i_{jk}, G^i_j, 0) \) respectively.

The \( h \) and \( v \)-covariant derivatives with respect to Cartan’s connection of any tensor field \( X^i_j \) are defined as

\[ X^i_{jk} = \delta_k X^i_j + X^m_j F^i_{mk} - X^i_m F^m_{jk}, \]
\[ X^i_{j\mid k} = \delta_k X^i_j + X^m_j C^i_{mk} - X^i_m C^m_{jk}. \]

The covariant derivative of any tensor \( X^i_j \) with respect to Berwald’s connection is given by

\[ X^i_{jk} = \partial_k X^i_j - (\partial_m X^i_j) G^m_k + X^m_j G^i_{mk} - X^i_m G^m_{jk}. \]
1.3 Curvature and Torsion Tensors of Finsler Connection

The Berwald's curvature tensor, Cartan's first, second and third curvature tensors are respectively given by

\[ H^i_{hjk} = \partial_k G^i_{hj} + G^m_{hj} G^i_{mk} - j/k, \]

\[ S^i_{hjk} = C^i_{mj} C^m_{hk} - j/k, \]

\[ P^i_{hjk} = \partial_k F^i_{hj} - \partial_j C^i_{hk} + F^c_{hj} C^i_{ck} - C^c_{hk} F^i_{cj} + \partial_h N^i_j C^c_{cj}, \]

\[ R^i_{hjk} = \delta_k F^i_{hj} + F^m_{hj} F^i_{mk} - j/k + C^i_{mk} R^m_{hj}, \]

where the indices j and k in the foregoing terms and substraction.

We have the following notations and identities:

1. \( P^i_{jk} = C^i_{jk}/0, \)
2. \( G_{ijk} = -2P_{ijk}, \)
3. \( C^i_{kj} = 2P^i_{kj}, \)
4. \( H^i_{jk} = \delta_k G^i_{j} - j/k, \)
5. \( H^i_{jk} = R^i_{0jk} = H^i_{0jk}, \)
6. \( H^i_{k} = 2\partial_k G^i - \partial_k \partial_k G^i y^h + 2G^i_{ki} y^i - \partial_i G^i \partial_k G^i, \)
7. \( H^i_{k} = \frac{1}{3} (\partial_j H^i_{k}) - j/k, \)
8. \( H^i_{k} = H^i_{0k}, \quad H^i_{jk} = H^i_{jki}, \quad H^i_{i} = (n - 1)H \quad H^i_{0} = 0. \)

The \( ki \)-curvature tensor \( G^i_{hjk} \) satisfies

\[ G^i_{hjk} = G^i_{jkh} = G^i_{khj}, \]

\[ G^i_{0jk} = G^i_{j0k} = G^i_{jko}. \]
Also we have the following:

\[ a) \quad R^i_{jkh}g_{ir} = R_{jkhi}, \]
\[ b) \quad H^i_{jkh}g_{ir} = H_{jkhi}, \]
\[ c) \quad R^i_{jki} = R_{jki}. \]

For any tensor \( T^i_j \), the Ricci identities are given by

\[ T^i_{h:j:k} - T^i_{h:k:j} = H^i_{mjk}T^m_h - H^m_{hjk}T^i_m - H^m_{jk}\partial_m T^i_h, \]
\[ \hat{\partial}_h(T^i_{h:j}) - (\hat{\partial}_h T^i_{h:j}) = C^i_{mjk}T^m_h - C^m_{hjk}T^i_h. \]

The Cartan’s second curvature tensor satisfies the following relations:

\[ a) \quad P_{hijk} = C_{ijk}h + C^m_{hj}P_{mik} - h/i, \]
\[ b) \quad P_{i0jk} = P^i_{jk}, \]
\[ c) \quad P^i_{h0k} = 0, \]
\[ d) \quad P^i_{hj0} = 0. \]

### 1.4 Conformal and C-conformal Change on Finsler Spaces

**Definition 1.4.1.** Let \( F^n = (M^n, L(x,y)) \) and \( \overline{F}^m = (M^m, \overline{L}(x,y)) \) be two Finsler spaces on a same underlying manifold \( M^n \). If the angle in \( F^n \) is equal to that in \( \overline{F}^m \) for any tangent vectors, then \( F^n \) is called conformal to \( \overline{F}^m \) and the change \( L \rightarrow \overline{L} = e^{e(x)L} \) of the metric is called a conformal change.

In other words, the two metrics resulting from \( F^n \) and \( \overline{F}^m \) are called conformal if the corresponding metric tensors \( g_{ij} \) and \( \overline{g}_{ij} \) are proportional to each other. M.S. Knebelman
has proved that the factor of proportionality between them is at most a point function. Thus we have

\[ \bar{g}_{ij} = e^{2\sigma} g_{ij}, \quad \bar{g}^{ij} = e^{-2\sigma} g^{ij}. \]

where \( \sigma = \sigma(x) \).

The Christoffel's symbol of the second kind under the conformal change is given by

\[ \bar{\Gamma}^{h}_{ij} = \gamma^{h}_{ij} + 2\delta^{h}_{j} \sigma_{i} - g^{hk} g_{ij} \sigma_{k}, \]

where \( \sigma_{i} = \frac{\partial \sigma}{\partial x^{i}} \).

Various geometric entities under conformal change are given by:

- \( \bar{C}_{ijk} = e^{2\sigma} C_{ijk}, \quad \bar{C}^{i}_{jk} = C^{i}_{jk}, \quad \bar{C}^{i}_{ik} = C^{i}_{i} = C_{i} \)
- \( \bar{G}^{i} = G^{i} - B^{ir} \sigma_{r}, \quad \sigma_{r} = \partial_{r} \sigma, \quad \bar{h}_{ij} = e^{2\sigma} h_{ij} \)
- \( \bar{G}^{i}_{j} = G^{i}_{j} - B^{jr} \sigma_{r}, \quad \bar{G}^{i}_{jk} = G^{i}_{jk} - B^{jr}_{jk} \sigma_{r}, \quad \bar{G}^{i}_{jkl} = G^{i}_{jkl} - B^{ir}_{jkl} \sigma_{r} \)
- \( \bar{I}^{i} = e^{-\sigma} I^{i}, \quad \bar{I}_{i} = e^{\sigma} I_{i}, \quad \bar{y}_{i} = e^{2\sigma} y_{i} \)
- \( \bar{G}^{i}_{jk} = G^{i}_{jk} - g_{jk} \sigma^{i} + \sigma_{k} \delta_{j}^{i} + \sigma_{j} \delta_{k}^{i} \) \quad \text{(1.4.1)}
- \( \bar{N}^{i}_{j} = N^{i}_{j} - y_{j} \sigma^{i} + \sigma_{0} \delta_{j}^{i} + \sigma_{j} y^{i} \)
- \( \bar{F}^{i}_{jk} = F^{i}_{jk} - g_{jk} \sigma^{i} + \delta_{j}^{i} \sigma_{k} + \delta_{k}^{i} \sigma_{j} + \sigma_{0} C^{i}_{jk} \)
- \( \bar{G}^{i}_{hjk} = G^{i}_{hjk} - 2C_{hjk} \sigma^{i} \)
- \( \bar{A}^{i}_{jk} = e^{\sigma} A^{i}_{jk}, \quad \bar{A}_{ijk} = e^{3\sigma} A_{ijk} \)

M. Hashiguchi [5] introduced the special change called C-conformal change, which is non-homothetic conformal change satisfying

\[ C_{jk}^{i} \sigma^{j} = 0, \] \quad \text{(1.4.2)}

where \( \sigma^{i} = g^{ij} \sigma_{j} \) and \( \sigma_{j} = \partial_{j} \sigma \).

From (1.4.2) and symmetry of lower indices of \( C_{ijk} \), we have

\[ C_{jk}^{i} \sigma^{j} = C_{jk}^{i} \sigma^{k} = C_{jk}^{i} \sigma_{i} = 0. \]
Under the C-conformal change, we have the followings:

\begin{align*}
\text{a)} & \quad \bar{g}_{ij} = e^{2\sigma} g_{ij}, \quad \bar{g}^{ij} = e^{-2\sigma} g^{ij}, \\
\text{b)} & \quad \bar{l}_i = e^{-\sigma} l_i, \quad \bar{l}_i = e^{\sigma} l_i, \quad \bar{h}_{ij} = e^{2\sigma} h_{ij}, \\
\text{c)} & \quad \bar{C}_{ijk} = C_{ijk}, \quad \bar{C}_{jk} = e^{2\sigma} C_{jk}, \quad \bar{C}_i = e^{-2\sigma} C_i, \\
\text{d)} & \quad \bar{\gamma}^i_{jk} = \gamma^i_{jk} + (\sigma_j \delta^i_k + \sigma_k \delta^i_j - g_{jk}\sigma^i), \\
\text{e)} & \quad \bar{G}^i = G^i - \frac{1}{2} L^2 \sigma^i + \sigma_0 y^i, \quad \text{(1.4.3)} \\
\text{f)} & \quad \bar{G}^i_{jk} = G^i_{jk} - g_{jk}\sigma^i + \sigma_k \delta^i_j + \sigma_j \delta^i_k, \\
\text{g)} & \quad \bar{N}^i_j = N^i_j - y_j \sigma^i + \sigma_0 \delta^i_j + \sigma_j y^i, \quad \bar{N}^i = e^{-\sigma} N^i, \\
\text{h)} & \quad \bar{F}^i_{jk} = F^i_{jk} - g_{jk}\sigma^i + \delta^i_j \sigma_k + \delta^i_k \sigma_j + C^i_{jk}\sigma_0, \\
\text{i)} & \quad \bar{G}_{hijk} = G^i_{hjk} - 2C_{hjk}\sigma^i, \\
\text{j)} & \quad \bar{P}_{hijk} = P_{hijk} + \sigma_0 (C^m_{hj}C_{mik} - C^m_{ij}C_{mhk}), \\
\text{k)} & \quad \bar{S}_{hijk} = S_{hijk}. 
\end{align*}

### 1.5 Projective Change on Finsler Spaces

**Definition 1.5.1.** Let $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ be two Finsler spaces on a common underlying manifold $M^n$. If any geodesic on $F^n$ is also a geodesic on $\bar{F}^n$ and the converse is true, then the change $L \rightarrow \bar{L}$ of the metric is called a projectively change.

By a projective change, we have

$$
\bar{G}^i(x,y) = G^i(x,y) - P(x,y)y^i,
$$

where $P(x,y)$ is a Finsler scalar field, which is positively homogeneous of degree one called the projective factor.
By the homogeneity of $P(x, y)$, we have

\begin{align*}
  a) & \quad y^j \partial_i P = P, \\
  b) & \quad y^j \partial_i \partial_j P = 0.
\end{align*}

### 1.6 Subspace of Finsler Spaces

We consider an $n_1$-dimensional Finsler subspace $F^m$ of Finsler space $F^n$ may be parametrically represented by the equation

$$x^i = x^i(u^\alpha),$$

where $\alpha = 1, \ldots, m$ and $u^\alpha$ are the Gaussian coordinates of $F^n$.

Suppose that the matrix of the projection factor $B^i_\alpha = \partial x^i / \partial u^\alpha$ is of rank $m$. The element of the support $X^i$ of $F^m$ is taken to be tangential to $F^m$, i.e., $X^i = B^i_\alpha(u)X^\alpha$.

Thus $X^\alpha$ is the element of support $F^m$ at a point $u^\alpha$. The metric tensor $g_{\alpha\beta}$ and Cartan's C-tensor $C_{\alpha\beta\gamma}$ of $F^m$ are given by

\begin{align*}
  a) & \quad g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta, \\
  b) & \quad C_{\alpha\beta\gamma} = C_{ijk}B^i_\alpha B^j_\beta B^k_\gamma, \quad (1.6.1)
\end{align*}

where $B^i_\alpha B^j_\beta B^k_\gamma = B^i_\alpha B^j_\beta \cdots$.

Since the rank of the matrix $(B^i_\alpha)$ is $m$, it follows that there exists a field of $(n - m)$ linearly independent vectors $N^i_{(\mu)}$ normal to $F^m$ and they are given by the relation

$$g_{ij}N^i_{(\mu)}B^j_\alpha = 0, \quad (\mu = m + 1, \ldots, n).$$

These vectors are normalized by means of relations:

\begin{align*}
  a) & \quad g_{ij}N^i_{(\mu)}N^j_{(\gamma)} = \delta_{\mu\gamma}, \\
  b) & \quad N^i_{(\mu)} = g^{ij}N^j_{(\mu)}.
\end{align*}
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\[ B_i^\alpha B_j^\beta = \delta^\beta_i, \quad B_i^\alpha N_i(\mu) = 0, \quad N_i(\mu) B_i^\alpha = 0, \quad N_\mu N_i(\mu) = 1. \]

Further

\[ B_i^\alpha B_j^\beta + N_i(\mu) N_j(\mu) = \delta^\beta_j. \]

Making use of the inverse matrix \((g_\alpha^\beta)\) of \((g_\alpha^\beta)\), we get

\[ B_i^\alpha = g_\alpha^\beta g_{ij} B_j^\beta, \quad N_i(\mu) = g_{ij} N_j(\mu). \]

1.7 Hypersurface of Finsler Spaces

Finsler hypersurface \(F^{n-1} = (M^{n-1}, L(u, v))\) of a Finsler space \(F^n = (M^n, L(x, y))\) may be parametrically represented by the equation

\[ x^i = x^i(u^\alpha), \]

where \(u^\alpha\) are Gaussian coordinates on \(F^{n-1}\) and Greek indices take values 1 to \(n-1\). The fundamental metric tensor \(g_\alpha^\beta\) and Cartan’s C-tensor \(C_\alpha^\beta\) of \(F^{n-1}\) are given by

\[ a) \quad g_\alpha^\beta(u, v) = g_{ij}(x, \dot{x}) B_i^\alpha B_j^\beta, \]

\[ b) \quad C_\alpha^\beta = C_{ijk} B_i^\alpha B_j^\beta B_k^\gamma, \]

where \(B_i^\alpha = \partial x^i/\partial u^\alpha\) is of rank \((n-1)\). The following notations are also employed

\[ B_i^\alpha = \partial^2 x^i/\partial u^\alpha \partial u^\beta, \quad B_{0\beta} = u^\alpha B_i^\alpha B_{0\beta}, \quad B_{a\beta\gamma\delta\epsilon} = B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta B_m^\epsilon. \]

If the supporting element \(y^i\) at a point \((u^\alpha)\) of \(M^{n-1}\) is assumed to be tangential to \(M^{n-1}\), we may then write

\[ y^i = B_i^\alpha(u) u^\alpha, \]

(1.7.2)
so that $v^\alpha$ is thought of as the supporting element of $M^{n-1}$ at a point $(u^\alpha)$. Since the function $L(u, v) = L(x(u), y(u, v))$ gives rise to a Finsler metric of $M^{n-1}$, we get a $(n-1)$-dimensional Finsler space $F^{n-1} = (M^{n-1}, L(u, v))$.

At each point $(u^\alpha)$ of $F^{n-1}$, the unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}B^i_\alpha N^j = 0, \quad g_{ij}N^i N^j = 1.$$  \hfill (1.7.3)

As for the angular metric tensor $h_{ij}$, we have

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta, \quad h_{ij}B^i_\alpha N^j = 0, \quad h_{ij}N^i N^j = 1.$$  \hfill (1.7.4)

If $(B^\alpha_i, N_i)$ is the inverse matrix of $(B^i_\alpha, N^i)$, we have

$$B^\alpha_i B^i_\alpha = \delta^\alpha_\beta, \quad B^i_\alpha N_i = 0, \quad N^i B^\alpha_i = 0, \quad N^i N_i = 1.$$  

Further

$$B^\alpha_i B^i_\alpha + N^i N_i = \delta^i_j.$$  \hfill (1.7.5)

Making use of the inverse $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^i_\beta, \quad N_i = g_{ij}N^j.$$  

For the induced Cartan's connections $\Gamma^{\alpha\beta\gamma}_\delta = (F^{\alpha}_{\beta\gamma}, N^\alpha_\beta, C^{\alpha}_{\beta\gamma})$ on $F^{n-1}$, the second fundamental h-tensor $H_{\alpha\beta}$ and the normal curvature tensor $H_\alpha$ are given by

\begin{align*}
\text{a)} & \quad H_{\alpha\beta} = N_i(B^i_{\alpha\beta} + F^i_{\beta k}B^j_{\alpha\beta}) + M_\alpha H_\beta, \\
\text{b)} & \quad H_\alpha = N_i(B^i_\alpha + N^j B^j_\alpha)
\end{align*}  \hfill (1.7.6)

respectively, where

$$M_\alpha = C_{ijk}B^i_\alpha N^j N^k \quad \text{and} \quad B^i_\alpha = B^i_{\beta\alpha}v^\beta.$$  \hfill (1.7.7)

Transvecting $H_{\beta\alpha}$ by $v^\beta$, we get

$$H_{\beta\alpha} = H_{\beta\alpha}v^\beta = H_\alpha.$$  \hfill (1.7.8)
Further we put

\[ M_{\alpha\beta} = C_{ijk}B^{ij}_{\alpha\beta}N^k. \]  

(1.7.9)

The relative \( h \) and \( v \)-covariant derivatives of projection factor \( B^i_{\alpha} \) with respect to ICT are given by

\[ B^i_{\alpha|\beta} = H_{\alpha\beta}N^i, \quad B^i_{\alpha|\beta} = M_{\alpha\beta}N^i. \]  

(1.7.10)

The equation (1.7.6) shows that \( H_{\alpha\beta} \) is generally not symmetric and

\[ H_{\beta\gamma} - H_{\gamma\beta} = M_{\beta\gamma}H_{\gamma} - M_{\gamma\beta}H_{\beta}. \]  

(1.7.11)

The above equations yield

\[ H_{\beta\gamma} = H_{\gamma}, \quad H_{\gamma0} = H_{\gamma} + M_{\gamma}H_{0}. \]  

(1.7.12)

The Gauss equation with respect to ICT is written as

\[ R_{\alpha\beta\gamma\delta} = R_{ijkl}B^{ijkl}_{\alpha\beta\gamma\delta} + P_{ijkl}(B^k_{\alpha\beta}H_{\delta} - B^k_{\alpha\delta}H_{\beta})B^{ij}_{\alpha\beta}N^k + (H_{\alpha\gamma}H_{\beta\delta} - H_{\alpha\delta}H_{\beta\gamma}). \]  

(1.7.13)

We use the following notations of Finsler hypersurface:

\[ a) \quad g^{\alpha\beta} = g^{ij}B^{\alpha\beta}_{ij}, \]

\[ b) \quad B^i_{\alpha} = g^{\alpha\beta}g_{ij}B^{ij}_{\beta}, \]

\[ c) \quad C_{\alpha} = B^{i}_{\alpha}C_{i}, \quad C^{\alpha} = B^{i}_{\alpha}C^{i}, \]  

(1.7.14)

\[ d) \quad C^{\alpha}_{\beta\gamma} = B^{i}_{\alpha}C^{i}_{jk}B^{jk}_{\beta\gamma}, \]

\[ e) \quad h_{\alpha\beta} = g_{\alpha\beta} - l_{\alpha}l_{\beta}, \quad h_{\alpha\beta} = h_{ij}B^{ij}_{\alpha\beta}, \]

\[ f) \quad l_{\alpha} = B^{i}_{\alpha}l_{i}. \]

The following are the important Definitions and results in \( F^{n-1} \):

**Definition 1.7.1.** If each path of a hypersurface \( F^{n-1} \) with respect to the induced connection is also a path of the enveloping space \( F^n \), then \( F^{n-1} \) is called a hyperplane of the 1st kind.
**Definition 1.7.2.** If each h-path of a hypersurface $F^{n-1}$ with respect to the induced connection is also a h-path of the enveloping space $F^n$, then $F^{n-1}$ is called a hyperplane of the 2nd kind.

**Definition 1.7.3.** If the unit normal vector of $F^{n-1}$ is parallel along each curve of $F^{n-1}$, then $F^{n-1}$ is called a hyperplane of the 3rd kind.

**Theorem 1.7.1.** The normal curvature $H_0 = H_\alpha v^\alpha$ vanishes if and only if the normal curvature vector $H_\alpha$ vanishes.

**Theorem 1.7.2.** A hypersurface $F^{n-1}$ is a hyperplane of the 1st kind if and only if $H_\alpha = 0$.

**Theorem 1.7.3.** A hypersurface $F^{n-1}$ is a hyperplane of the 2nd kind with respect to the connection $C^T$ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.

**Theorem 1.7.4.** A hypersurface $F^{n-1}$ is a hyperplane of the 3rd kind if and only if $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$.

### 1.8 Special Finsler Spaces

**Definition 1.8.1.** A Finsler space $F^n(n > 2)$ is called a quasi-C-reducible, if the torsion tensor satisfies the equation

$$C_{ijk} = A_{ij} C_k + A_{jk} C_i + A_{ki} C_j,$$

where $A_{ij}$ is a symmetric Finsler tensor field and satisfies $A_{i0} = A_{ij} y^j = 0$.

**Definition 1.8.2.** A Finsler space $F^n(n > 2)$ is said to be C-reducible, if it satisfies the equation

$$(n + 1)C_{ijk} = h_{ij} C_k + h_{jk} C_i + h_{ki} C_j,$$
where $C_i = g^{jk}C_{ijk}$.

**Definition 1.8.3.** A Finsler space $F^n(n > 2)$ with non-zero length $C$ of the torsion vector $C_i$ is said to be semi-$C$-reducible, if the torsion tensor $C_{ijk}$ is of the form

$$C_{ijk} = P(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j)/(n + 1) + qC_iC_jC_k/C^2,$$

where $C_2 = g^{ij}C_iC_j = C_iC^i$ and $p + q = 1$.

**Definition 1.8.4.** A Finsler space $F^n(n > 2)$ with $C^2 = C_iC^i \neq 0$ is called $C^2$-like, if the torsion tensor $C_{ijk}$ satisfies the equation

$$C_{ijk} = C_iC_jC_k/C^2.$$

**Definition 1.8.5.** A Finsler space $F^n(n > 2)$ is $P^2$-like, if it is characterized by

$$P_{hijk} = K_hC_{ijk} - K_iC_{hjk},$$

where $K_h = K_h(x, y)$ is a covariant vector field.

**Definition 1.8.6.** A Finsler space $F^n$ is called a P-reducible, if the torsion tensor $P_{ijk}$ is written as

$$P_{ijk} = (h_{ij}P_k + h_{jk}P_i + h_{ki}P_j)/(n + 1),$$

where $P_i = P_{im} = C_{i/0}$.

**Definition 1.8.7.** A Finsler space $F^n$ is called $S^3$-like, if the curvature tensor $S_{hijk}$ satisfies the equation

$$L^2S_{hijk} = S(h_{hj}h_{ik} - h_{hk}h_{ij}),$$

where the scalar curvature $S = S_{hijk}g^{hj}g^{ik}$ is a function of position alone.
**Definition 1.8.8.** A Finsler space $F^n$ is called $S_4$-like, if the curvature tensor $S_{hijk}$ satisfies the equation

$$L^2 S_{hik} = h^m M_{ik} + h_{ik} M^m_h - h_{hk} M^m_i - h^m M_{hk}.$$  \hfill (1.8.8)

**Definition 1.8.9.** A Finsler space $F^n(n > 2)$ will be called $C^h$-recurrent, if the torsion tensor $C_{ijk}$ satisfies the equation

$$C_{ijk/l} = K_l C_{ijk},$$  \hfill (1.8.9)

where $K_l = K_l(x,y)$ is a covariant vector field.

**Definition 1.8.10.** A Finsler space $F^n(n > 2)$ is said to be $h$-isotropic, if the curvature tensor $R_{hijk}$ is written as

$$R_{hijk} = K(g_{hj} g_{ik} - g_{hk} g_{ij}),$$  \hfill (1.8.10)

where $K$ is a Finsler scalar.

**Definition 1.8.11.** A Finsler space $F^n$ is said to satisfy T-condition, if the completely symmetric T-tensor $T_{hijk}$ given below vanishes identically

$$T_{hijk} = L C_{hijkl} + l_h C_{ijk} + l_i C_{hjk} + l_j C_{hik} + l_k C_{hij}.$$  \hfill (1.8.11)

**Definition 1.8.12.** A Finsler space $F^n$ is said to satisfy generalized T-condition, if the contracted T-tensor given below vanishes

$$T_{ij} = T_{ijrs} g^{rs} = L C_{ij} + l_i C_j + l_j C_i,$$  \hfill (1.8.12)

where $C_i = C_{ijk} g^{jk}$ being the torsion vector.

**Deicke’s theorem:[28]**

**Theorem 1.8.1.** If the torsion vector $A_i = L C_i, C_i = C^r_i$, vanishes identically, the space is Riemannian, provided that the fundamental function $L(x,y)$ is positive and $C^4$-differentiable for any non-zero $y^i$. 

1.9 Finsler Spaces with \((\alpha, \beta)\)-metric

Let \(F^n = (M^n, L)\) be an \(n\)-dimensional Finsler space, that is, an \(n\)-dimensional differential manifold \(M^n\) equipped with a fundamental function \(L(x, y)\). The concept of an \((\alpha, \beta)\)-metric \(L(\alpha, \beta)\) was introduced by M. Matsumoto. A Finsler metric \(L(x, y)\) is called an \((\alpha, \beta)\)-metric \(L(\alpha, \beta)\), if \(L\) is a positively homogeneous function of \(\alpha\) and \(\beta\) of degree one, where \(\alpha^2 = a_{ij}(x)y^iy^j\) is a Riemannian metric and \(\beta = b_i(x)y^i\) is a 1-form on \(M^n\).

The derivative of the \((\alpha, \beta)\)-metric with respect to \(\alpha\) and \(\beta\) are given by

\[
L_\alpha = \frac{\partial L}{\partial \alpha},
L_\beta = \frac{\partial L}{\partial \beta},
L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha},
L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta},
L_{\alpha\beta} = \frac{\partial L_{\alpha\beta}}{\partial \beta},
\]

Then the normalized element of support \(l_i = \hat{\partial}L\) is given by

\[
l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i,
\]

where \(Y_i = a_{ij}y^j\). The angular metric tensor \(h_{ij} = L\hat{\partial}_i\hat{\partial}_jL\) is given by

\[
h_{ij} = p a_{ij} + q_0 b_i b_j + q_1 (b_i Y_j + b_j Y_i) + q_2 Y_i Y_j,
\]

where

\[
p = LL_\alpha \alpha^{-1},
q_0 = LL_{\beta\beta},
q_1 = LL_{\alpha\beta} \alpha^{-1},
q_2 = L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1}).
\]
The fundamental tensor \( g_{ij} = \frac{1}{2} \partial_i \partial_j L^2 \) is given by

\[
g_{ij} = p a_{ij} + p_0 b_i b_j + p_1 (b_i Y_j + b_j Y_i) + p_2 Y_i Y_j,
\]

where

\[
\begin{align*}
p_0 &= q_0 + L^2, \\
p_1 &= q_1 + L^{-1} p L, \\
p_2 &= q_2 + p^2 L^{-2}.
\end{align*}
\]

Moreover, the reciprocal tensor \( g^{ij} \) of \( g_{ij} \) is given by

\[
g^{ij} = p^{-1} a^{ij} + S_0 b^i b^j + S_1 (b^i y^j + b^j y^i) + S_2 y^i y^j,
\]

where

\[
\begin{align*}
b^i &= a^{ij} b_j, \\
S_0 &= (p p_0 + (p_0 p_2 - p_1^2) \alpha^2)/\zeta, \\
S_1 &= (p p_1 + (p_0 p_2 - p_1^2) \beta)/\zeta p, \\
S_2 &= (p p_2 + (p_0 p_2 - p_1^2) b^2)/\zeta p, \\
\zeta &= p(p + p_0 b^2 + p_1 \beta) + (p_0 p_2 - p_1^2)(\alpha^2 b^2 - \beta^2).
\end{align*}
\]

The \( h \)-torsion tensor \( C_{ijk} = \frac{1}{2} \partial_k g_{ij} \) is given by

\[
2p C_{ijk} = p_1 (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \gamma_1 m_i m_j m_k,
\]

where

\[
\gamma_1 = p \frac{\partial p_0}{\partial \beta} - 3 p_1 q_0, \\
m_i = b_i - \alpha^{-2} \beta Y_i.
\]

Note that the covariant vector \( m_i \) is a non-vanishing one and it is orthogonal to the element of support \( y^i \).
Let \( \left\{ \begin{array}{c} i \\ jk \end{array} \right\} \) be the components of Christoffel’s symbol of the associated Riemannian space \( R^n \) and \( \nabla_k \) be covariant differentiation with respect to \( x^k \) relative to this Christoffel’s symbol. We shall use the following tensors,

\[
2E_{ij} = b_{ij} + b_{ji}, \quad 2F_{ij} = b_{ij} - b_{ji},
\]

where \( b_{ij} = \nabla_j b_i \).

If we denote the Cartan’s connection \( C \Gamma \) as \( (\Gamma^i_{jk}, \Gamma^i_{0k}, C^i_{jk}) \), then the difference tensor \( D^i_{jk} = \Gamma^i_{jk} - \left\{ \begin{array}{c} i \\ jk \end{array} \right\} \) of the special Finsler space is given by

\[
D^i_{jk} = B^iE_{jk} + F^i_jB_j + F^i_kB_k + B^i_jb_{0k} + B^i_kb_{0j}
- b_{0m}g^{mn}B_{jk} - C^i_{jm}A^m_k - C^i_{km}A^m_j + C_{jkm}A^m_n g^{ns} + \lambda^s(C^i_{jm}C^m_{sk} + C^i_{km}C^m_{sj} - C^i_{jk}C^i_{ms}),
\]

where

\[
B_k = p_0b_k + p_1Y_k, \quad b^j = g^{ij}B_j, \quad F^k_i = g^{kj}F_{ji},
B_{ij} = \left\{ p_1(a_{ij} - \alpha^{-2}Y_iY_j) + \frac{\partial p_0}{\partial \beta}m_im_j \right\} / 2,
B^k_i = g^{kj}B_{ji},
A^m_k = b^m_kE_{00} + B^mE_{k0} + B_kF^m_0 + B_0F^m_k,
\lambda^m = B^mE_{00} + 2B_0F^m_0, \quad B_0 = B_1y^1.
\]

Here and in the following we denote \( 0 \) as contraction with \( y^i \) except for the quantities \( p_0, q_0 \) and \( S_0 \).
From the differential 1-form $\beta(x, y) = b_i(x) y^i$, we define

\begin{align*}
a) & \quad 2r_{ij} = b_{k,j} + b_{j,i}, \\
b) & \quad 2s_{ij} = b_{vi,j} - b_{j,i} = (\partial_j b_i - \partial_i b_j), \\
c) & \quad s^i_j = a^{ir} s_{rj}, \\
d) & \quad b^i = a^{ir} b_r, \\
e) & \quad b^2 = a^{rs} b_r b_s. \\
\end{align*}