CHAPTER 1
Chapter 1

Introduction and basic concepts

1.1 Introduction to Fluid Mechanics

1.1.1 Introduction

Fluid Mechanics is a science which deals with the behavior of fluids when subjected to a system of forces and involves the study of kinematics, dynamics and statics of fluids. The term fluid can be defined as substance which deforms continuously due to shear stress. The fluid dynamics involves the problems of fluid flow in different physical conditions in several fields e.g. space science, aerodynamics, rocket propulsion, ship motion on water, power generation by nuclear reactors and hydroelectric power generation etc.

The applications of fluid flow are very wide as it plays a very important role in the industries of steel, plastic, electric wire, glasses etc. It also helps in maintaining the temperature of computer chips, vehicle engines and high power machines. In our daily life, lubrication involves the presence of a thin liquid layer that greatly reduces friction and can eliminate squeaks in door hinges, make wheels to turn more easily and prevent engine parts from rubbing each other and thus save them from destruction and energy loss.
Under different conditions, the knowledge of fluid motion helps us to solve many types of problems in dairy-plants, solar science, motion of earth core, soil erosion and sedimentation, meteorology, cosmic fluid dynamics, flood control, water purification system, oil refining, environmental pollution, ocean science, shock waves, drainage system and biotechnology etc. In observing the operation of cyclones, swirl-burners and scrubbers, the knowledge of aerosols in curvilinear fluid flow is very necessary. As most of the fluids flow in magnetic field are involving multiphase flow, so we have considered dusty incompressible fluid flowing through different types of channels placed in a transversely applied magnetic field. Thus observing the wide applications of fluid flow, it is necessary to thrust on new researches of this area for its better applications in the above sectors. So we have concentrated our study on the flow of dusty incompressible fluid through the channels viz. triangular, co-axial cylindrical, and parallel plates placed in a magnetic field.

**Compressible and Incompressible fluid:** Fluids undergo density changes when temperature and pressure variations occur in them. They are consider compressible when density changes are significant. For several flows situations, however density changes are negligible and the fluid may be treated as incompressible. Therefore, flow of liquids are treated as incompressible for small pressure and temperature variations.

**Newtonian and Non-Newtonian fluid:** A fluid which obeys Newton's law of velocity is known as Newtonian fluid, and the fluid which doesn't obeys the Newton's law of viscosity is known as Non-Newtonian fluid.
Some Important Types of Flow:

**Steady and Unsteady flow:** A flow in which properties and conditions of fluid motions do not change with change of time such that the flow pattern is not affected with time is called steady, i.e. $\frac{\partial \phi}{\partial t} = 0$, where $\phi$ may be velocity, temperature, pressure, density etc. On the other hand, if flow pattern depends upon time, it is called unsteady.

**Laminar and Turbulent Flow:** In a laminar flow, every fluid particle traces out a definite curve and the curves traced out by any two different particles never intersect and in turbulent flow, every fluid particle does not trace out a definite curve and the curves traced out by the particles intersect.

**Rotational and Irrotational Flow:** The flow in which the fluid particles rotate about their own axis is called rotational and the flow in which the fluid particle does not rotate about their own axis is called irrotational.

**Uniform and Non-uniform Flow:** A flow in which the velocities of fluid particles are equal at each section of the channel is called uniform and a flow in which the velocities of fluid particles are different at each section of the channel is called non-uniform.

**Magnetohydrodynamic Flow (MHD):** The flow of electrically-conducting fluid under influence of applied magnetic field is called MHD flow.

**Multiphase Fluid Flow:** It is a commingled flow of different phase fluids, such as water, oil and gas. Multiphase fluid flow is complex which is used in optimizing production in both oil and gas wells. Four multiphase fluid flow regimes are recognized while describing flow in oil and gas wells, bubble flow, slug flow, transition flow and mist flow.
No-Slip Condition of Viscous Fluid:

When a viscous fluid flows over a solid surface, the fluid elements adjacent to the surface attain the velocity of the surface; in other words, the relative velocity between the solid surface and the adjacent fluid particles is zero. This phenomenon is known as the 'no-slip' condition.

Viscosity: Viscosity is a measure of the resistance of a fluid which is being deformed by either shear stress or tensile stress. Generally speaking the term viscosity is thickness of the fluid. Thus, water is “thin”, having a lower viscosity, while honey is “thick”, having a higher viscosity. Viscosity describes a fluid's internal resistance to flow and may be thought of as a measure of fluid friction. For example, high-viscosity magma will create a tall, steep stratovolcano, because it cannot flow far before it cools, while low-viscosity lava will create a wide, shallow-sloped shield volcano. All real fluids except super fluids have some resistance to stress, but a fluid which has no resistance to shear stress is known as an ideal fluid or inviscid fluid.

Plane Poiseuille's Flow:

A steady laminar flow of a viscous incompressible fluid in the space between two infinite stationary surfaces in the absence of external force. The flow is due to constant pressure gradient. This type of flow is called as Plane Poiseuille's flow.

Generalized Couette Flow:

Generalized Couette flow refers to a steady laminar flow of a viscous incompressible fluid in the space between two surfaces, one of which is moving relative to the other and also the pressure gradient is non-zero. This type of flow is named in honor of Maurice
Marie Alfred Couette, a Professor of Physics at the French university of Angers in the late 19th century.

**Dimensionless Parameters:**

In dimensional analysis, a dimensionless quantity (or more precisely, a quantity with the dimension of 1) is a quantity without any physical unit and thus a pure number. Such a number is typically defined as a product or ratio of quantities which do have units, in such a way that all units cancel. The dimensionless parameters, which appear in the resulting equations, are the parameters of solutions and are the key factors in determining the qualitative and quantitative nature of the flow phenomenon.

**Reynolds Number:**

The Reynolds number is the ratio of inertial forces to viscous forces and consequently it quantifies the relative importance of these two types of forces for given flow conditions. Thus, it is used to identify different flow regimes, such as laminar or turbulent flow.

It is one of the most important dimensionless numbers in fluid dynamics and is used, usually along with other dimensionless numbers, to provide a criterion for determining dynamic similitude. It is named after Osborne Reynolds (1842-1912), who first introduced this number while discussing boundary layer theory in 1883. Typically it is given as follows:

\[ Re = \frac{\rho U^2 h}{\mu U / h^2} = \frac{\rho U h}{\mu} = \frac{U h}{\nu} \]

where \( U \) is some characteristic velocity, \( h \) is some characteristic length and \( \nu = \frac{k}{\rho} \) is kinematic fluid viscosity and \( \rho \) fluid density.
Hartmann Number: Hartmann number is the ratio of electromagnetic force to the viscous force. It was first introduced by Hartmann and is defined as:

\[ H_a = BL \sqrt{\frac{\sigma}{\mu}} \]

where

\( B \) - the magnetic field,
\( L \) - the characteristic length scale,
\( \sigma \) - the electrical conductivity,
\( \mu \) - the viscosity.

Prandtl Number (Pr): Prandtl number is a dimensionless number which is defined as the ratio of momentum diffusivity (kinematic viscosity) to thermal diffusivity.

Number Density: Number density is an intensive quantity used to describe the degree of concentration of countable objects in the three-dimensional physical space, or Number density is the number of specified objects per volume i.e., \( n = N/V \).

Ekman Number: It is the ratio of viscous forces in a fluid to the fictitious forces arising from planetary rotation.

Radiation parameter: Radiation is a process in which energetic particles or energy or waves travel through a medium or space. There are two distinct types of radiation; ionizing and non-ionizing. The word radiation is commonly used in reference to ionizing radiation only, but it may also refer to non-ionizing radiation (e.g., radio waves or visible light). The energy radiates (i.e., travels outward in straight lines in all directions) from its source. This geometry naturally leads to a system of measurements and physical units that are equally applicable to all types of radiation.
**Hall effect:** The Hall effect is the production of a voltage difference (the Hall voltage) across an electrical conductor, transverse to an electric current in the conductor and a magnetic field perpendicular to the current.

**Shear stress/Skin Friction:**

Any real fluids (liquids and gases include) moving along solid boundary will incur a shear stress on that boundary. The no-slip condition dictates that the speed of the fluid at the boundary (relative to the boundary) is zero, but at some height from the boundary the flow speed must equal that of the fluid. The region between these two points is aptly named the boundary layer. For all Newtonian fluids in laminar flow the shear stress is proportional to the strain rate in the fluid, where the viscosity is the constant of proportionality. However for Non Newtonian fluids, this is no longer the case as for these fluids the viscosity is not constant. The shear stress is imparted onto the boundary as a result of this loss of velocity. The shear stress, for a Newtonian fluid, at a surface element parallel to a flat plate, at the point $y$, is given by:

$$
\tau(y) = \mu \frac{\partial u}{\partial y}
$$

where

- $\mu$ is the dynamic viscosity of the fluid,
- $u$ is the velocity of the fluid along the boundary, and
- $y$ is the height of the boundary.
Specifically, the wall shear stress is defined as:

$$\tau_w \equiv \tau(y = 0) = \mu \frac{\partial u}{\partial y} \bigg|_{y=0}.$$ 

In case of wind, the shear stress at the boundary is called wind stress.

1.2 Boundary-layer theory:

At the beginning of the 20th century the Ludwig Prandtl introduced the concept of boundary-layer theory. He showed that the flow past a body can be divided into two regions: a very thin layer close to the body (boundary-layer) where the viscosity is important, and the remaining region outside this layer where the viscosity can be neglected.

In physics and fluid mechanics, a boundary layer is that layer of fluid in the immediate vicinity of a bounding surface where effects of viscosity of the fluid are considered in detail. The boundary layer effect occurs at the field region in which all changes occur in the flow pattern. The boundary layer distorts surrounding non-viscous flow. It is a phenomenon of viscous forces. This effect is related to the Reynolds number.

Initially boundary-layer theory was developed mainly for the laminar flow of an incompressible fluid. The theory was extended to the practically important turbulent incompressible boundary-layer flow. One of the most important applications of the boundary-layer theory is the calculation of the friction drag of bodies in a flow. In the Earth’s atmosphere, the planetary boundary-layer is the air layer near the ground affected by diurnal heat, moisture or momentum transfer to or from the surface. On an aircraft wing the boundary-layer is the part of the flow close to the wing. In Naval architecture, many of the principles that apply to aircraft also apply to ships and submarines.
1.3 Frenet frame field system:

The tools of differential geometry and, in particular, Riemannian geometry have been proved very useful in handling problems in areas of Fluid dynamics. Differential geometry of curves and surfaces has appeared in several areas of physics, ranging from liquid crystals to plasma physics, and from solutions to general relativity, or even in high energy strings and thermodynamics. In all these applications one important common feature arises, which is the application of the Serret-Frenet frame to the motion of curves.

Frenet frames are a central construction in modern differential geometry, in which structure is described with respect to an object of interest rather than with respect to external coordinate systems.

They are the formulae, which help to measure the turning and twisting of a curve in an Euclidean Space $E^3$. If $\beta : I \to E^3$ is a unit-speed curve then $\vec{s} = \beta'$ is a unit tangent vector to the curve $\beta$. Since $\vec{s}$ has constant length 1, its derivative $\vec{s}' = \beta''$ measures the way the curve is turning $E^3$. We call $s'$ the curvature vector field of $\beta$. The length of curvature vector field $s'$ gives a numerical measurement of the turning of $\beta$. The real-valued function $k$ such that $k(t) = ||s'(t)||$ for all $t$ in $I$ is called the curvature function of $\beta$. Thus $k \geq 0$, and the larger $k$ is, the sharper the turning of $\beta$. Now we impose that $k$ is never zero, so $k > 0$. Then the unit-vector field $\vec{n} = \frac{\vec{s}'}{k}$ on $\beta$ tells the direction in which $\beta$ is turning at each point. $\vec{n}$ is called the principal normal vector field of $\beta$. 
The vector field \( \mathbf{b} = \mathbf{s}' \times \mathbf{n} \) on \( \beta \) is then called the binormal vector field of \( \beta \). These \( \mathbf{s}' \), \( \mathbf{n} \), \( \mathbf{b} \) on \( \beta \) are unit-vector fields and are mutually orthogonal at each point. We call \( \mathbf{s}' \), \( \mathbf{n} \), \( \mathbf{b} \) the Frenet frame field on \( \beta \). \((\mathbf{s}', \mathbf{n})\) is the osculating plane of the curve and \((\mathbf{n}, \mathbf{b})\) is the normal plane (see Fig. 1.1).

In 1951 Serret expressed the derivatives \( \mathbf{s}', \mathbf{n}', \mathbf{b}' \) in terms of \( \mathbf{s}', \mathbf{n}, \mathbf{b} \) as

\[
\mathbf{s}' = k \mathbf{n}, \quad \mathbf{n}' = -k \mathbf{s}' + \tau \mathbf{b} \quad \text{and} \quad \mathbf{b}' = -\tau \mathbf{b}.
\]

where \( k \) and \( \tau \) are the curvature and torsion of the curve. In 1952 Frenet independently proved the same results. Hence they are named as Serret-Frenet formulae.

This system is superior to the other system such as Cartesian co-ordinate system or Curvilinear co-ordinate system because in the usual co-ordinate system the solution of the flow will be obtained in the space prescribed whereas Frenet frame system gives the description of the flow from point to point in the region of flow.
Geometrical relations are given by Frenet formulae [10]

\[ \begin{align*}
\text{i}) \quad & \frac{\partial \vec{n}}{\partial s} = k_s \vec{n}, \quad \frac{\partial \vec{b}}{\partial s} = \tau_s \vec{s} - k_s \vec{n}, \quad \frac{\partial \vec{b}}{\partial s} = -\tau_s \vec{n} \\
\text{ii}) \quad & \frac{\partial \vec{n}}{\partial n} = k_n \vec{s}, \quad \frac{\partial \vec{b}}{\partial n} = -\sigma_n \vec{s}, \quad \frac{\partial \vec{s}}{\partial n} = \sigma_n \vec{n} - k_n \vec{n} \\
\text{iii}) \quad & \frac{\partial \vec{b}}{\partial b} = k_b \vec{s}, \quad \frac{\partial \vec{n}}{\partial b} = -\sigma_b \vec{s}, \quad \frac{\partial \vec{s}}{\partial b} = \sigma_b \vec{n} - k_b \vec{b} \\
\text{iv}) \quad & \nabla \cdot \vec{s} = \theta_{ns} + \theta_{bs}, \quad \nabla \cdot \vec{n} = \theta_{bn} - k_s, \quad \nabla \cdot \vec{b} = \theta_{nb}
\end{align*} \]

(1.3.1)

where \(\partial/\partial s, \partial/\partial n\) and \(\partial/\partial b\) are the intrinsic differential operators along fluid phase velocity (or dust phase velocity) lines, principal normal and binormal. The functions \((k_s, k_n, k_b)\) and \((\tau_s, \sigma_n, \sigma_b)\) are the curvatures and torsions of the above curves. \(\theta_{ns}\) and \(\theta_{bs}\) are normal deformations of these spatial curves along their principal normal and binormal respectively and they are given by

\[ \theta_{ns} = \vec{n} \cdot \frac{\partial \vec{s}}{\partial n}, \quad \theta_{bs} = \vec{b} \cdot \frac{\partial \vec{s}}{\partial b}, \]

and the gradient operator is

\[ \nabla = \frac{\partial}{\partial s} \vec{s} + \frac{\partial}{\partial n} \vec{n} + \frac{\partial}{\partial b} \vec{b}. \]

In Frenet frame field system the operators Gradient, Divergence, Curl and Laplacian
are defined as

\[ \text{Grad } \phi = \nabla \phi = \frac{\partial \phi}{\partial s} \vec{s} + \frac{\partial \phi}{\partial n} \vec{n} + \frac{\partial \phi}{\partial b} \vec{b}, \]

\[ \text{Div } \vec{u} = \nabla \cdot \vec{u} = \frac{\partial u_s}{\partial s} + \frac{\partial u_n}{\partial n} + \frac{\partial u_b}{\partial b}, \]

\[ \text{Curl } \vec{u} = \nabla \times \vec{u} \left[ \begin{array}{ccc} \vec{s} & \vec{n} & \vec{b} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial n} & \frac{\partial}{\partial b} \\ u_s & u_n & u_b \end{array} \right], \]

\[ \nabla^2 = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial n^2} + \frac{\partial^2}{\partial b^2}, \]

where \( \vec{u} = u_s \vec{s} + u_n \vec{n} + u_b \vec{b} \) and \( \phi \) is a scalar function.

### 1.4 Laplace Transforms:

Laplace Transform is an essential mathematical tool which can be used to solve several problems in science and engineering. This technique becomes popular when Heaviside function applied to the solution of an ODE representing a problem in electrical engineering. Transforms are used to accomplish the solution of certain problems with less effort and in a simple routine way. The Laplace transform method reduces the solution of an ODE to the solution of an algebraic equation. Also, when the Laplace transform technique is applied to a PDE, it reduces the number of independent variables by one.

**Definition 1.** Let \( f(t) \) be a continuous and single-valued function of the real variable \( t \) defined for all \( t, 0 < t < \infty \), and is of exponential order. Then the Laplace transform of
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$f(t)$ is defined as a function $F(t)$ denoted by the integral

$$L[f(t)] = \int_0^\infty e^{-st} f(t) dt.$$  \hspace{1cm} (1.4.1)

**Error Function:**

The error function is defined as,

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$  \hspace{1cm} (1.4.2)

and $$\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$  \hspace{1cm} (1.4.3)

The Laplace transform of the error function is,

$$L[\text{erf}(x)] = \frac{1}{s} e^{s^2/4} \text{erfc}(s/2).$$

Some of the inverse transforms are,

$$L^{-1} \left\{ \frac{e^{(-k \sqrt{s+a})}}{s + a} \right\} = \frac{e^{-at}}{2} \left[ e^{-k \sqrt{\alpha-a}} \text{erfc}(\phi_1) + e^{-k \sqrt{\alpha-a}} \text{erfc}(\phi_2) \right],$$

where $$\phi_1 = \frac{k}{2 \sqrt{t}} - \sqrt{\alpha - a} t, \quad \phi_2 = \frac{k}{2 \sqrt{t}} + \sqrt{\alpha - a} t,$$

$$L^{-1} \left\{ \frac{e^{(-k \sqrt{s+a})}}{(s + a)(s + b)} \right\} = \frac{k}{2 \sqrt{\pi} (b - a)} \left[ e^{-at} I(a, t) - e^{-bt} I(b, t) \right],$$
where \( I(x, t) = \int_0^t \tau^{-3/2} \exp \left[ \frac{-k^2}{4\tau} - (\alpha - x)\tau \right] d\tau \)

and \( x=a \) or \( b \). On integration then the above inverse Laplace transform is,

\[
L^{-1} \left\{ \frac{e^{-k\sqrt{s+a}}}{(s + a)(s + b)} \right\} = \frac{1}{b - a} \left[ T(a, t) - T(b, t) \right],
\]

where \( T(x, t) = \frac{e^{-xt}}{2} \left[ e^{-(k\sqrt{\alpha-x})} \text{erfc} \left( \frac{k}{2\sqrt{t}} - \sqrt{(\alpha - x)t} \right) \right.
+ \left. e^{(k\sqrt{\alpha-x})} \text{erfc} \left( \frac{k}{2\sqrt{t}} + \sqrt{(\alpha - x)t} \right) \right],
\]

\[
L^{-1} \left\{ \frac{e^{-k\sqrt{s+a}}}{(s + a)(s + b)(s + c)} \right\} = -\sum \frac{T(a, t)}{(c - a)(a - b)},
\]

\[
L^{-1} \left\{ \frac{e^{-k\sqrt{s+a}}}{(s + a)} \right\} = e^{-at} \text{erfc} \left( \frac{k}{2\sqrt{t}} \right),
\]

\[
L^{-1} \left\{ \frac{e^{-k\sqrt{s+a}}}{(s + a)(s + a)} \right\} = \frac{T(a, t)}{(\alpha - a)} - \frac{e^{-at}}{(\alpha - a)} \text{erfc} \left( \frac{k}{2\sqrt{t}} \right),
\]

\[
L^{-1} \left\{ \frac{e^{-k\sqrt{s+a}}}{(s + a)^2} \right\} = \frac{e^{-at}}{2} \left[ \left( t - \frac{k}{2\sqrt{\alpha-a}} \right) e^{-k\sqrt{\alpha-a}} \text{erfc}(\phi_1) \right.
+ \left. \left( t + \frac{k}{2\sqrt{\alpha-a}} \right) e^{k\sqrt{\alpha-a}} \text{erfc}(\phi_2) \right].
\]
1.4.1 Complex Inversion Formula/Mellin-Fourier integral:

In solving partial differential equations using Laplace transform method, complex variable theory may come in handy for finding inverse transform. Inverse Laplace transform can be expressed as an integral which is known as inverse integral and this integral can be evaluated by using contour integration methods.

The inverse Laplace Transforms of $U$, $V$ are $u$, $v$ respectively and are given by the integrals

$$u = \frac{1}{2i\pi} \int_{r-i\infty}^{r+i\infty} e^{zt}Udt \quad \text{and} \quad v = \frac{1}{2i\pi} \int_{r-i\infty}^{r+i\infty} e^{zt}Vdt$$ (1.4.4)

Which can be evaluated by means of contour integration. Since there is no branch point, the contour chosen is the closed curve ABC formed by the line $x = r$ and a semi circle C with origin as center and radius $R$ (See figure 1.2) so that

Figure 1.2: Cantour formed by line $x = r$ and a semi-circle $C$

with origin as center and radius $R$. 


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\[ \int_{r-i\infty}^{r+i\infty} e^{zt} dt = \lim_{R \to \infty} \int_{A}^{B} e^{zt} dt \]

\[ = \lim_{R \to \infty} \left[ \int_{A}^{B} e^{zt} dt - \int_{C} e^{zt} dt \right] \]

Using Cauchy's theorem of residues and Jordan's lemma, we have

\[ u = \frac{1}{2\pi} \int_{r-i\infty}^{r+i\infty} e^{zt} dt = \text{sum of residues of } \left\{ e^{zt} \right\} \text{ at its poles} \]

Similarly,

\[ v = \frac{1}{2\pi} \int_{r-i\infty}^{r+i\infty} e^{zt} dt = \text{sum of residues of } \left\{ e^{zt} \right\} \text{ at its poles.} \]

Some Laplace transform formulae are to be listed.

<table>
<thead>
<tr>
<th>Sl.No.</th>
<th>Function</th>
<th>Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{e^{-a\sqrt{t}}}{\sqrt{\pi t}} )</td>
<td>( \frac{e^{-a\sqrt{s}}}{\sqrt{s}} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{ae^{-a^2/4t}}{2\sqrt{\pi t^3}} )</td>
<td>( e^{-a\sqrt{s}} )</td>
</tr>
<tr>
<td>3</td>
<td>( \text{erfc} \left( \frac{a}{2\sqrt{t}} \right) )</td>
<td>( \frac{e^{-a\sqrt{s}}}{s} )</td>
</tr>
<tr>
<td>4</td>
<td>( 2\sqrt{\frac{1}{\pi}} e^{-a^2/4t} - a \text{ erfc} \left( \frac{a}{2\sqrt{t}} \right) )</td>
<td>( \frac{e^{-a\sqrt{s}}}{\sqrt{s}\sqrt{s+b}} )</td>
</tr>
<tr>
<td>5</td>
<td>( e^{at}e^{bt} \text{ erfc} \left( b\sqrt{t} + \frac{a}{2\sqrt{t}} \right) )</td>
<td>( \frac{e^{-a\sqrt{s}}}{\sqrt{s}\sqrt{s+b}} )</td>
</tr>
<tr>
<td>6</td>
<td>( e^{at}e^{bt} \text{ erfc} \left( b\sqrt{t} + \frac{a}{2\sqrt{t}} \right) + \text{erfc} \left( \frac{a}{2\sqrt{t}} \right) )</td>
<td>( \frac{be^{-a\sqrt{s}}}{\sqrt{s}\sqrt{s+b}} )</td>
</tr>
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</table>
1.5 Numerical Methods:

Numerical methods are the way to do higher mathematics problems on a computer, a technique widely used by scientists and engineers to solve their problems. A major advantage for numerical analysis is that a numerical answer can be obtained even when a problem has no "analytical" solution. It is important to realize that a numerical analysis solution is always numerical. Analytical methods usually give a result in terms of mathematical functions that can then be evaluated for specific instances. There is thus advantage to the analytical results, in that the behavior and properties of the function are often apparent. However, numerical results can be plotted to show some of the behavior of the solution.

Another important distinction is that the result from numerical method is an approximation, but results can be made as accurate as desired. To achieve high accuracy many, many separate operations must be carried out. Here are some of the operations that numerical methods can do:

- Solve for the roots of a nonlinear equation.
- Solve large systems of linear equations.
- Get the solutions of a set of nonlinear equations.
- Interpolate to find intermediate values within a table of data.
- Solve ODE when given initial values for the variables.
- Solve boundary-value problems and determine eigenvalues and eigenvectors.
- Obtain numerical solutions to all types of partial differential equations and so on.
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In connection with numerical analysis many symbolic algebraic programmes are available, namely Mathematica, DERIVE, Maple, MathCad, MATLAB, and MacSyma.

The MATLAB PDE solver, \( pdepe \) solves initial-boundary value problems for systems of parabolic and elliptic PDEs in the one space variable \( x \) and time \( t \). There must be at least one parabolic equation in the system. The \( pdepe \) solver converts the PDEs to ODEs using a second-order accurate spatial discretization based on a fixed set of user-specified nodes. The discretization method is described in [103].

1.5.1 Finite Difference Method:

The Finite-difference method provide a powerful approach to solve differential equations and are widely used in any field of applied science. Equations with variable coefficients and even non linear problems can be treated by these techniques. Generally the error of an approximating solution can be made arbitrary small. Rounding errors, which inevitably arise during the computational process, can be controlled by a preliminary analysis of the numerical stability of finite difference schemes. Further, more numerical solutions can give suggestions to more general equations.

Function discretization:

Let us consider a function \( u(x, t) \) depending on two variables \( x \in [0, L] \) and \( t \in [0, T] \). A discretization of a function \( u \) is obtained by considering only the values \( u_{i,j} \) on finite number of points \((x_i, t_j)\).
Finite Difference approximation of derivatives:

- Forward difference approximation for the partial derivative of $u$ with respect to $t$ is,

$$ (u_t)_{i,j} \approx \frac{u_{i,j+1} - u_{i,j}}{\Delta t}. $$

- Forward difference approximation for the partial derivative of $u$ with respect to $x$ ,

$$ (u_x)_{i,j} \approx \frac{u_{i+1,j} - u_{i,j}}{\Delta x}. $$

- Backward difference approximation for the partial derivative of $u$ with respect to $t$ ,

$$ (u_t)_{i,j} \approx \frac{u_{i,j} - u_{i,j-1}}{\Delta t}. $$

- Backward approximation for the partial derivative of $u$ with respect to $x$ ,

$$ (u_x)_{i,j} \approx \frac{u_{i,j} - u_{i-1,j}}{\Delta x}. $$

- Central difference approximation for the partial derivative of $u$ with respect to $t$ and $x$ respectively as ,

$$ (u_t)_{i,j} \approx \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta t}, $$

$$ (u_x)_{i,j} \approx \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}. $$

- Second order finite difference approximation with respect to $x$ and $t$ respectively as ,

$$ (u_{xx})_{i,j} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2}, $$

$$ (u_{tt})_{i,j} \approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta t)^2}. $$
1.6 Vector Identities:

In various applications of vector analysis we shall have the occasion to use the following
vector identities frequently;

1) \( \nabla (\phi + \varphi) = \nabla \phi + \nabla \varphi \)

2) \( \nabla (\phi \varphi) = \varphi \nabla \phi + \phi \nabla \varphi \)

3) \( \nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B} \)

4) \( \nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B} \)

5) \( \nabla \left( \vec{A} \cdot \vec{B} \right) = \vec{A} \times (\nabla \times \vec{B}) + (\vec{A} \cdot \nabla) \vec{B} + \vec{B} \times (\nabla \times \vec{A}) + (\vec{B} \cdot \nabla) \vec{A} \)

6) \( \nabla \cdot (\phi \vec{A}) = \phi \nabla \cdot \vec{A} + \vec{A} \cdot \nabla \phi \)

7) \( \nabla \times (\phi \vec{A}) = \phi (\nabla \times \vec{A}) + (\nabla \phi) \times \vec{A} \)

8) \( \nabla \cdot (\nabla \times \vec{A}) = 0 \)

9) \( \nabla \times (\nabla \phi) = 0 \)

10) \( \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \)

11) \( \nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} + \vec{A} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \vec{A}) \)

12) \( \nabla \times \nabla \times \vec{A} = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \)

13) \( \nabla \times f(\phi) = \frac{df}{d\phi} \nabla \phi. \)