Publications


5. B.J.Gireesha, G.K.Ramesh, C.S.Bagewadi and Mahesha,  *Flow of an unsteady dusty fluid through a channel having triangular cross-section in frenet frame field system under varying pulsatile pressure gradient*, COMMUNICATED.

7. C.S.Bagewadi, B.J.Gireesha, Mahesha and G.S.Roopa, *Effect of radiation on hydromagnetic flow and heat transfer of a dusty fluid between two parallel plates*, COMMUNICATED.
REPRINTS
Flow of an Unsteady Conducting Dusty Fluid between a Non-torsional Oscillating Plate and a Long Wavy Wall

Mahesha, B.J. Gireesh*, G.K. Ramesh and C.S. Bagewadi

Department of Mathematics, Kuvempu University, Shankaraghatta-577 451, Shimoga, Karnataka, India.
*Corresponding Author E-mail: bjgireeshu@rediffmail.com

Abstract

The geometry of flow of a dusty viscous conducting fluid between a non-torsional oscillating parallel plate and a long wavy wall in an anholonomic coordinate system has been studied. The velocity distribution of fluid and dust for different pressure gradients is obtained analytically. The effect of strength of magnetic field on velocity profiles at fixed time has been discussed with the help of graphs. Finally the skin fraction at the boundaries is calculated.

Keywords: Frenet frame field system; non-torsional oscillating plate, long wavy wall, conducting dusty fluid; velocity of dust phase and fluid phase, magnetic field, laminar flow, skin friction.

AMS Subject Classification (2000): 76T10, 76T15;

Introduction

The study of the flow of dusty fluids has attracted many researchers to his applications in the fields of fluidization, combustion, use of dust in gas cooling systems, centrifugal separation of matter from fluid, petroleum industry, purification of crude oil, electrostatic precipitation, polymer technology, high-energy solid rocket propellant, fluid-droplet sprays, the electrostatic precipitation of dust, blood flow fluid droplets sprays and so on.

The importance of study of fluids having non-conducting fine dust particles was traced out by many mathematicians. In 1962 Saffman [16] has given the equations describing the motion of gas containing the small dust particles. On basis of these equations Rossow [15] studied the flow of a viscous, incompressible and electrically
conducting fluid in presence of an external magnetic field due to the impulsive motion of an infinite flat plate.

Srinivasacharya, G. Radhakrishnamacharya and Ch. Srinivasulu [19] has discussed the effects of wall properties on peristaltic transport of a dusty fluid, Liu [11], Michael and Miller [13] have studied the flow produced by the motion of an infinite plane in a steady fluid occupying the semi-infinite space above it. A. Eric et al. [5] has studied the quantitative assessment of steady and pulsatile flow fields in a parallel plate flow chamber. Thierry Feraille et al. [17] discussed the channel flow induced by wall injection of fluid and particles.

During the second part of the 20th century, some researchers like Kanwal [10], Truesdell [18], Indrasena [9], Purushotham [14], Bagewadi and Gireesha [1], [2] have applied differential geometry techniques to investigate the kinematical properties of fluid flows in the field of fluid mechanics. Further, recently the authors [6], [7], have studied two-dimensional dusty fluid flow in Frenet frame field system. The paper deals with the study of flow of an electrically conducting viscous incompressible fluid which suspended non-conducting small spherical dust particles between a non-torsional oscillating plate and a long wavy wall. The flow is due to the presence of a uniform transverse magnetic field, non-torsional oscillations of the plate and time dependent pressure gradient. Initially it is assumed that both the conducting fluid and the non-conducting dust particles are to be at rest. Applying Laplace transform technique, the velocity fields for fluid and dust particles have been obtained. Also the skin friction at both the walls has been calculated. Finally the graphs are plotted for different values of Hartmann number and number density.

Equations of Motion
The modified Saffman's [16] equations for the conducting dusty gas and non-conducting dust particle are:

For fluid phase

\[ \nabla \cdot \vec{u} = 0 \]  
(Continuity)

\[ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\rho^{-1} \nabla p + \nu \nabla^2 \vec{u} + \frac{kN}{\rho} (\vec{v} - \vec{u}) + \frac{1}{\rho} (\vec{j} \times \vec{B}) \]  
(Linear Momentum)

For dust phase

\[ \nabla \cdot \vec{v} = 0 \]  
(Continuity)

\[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = \frac{k}{m} (\vec{u} - \vec{v}) \]  
(Linear Momentum)

We have following nomenclature:
\( \vec{u} \) - velocity of the fluid phase, \( \vec{v} \) - velocity density of dust phase, \( \rho \) - density of the
Flow of an Unsteady Conducting Dusty Fluid

Gas, $p$ - pressure of the fluid, $N$ - number of density of dust particles, $\nu$ - Kinematic viscosity, $K = 6\pi \alpha u$ - Stoke’s resistance (drag coefficient), $a$ - spherical radius of dust particle, $m$ - mass of the dust particle, $\mu$ - the co-efficient of viscosity of fluid particles, $t$ - time and $\vec{J}$ and $\vec{B}$ - given by Maxwell’s equations and Ohm’s law, namely,  

\[ \nabla \times \vec{H} = 4\pi \vec{J}, \nabla \times \vec{B} = 0, \nabla \times \vec{E} = 0, \vec{J} = \sigma \vec{J} + \vec{u} \times \vec{B} \]

Here $\vec{H}$ - magnetic field, $\vec{J}$ - current density, $\vec{B}$ - magnetic Flux, $\vec{E}$ - electric field.

It is assumed that the effect of induced magnetic fields produced by the motion of the electrically conducting gas is negligible and no external electric field is applied. With those assumptions the magnetic field $\vec{J} \times \vec{B}$ of the body force in (2.2) reduces simply to $-\sigma B_0^2 \vec{u}$ where $B_0$ - the intensity of the imposed transverse magnetic field.

**Frenet Frame Field System**

Let $\vec{s}, \vec{n}, \vec{b}$ be triply orthogonal unit vectors tangent, principal normal, binormal respectively to the spatial curves of congruence’s formed by fluid phase velocity and dusty phase velocity lines respectively as shown in the figure-1.

![Figure 1: Frenet Frame Field System.](image)

Geometrical relations are given by Frenet formulae [3]

\[
\frac{\partial \vec{s}}{\partial s} = k_s \vec{n}, \frac{\partial \vec{n}}{\partial s} = \tau_s \vec{b}, \frac{\partial \vec{b}}{\partial s} = -\tau_s \vec{n},
\]

\[
\frac{\partial \vec{n}}{\partial n} = k_n \vec{s}, \frac{\partial \vec{s}}{\partial n} = -\sigma_s \vec{n}, \frac{\partial \vec{b}}{\partial n} = -\sigma_n \vec{b} - k_n \vec{n},
\]

\[
\frac{\partial \vec{b}}{\partial b} = k_b \vec{s}, \frac{\partial \vec{s}}{\partial b} = -\sigma_b \vec{b}, \frac{\partial \vec{n}}{\partial b} = \sigma_b \vec{n} - k_b \vec{b}.
\]
where \( \frac{\partial}{\partial s}, \frac{\partial}{\partial n} \) and \( \frac{\partial}{\partial b} \) are the intrinsic differential operators along fluid phase velocity (or dust phase velocity) lines, principal normal and binormal. The functions \( (k_s, k'_n, k'_b) \) and \( (\tau_s, \sigma'_n, \sigma'_b) \) are the curvatures and torsions of the above curves and \( \theta_{ns} \) and \( \theta_{bs} \) are normal deformations of these spatial curves along their principal normal and binormal respectively.

**Formulation and Solution of the Problem**

Consider a flow of an unsteady viscous incompressible, dusty fluid between a non-torsional oscillating plate and a long wavy wall as shown in the figure-2. Both the fluid and the dust particle clouds are supposed to be static at the beginning. The number density of the dust particles is taken as a constant throughout the flow and these are assumed to be spherical in shape, uniform in size and electrically non-conducting. The flow is due to magnetic field of uniform strength \( B_0 \), non-torsional oscillations of the plate and under the influence of time dependent pressure gradient. Under these assumptions the flow will be a parallel flow in which the streamlines are along the tangential direction and the velocities are varies with binormal direction and time \( t \), since we extended the fluid to infinity in the principal normal direction.

For the above described flow the velocities of fluid and dust phase are of the form

\[
\vec{u} = u_s \hat{s}, \quad \vec{v} = v_s \hat{s}.
\]

where \((u_s, u_n, u_b)\) and \((v_s, v_n, v_b)\) are velocity components of fluid and dust particles respectively.

By virtue of system of equations (3.1) the intrinsic decomposition of equations (2.2) and (2.4) give the following forms

(4.1) \[
\frac{\partial u_s}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial s} + \nu \left[ \frac{\partial^2 u_s}{\partial b^2} - C_T u_s \right] + \frac{kN}{\rho} (v_s - u_s) - \vec{E} \cdot \vec{u}_s,
\]
Flow of an Unsteady Conducting Dusty Fluid

\[ 2u_s^2 k_s = -\frac{1}{\rho} \frac{\partial p}{\partial n} + \nu \left[ 2\sigma \rho \frac{\partial u_s}{\partial b} - 2k_b \frac{\partial u_s}{\partial b} \right], \]

\[ 0 = -\frac{1}{\rho} \frac{\partial p}{\partial b} + \nu [u_s k_s \tau_s] , \]

\[ \frac{\partial v_s}{\partial t} = \frac{k}{m} (u_s - v_s) , \]

\[ 2v_s^2 k_s = 0 , \]

where \( E = \frac{\sigma B^2}{\rho} \) and \( C_r = k_s^2 + \sigma_n^2 + k_n^2 + \sigma_1^2 b + k_{1b}^2 \) is called curvature number \[2\].

From equation (4.5) we see that \( v_s^2 k_s = 0 \) which implies either \( v_s = 0 \) or \( k_s = 0 \).

The choice \( v_s = 0 \) is impossible, since if it happens then \( u_s = 0 \), which shows that the flow doesn’t exist. Hence \( k_s = 0 \), it suggests that the curvature of the streamline along tangential direction is zero. Thus no radial flow exists.

Let us consider the following non-dimensional quantities,

\[ u_s^* = \frac{u_s h}{U} , \quad v_s^* = \frac{v_s h}{v} , \quad b^* = \frac{b}{h} , \quad t^* = \frac{t U}{h^2} , \quad p^* = \frac{p h^2}{\rho U^2} , \quad s^* = \frac{s}{h} , \]

where \( U \) is the characteristic velocity and \( h \) is the characteristic length.

Using the above non-dimensional quantities we get the non-dimensionalised form of the equations are as follows

\[ \frac{\partial u_s}{\partial t} = -\frac{\partial p}{\partial s} + \frac{h}{Re} \frac{\partial^2 u_s}{\partial b^2} - \frac{h^3 C_r}{Re} u_s + \frac{k h^2}{\rho U} (v_s - u_s) - M u_s , \]

\[ \frac{\partial v_s}{\partial t} = \frac{k h^2}{m U} (u_s - v_s) , \]

where \( Re = U h / v \) is the Reynold’s number, and \( M = \frac{E h^2}{U} \) is the Hartmann number, \( \alpha = a^*/h \) is the non-dimensional amplitude parameter and \( \beta = \beta^* h \) is the non-dimensional frequency parameter.

\[
\begin{cases}
\text{Initial condition: at } t = 0, u_s = 0, v_s = 0 \\
\text{Boundary condition for } t > 0, u_s = f(t), \text{ at } b = 0 \\
\text{and } u_s = -\alpha \sin(t + \beta), \text{ at } b = 1 - \epsilon \cos t \\
\text{where } \epsilon \text{ is a constant.}
\end{cases}
\]

We define Laplace transformations of \( u_s \) and \( v_s \) as

\[ U_s = \int_0^\infty e^{-xt} u_s dt \quad \text{and} \quad V_s = \int_0^\infty e^{-xt} v_s dt , \]

By applying Laplace transform to the equations (4.6) and (4.7) one can obtains

\[ x U_s = P(x) + \frac{h}{Re} \frac{d^2 u_s}{db^2} = \frac{h^3 C_r}{Re} U_s + \frac{h^2}{U_c} (V_s - U_s) - M U_s , \]
\[ (4.11) \quad xV_5 = \frac{h^2}{\nu} (U_s - V_5), \]

where \( \kappa = \frac{m_N}{p} \) and \( \tau = \frac{m}{\kappa} \). Equation (4.11) implies

\[ (4.12) \quad V_5 = \frac{h^2}{(h^2 + x\nu)} U_s, \]

Eliminating \( V_5 \) from (4.10) using (4.12) we obtain the following equation

\[ (4.13) \quad \frac{d^2 U_s}{d t^2} - Q^2 U_s = -\frac{Re}{h} P(x), \]

where \( Q^2 = h^2 C_r + \frac{MRe}{h} + \frac{xRe}{h} (1 + \frac{h^2}{(x\nu + h^2)}). \)

**Case 1: Impulsive Motion**

Suppose \( \frac{\partial}{\partial t} U_s = p_0 \delta(t) \), is imposed on the flow and \( f(t) = u_0 \delta(t) \), where \( p_0 \) and \( u_0 \) are constant and \( \delta(t) \) is the Dirac delta function. Now, solving the equation (4.13) with the boundary conditions (4.8), one can obtain the fluid and dust phase velocities as follows,

\[
\begin{align*}
  u_s &= \frac{u_0 \sinh((b_2-b)X_1)}{\sinh(bX_1)} - \frac{2h u_0 \pi}{Re b_2} \sum_{r_1=0}^{\infty} (-1)^{r_1} r_1 \sin \left( \frac{r_1 \pi ((b_2-b)X_1)}{b_2} \right) \\
  &\times \left[ \frac{e^{\pi t} (h^2 + x_3 \nu)^2}{x_3 [(h^2 + x_3 \nu + h^4)]} + \frac{e^{\pi t} (h^2 + x_4 \nu)^2}{x_4 [(h^2 + x_4 \nu + h^4)]} \right] \\
  &- \alpha \left[ \frac{k_3 \sin t - k_1 \cos t}{(C^2 + D^2)} \right] + \frac{2h \alpha \pi}{Re b_2} \sum_{r_1=0}^{\infty} (-1)^{r_1} r_1 \sin \left( \frac{r_1 \pi b}{b_2} \right) \\
  &- \frac{2p_0}{\pi} \sum_{r_1=0}^{\infty} \left[ (-1)^{r_1} \sin \left( \frac{r_1 \pi b}{b_2} \right) \right] \frac{e^{\pi t} (h^2 + x_3 \nu)^2}{x_3 [(h^2 + x_3 \nu + h^4)]} + \frac{e^{\pi t} (h^2 + x_4 \nu)^2}{x_4 [(h^2 + x_4 \nu + h^4)]} \\

v_5 &= \frac{\delta(t) u_0 \sinh((b_2-b)X_1)}{\sinh(bX_1)} - \frac{2h u_0 \pi}{Re b_2} \sum_{r_1=0}^{\infty} (-1)^{r_1} r_1 \sin \left( \frac{r_1 \pi ((b_2-b)X_1)}{b_2} \right) \\
  &\times \left[ \frac{e^{\pi t} (h^2 + x_3 \nu)^2}{x_3 [(h^2 + x_3 \nu + h^4)]} + \frac{e^{\pi t} (h^2 + x_4 \nu)^2}{x_4 [(h^2 + x_4 \nu + h^4)]} \right] \\
  &- \alpha \left[ \frac{L_2 \sin t - L_1 \cos t}{(C^2 + D^2)(h^2 + \tau^2 \nu^2)} \right] + \frac{2h^2 \alpha \pi}{Re b_2} \sum_{r_1=0}^{\infty} (-1)^{r_1} r_1 \sin \left( \frac{r_1 \pi b}{b_2} \right) \\
  &- \frac{2p_0}{\pi} \sum_{r_1=0}^{\infty} \left[ (-1)^{r_1} \sin \left( \frac{r_1 \pi b}{b_2} \right) \right] \frac{e^{\pi t} (h^2 + x_3 \nu)^2}{x_3 [(h^2 + x_3 \nu + h^4)]} + \frac{e^{\pi t} (h^2 + x_4 \nu)^2}{x_4 [(h^2 + x_4 \nu + h^4)]} \\
  &- \frac{Re p_0}{b_2^2} \left[ \sinh(bX_1) + \sinh((b_2-b)X_1) - \sinh(bX_1) \right] - \sinh(bX_1),
\end{align*}
\]
Flow of an Unsteady Conducting Dusty Fluid

Shearing Stress (Skin Friction): The Shear stress at the boundaries \( b = 0 \) and \( b = 1 - \epsilon \cos t \) are given by

\[
D_0 = \frac{-u_0X_1 \cosh((b_2X_1))}{\sinh(b_2X_1)} + \frac{2hu_0\pi^2}{\text{Reb}_2^2} \sum_{r_1=0}^{\infty} (-1)^{r_1} r_1 \sin \left( \frac{r_1\pi b}{b_2} \right) \left[ \frac{e^{x_3t}(h^2+x_3U_t)}{x_3[(h^2+x_3U_t)^2+l^2h^4]} + \frac{e^{x_4t}(h^2+x_4U_t)}{x_4[(h^2+x_4U_t)^2+l^2h^4]} \right]
\]

\[
D_1 = \frac{-u_0X_1}{\sinh(b_2X_1)} + \frac{2hu_0\pi^2}{\text{Reb}_2^2} \sum_{r_1=0}^{\infty} (-1)^{r_1} r_1 \sin \left( \frac{r_1\pi b}{b_2} \right) \left[ \frac{e^{x_3t}(h^2+x_3U_t)}{x_3[(h^2+x_3U_t)^2+l^2h^4]} + \frac{e^{x_4t}(h^2+x_4U_t)}{x_4[(h^2+x_4U_t)^2+l^2h^4]} \right]
\]

Case 2: Transition Motion: Suppose \(-\frac{\partial p}{\partial r} = p_0 H(t) e^{-\lambda_1 t}\), where \(p_0\) and \(\lambda_1\) are constants and \(H(t)\) is the Heaviside unit step function. By solving the equation \(4.13\) when subject to the boundary conditions \(4.8\), with \(f(t) = u_0 H(t) e^{-\lambda_1 t}\),

For this case one can obtain the expressions for both fluid and dust velocities as

\[
u_s = \frac{H(t)u_0 e^{-\lambda_1 t} \sinh(b_2-y) Y}{\sinh(b_2Y)} \times \left[ \frac{e^{x_3t}(h^2+x_3U_t)}{(x_3+\lambda_1)[(h^2+x_3U_t)^2+l^2h^4]} + \frac{e^{x_4t}(h^2+x_4U_t)}{(x_4+\lambda_1)[(h^2+x_4U_t)^2+l^2h^4]} \right]
\]

\[
-\alpha \frac{k_2 \sin t - k_1 \cos t}{(C^2+D^2)} + \frac{2h\pi}{\text{Reb}_2^2} \sum_{r_1=0}^{\infty} (-1)^{r_1} r_1 \sin \left( \frac{r_1\pi b}{b_2} \right) \left[ \frac{e^{x_3t}(h^2+x_3U_t)}{(x_3+\lambda_1)[(h^2+x_3U_t)^2+l^2h^4]} + \frac{e^{x_4t}(h^2+x_4U_t)}{(x_4+\lambda_1)[(h^2+x_4U_t)^2+l^2h^4]} \right]
\]

\[
- \frac{p_0 e^{-\lambda_1 t}}{hY} \sinh(b_2Y) + \sinh((b_2-y)Y) - \sinh(b_2Y)
\]

\[
- \frac{2p_0}{\pi} \sum_{r_1=0}^{\infty} \frac{(-1)^{r_1-1}}{r_1} \sin \left( \frac{r_1\pi b}{b_2} \right) \left[ \frac{e^{x_3t}(h^2+x_3U_t)}{x_3[(h^2+x_3U_t)^2+l^2h^4]} + \frac{e^{x_4t}(h^2+x_4U_t)}{x_4[(h^2+x_4U_t)^2+l^2h^4]} \right]
\]
\[ v_s = \frac{u_0 h^2 e^{-\alpha t} \sinh[(b_2 - b) t]}{(h^2 - \lambda_1) \sinh(b_2 t)} - 2 h^2 u_0 \pi e^{\pi t} \sum_{r_1=0}^{\infty} (-1)^{r_1} r_1 \sin \left( \frac{r_1 n b_1}{b_2} \right) \]

\[
\times \left[ \frac{e^{\pi t} (h^2 + x_3 U t)}{(x_3 + \lambda_1)(h^2 + x_3 U t)^2 + h^4)} + \frac{e^{\pi t} (h^2 + x_4 U t)}{(x_4 + \lambda_1)(h^2 + x_4 U t)^2 + h^4)} \right] \\
- \alpha h^2 \left[ \frac{L_2 \sin \left( \frac{1}{C^2 + D^2} \right)}{(h^2 - \lambda_1) U t} \right] + 2 h^2 \pi \sum_{r_1=0}^{\infty} (-1)^{r_1} r_1 \sin \left( \frac{r_1 n b_1}{b_2} \right) \]

\[
\times \left[ \frac{e^{\pi t} (h^2 + x_3 U t)}{(x_3 + \lambda_1)(h^2 + x_3 U t)^2 + h^4)} + \frac{e^{\pi t} (h^2 + x_4 U t)}{(x_4 + \lambda_1)(h^2 + x_4 U t)^2 + h^4)} \right] \\
- \frac{R_{p_0}}{h^2} e^{-\lambda t} \sinh(b_2 t) \sinh((b_2 - b) t) \sinh(b_2 t) \\
- \frac{2 p_0 h^2 \pi}{\sinh(b_2 t)} \sum_{r_1=0}^{\infty} \left[ (-1)^{r_1-1} \sin \left( \frac{r_1 n b_1}{b_2} \right) \right] \left[ \frac{e^{\pi t} (h^2 + x_3 U t)}{x_3 (h^2 + x_3 U t)^2 + h^4)} + \frac{e^{\pi t} (h^2 + x_4 U t)}{x_4 (h^2 + x_4 U t)^2 + h^4)} \right] \]

**Case 3: Periodic Motion for a Finite Time:** In this case, we take \(-\frac{\partial p}{\partial s} = p_0 \sin(\alpha t)\) and \(f(t) = u_0 \sin(\alpha t)\), where \(p_0\) and \(u_0\) are constants. Now one obtains the velocity profiles for both fluid and dust phase as, we obtain the fluid and dust phase velocities as

\[ u_s = \frac{u_0}{c_1^2 + D^2} [k_3 \sin(\alpha t) + k_4 \cos(\alpha t)] - 2 h u_0 \pi \alpha t \sum_{r_1=0}^{\infty} (-1)^{r_1} r_1 \sin \left( \frac{r_1 n b_1}{b_2} \right) \]

\[
\times \left[ \frac{e^{\pi t} (h^2 + x_3 U t)}{(x_3 + \alpha^2)(h^2 + x_3 U t)^2 + h^4)} + \frac{e^{\pi t} (h^2 + x_4 U t)}{(x_4 + \alpha^2)(h^2 + x_4 U t)^2 + h^4)} \right] \\
- \frac{R_{p_0}}{h^2} e^{-\lambda t} \sinh(b_2 t) \sinh(b_2 t) \\
- \frac{2 p_0 h^2 \pi}{\sinh(b_2 t)} \sum_{r_1=0}^{\infty} \left[ (-1)^{r_1-1} \sin \left( \frac{r_1 n b_1}{b_2} \right) \right] \left[ \frac{e^{\pi t} (h^2 + x_3 U t)^2}{x_3 (h^2 + x_3 U t)^2 + h^4)} + \frac{e^{\pi t} (h^2 + x_4 U t)^2}{x_4 (h^2 + x_4 U t)^2 + h^4)} \right] \]

\[ v_s = \frac{u_0 h^2}{c_1^2 + D^2} (h^2 + \alpha^2 U t^2) \sin(\alpha t) - \frac{2 h^2 u_0 \pi}{\sinh(b_2 t)} \sum_{r_1=0}^{\infty} (-1)^{r_1} r_1 \sin \left( \frac{r_1 n b_1}{b_2} \right) \]

\[
\times \left[ \frac{e^{\pi t} (h^2 + x_3 U t)}{(x_3 + \lambda_1)(h^2 + x_3 U t)^2 + h^4)} + \frac{e^{\pi t} (h^2 + x_4 U t)}{(x_4 + \lambda_1)(h^2 + x_4 U t)^2 + h^4)} \right] \\
- \frac{R_{p_0}}{h^2} e^{-\lambda t} \sinh(b_2 t) \sinh(b_2 t) \\
- \frac{2 p_0 h^2 \pi}{\sinh(b_2 t)} \sum_{r_1=0}^{\infty} (-1)^{r_1} r_1 \sin \left( \frac{r_1 n b_1}{b_2} \right) \]

\[
\times \left[ \frac{e^{\pi t} (h^2 + x_3 U t)}{(x_3 + \lambda_1)(h^2 + x_3 U t)^2 + h^4)} + \frac{e^{\pi t} (h^2 + x_4 U t)}{(x_4 + \lambda_1)(h^2 + x_4 U t)^2 + h^4)} \right] \\
- \frac{R_{p_0}}{h^2} e^{-\lambda t} \sinh(b_2 t) \sinh(b_2 t) \\
- \frac{2 p_0 h^2 \pi}{\sinh(b_2 t)} \sum_{r_1=0}^{\infty} (-1)^{r_1} r_1 \sin \left( \frac{r_1 n b_1}{b_2} \right) \]

\[
\times \left[ \frac{e^{\pi t} (h^2 + x_3 U t)}{(x_3 + \lambda_1)(h^2 + x_3 U t)^2 + h^4)} + \frac{e^{\pi t} (h^2 + x_4 U t)}{(x_4 + \lambda_1)(h^2 + x_4 U t)^2 + h^4)} \right] \\
- \frac{R_{p_0}}{h^2} e^{-\lambda t} \sinh(b_2 t) \sinh(b_2 t) \\
- \frac{2 p_0 h^2 \pi}{\sinh(b_2 t)} \sum_{r_1=0}^{\infty} (-1)^{r_1} r_1 \sin \left( \frac{r_1 n b_1}{b_2} \right) \]
Flow of an Unsteady Conducting Dusty Fluid

\[ \text{Re}_{p0} \left[ \frac{k_{11} \sin(\alpha t) - k_{12} \cos(\alpha t)}{c_{1}^2 + D_{1}^2} \right] \]

\[ - \frac{2 \rho_{0} \alpha}{\pi} \sum_{n=1}^{\infty} \left[ \frac{(-1)^{n-1}}{r_{1}} \sin \left( \frac{r_{1} n}{b_{2}} \right) \sin \left( \frac{x_{3} + \alpha^2}{(x_{3} + \alpha^2)^2 + \lambda n^2} \right) + \frac{e^{x_{3}^2}(h^2 + x_{3} \lambda n^2)}{(x_{3} + \alpha^2)^2 + \lambda n^4} \right] \]

where

\[ M = \frac{E h^2}{U}, \quad X_{1} = \sqrt{h^2 + \frac{M Re}{h}}, \quad b_{2} = 1 - \epsilon \cos t, \quad a_{1} = \text{Re} b_{2}^2 \tau U, \]

\[ a_{2} = h^2 C_{r} b_{2} + M \text{Re} b_{2} \tau U + \text{Re} b_{2}^2 h^2 + \text{Re} b_{2}^2 \tau U + \tau l_{2}^2 \eta^2, \]

\[ a_{3} = h^2 C_{r} b_{2} + M \text{Re} b_{2}^2 h^2 + r_{l}^2 \eta^2 h^3, \quad x_{3} = \frac{-a_{2} + \sqrt{a_{2}^2 - 4 a_{1} a_{3}}}{2 a_{1}}, \]

\[ x_{4} = \frac{-a_{2} - \sqrt{a_{2}^2 - 4 a_{1} a_{3}}}{2 a_{1}}, \quad y_{1} = h^2 C_{r} + \frac{M \text{Re}}{h} + \frac{\text{Re} \text{h} \tau U}{(h^2 + \tau U^2)}, \]

\[ z_{1} = \frac{\text{Re} \text{h} + \lambda \text{Re} U}{(h^2 + \tau U^2)}, \quad y_{2} = \sqrt{\frac{y_{1} + \sqrt{y_{1}^2 + z_{1}^2}}{2}}, \quad z_{2} = \sqrt{\frac{y_{1} + \sqrt{y_{1}^2 + z_{1}^2}}{2}}, \]

A = \sinh(y_{2} b), B = \cosh(y_{2} b) \sin(z_{2} b), C = \sinh(y_{2} b) \cos(z_{2} b),

D = \cosh(y_{2} b) \sin(z_{2} b), \quad k_{1} = -\cos \beta (BC - AD) + \sin \beta (AC + BD), \]

\[ k_{2} = \cos \beta (AC + BD) - \sin \beta (BC - AD), \quad L_{1} = k_{1} h^2 + k_{2} \tau U, \quad L_{2} = k_{12} h^2 - k_{2} \tau U, \]

\[ Y = \frac{\sqrt{h^2 C_{r} + \frac{M \text{Re}}{h} - \lambda \text{Re} \text{h} \tau U}}{(h^2 + \tau U^2)}, \quad Y_{3} = \sqrt{h^2 C_{r} + \frac{M \text{Re}}{h} + \frac{\text{Re} \text{h} \tau U}{(h^2 + \tau U^2)}}, \]

\[ z_{3} = \frac{\text{Rea} \text{h} + \lambda \text{Re} a_{3}}{(h^2 + \tau U^2)}, \quad Y_{4} = \sqrt{\frac{y_{2}^2 + \sqrt{y_{2}^2 + z_{3}^2}}{2}}, \quad z_{4} = \sqrt{\frac{y_{2}^2 + \sqrt{y_{2}^2 + z_{3}^2}}{2}}, \]

A_{1} = \sinh(y_{4} b_{1}) \cos(z_{4} b_{1}), B_{1} = \cosh(y_{4} b_{1}) \sin(z_{4} b_{1}), A_{2} = \sinh(y_{4} b) \cos(z_{4} b),

B_{2} = \cosh(y_{4} b) \sin(z_{4} b), \quad C_{1} = \sinh(y_{4} b_{2}) \cos(z_{4} b_{2}), D_{1} = \cosh(y_{4} b_{2}) \sin(z_{4} b_{2}),

k_{5} = A_{2} + A_{1} - C_{1}, k_{6} = B_{2} - B_{1} - D_{1}, k_{7} = y_{3}(k_{5} C_{1} + k_{6} D_{1}) + z_{3}(k_{6} C_{1} - k_{5} D_{1}),

k_{8} = y_{3}(k_{6} C_{1} - k_{5} D_{1}) - z_{3}(k_{5} C_{1} + k_{6} D_{1}), k_{9} = k_{3} h^2 - k_{4} \alpha \tau U, k_{10} = k_{4} h^2 + k_{3} \alpha \tau U,

k_{11} = k_{7} h^2 - k_{8} \alpha \tau U, k_{12} = k_{8} h^2 + k_{7} \alpha \tau U,

k_{1}^1 = \cos(\beta) (C_{2} - D_{2}), k_{1}^2 = \cos(\beta) (C_{2} + D_{2}),

k_{2}^1 = \cos(\beta) (C_{2} + D_{2}), \quad k_{2}^2 = \cos(\beta) (C_{2} - D_{2}).
Conclusion
One can observe the parabolic in nature of velocity profiles for the fluid and dust particles plotted as in graphs 3 to 8. It is observed that velocity of fluid particles is parallel to velocity of dust particles. It is evident from the graphs 3-5 that, as we increase the strength of the magnetic field, it has an appreciable effect on the velocities of fluid and dust particles, i.e., the magnetic effect has retarding influence. If we increase the number density of the dust particles it effect on the flow, i.e it decreases the velocities of both fluid and dust phase. Further one can can observe that if the dust is very fine i.e., mass of the dust particles is negligibly small then the relaxation time of dust particle decreases and ultimately as $\tau \to 0$ the velocities of fluid and dust particles will be the same.

**Figure 3:** Variation of fluid and phase velocity with $b$ (case-1).
Figure 4: Variation of fluid and phase velocity with $b$ (case-2).

Figure 5: Variation of fluid and phase velocity with $b$ (case-3).

Figure 6: Variation of fluid and phase velocity with $b$ (case-1).
Figure 7: Variation of fluid and phase velocity with b (case-2).

Figure 8: Variation of fluid and phase velocity with b (case-3).

References


Flow of an Unsteady Conducting Dusty Fluid


UNSTEADY FLOW OF A DUSTY FLUID THROUGH A CHANNEL HAVING TRIANGULAR CROSS-SECTION IN FRENET FRAME FIELD SYSTEM

MAHESHA, BIJANAL JAYANNA GIREESHA, GOSIKERE KENCHAPPA RAMESHA AND CHANNABASAPPA SHANTHAPPA BAGEWADI

ABSTRACT. An unsteady motion of a viscous liquid with uniform distribution of dust particles under the influence of time dependent pressure gradient through a channel having triangular cross-section has been considered. The intrinsic decomposition of flow equations are carried out in Frenet frame field system. The governing equations of the flow are solved using Laplace transform and variable separable method. The skin friction at the boundaries are calculated. Finally the conclusions are given on basis of the velocity profiles drawn for different values of time $t$ and number density $N$.

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1. INTRODUCTION

In recent years many researchers in fluid dynamics has been diverted towards study of the influence of dust particles on the motion of fluids. The presence of dust particles in fluids has certain influence on the motion of the fluids and such situation arise for instance, in the movement of dust-laden air in, combustion, polymer technology, and electrostatic precipitation. Other important application involving dust particles are fluidization, purification of crude oil, centrifugal separation of matter from fluid, petroleum industry and in the engineering problems concerned with atmospheric fallout, dust collection, nuclear reactor cooling, powder technology, performance of solid fuel rocket nozzles and paint spraying etc.

Saffman [16] investigated the effect of stability of the laminar flow of a dusty gas in which the dust particles are uniformly distributed. Further, Michael and Miller [13] investigated the motion of dusty gas with uniform distribution of the dust particles occupied in the semi-infinite space above a rigid plane boundary.
Chernyshov [6] obtained the exact solution for nonsteady two-dimensional problem of the motion of an incompressible viscous fluid in a rigid tube whose cross-section is a regular triangle. Rukmangadachari [15] has studied the solutions of dusty viscous flow through a cylinder of triangular cross-section. N.C.Ghosh, B.C.Ghosh & R.S.R.Gorla [10] have studied unsteady motion of a dusty viscoelastic Maxwell-type conducting fluid under arbitrary pressure gradient through a long uniform tube of rectangular cross section. Attia [1] studied the unsteady Hartmann flow of a dusty viscous incompressible electrically conducting fluid under the influence of an exponentially decreasing pressure gradient is studied without neglecting the ion slip. Chamkha [7] has obtained the analytical solution for unsteady flow of an electrically conducting dusty-gas in a channel due to an oscillating pressure gradient. Balasubramanian & Chen [5] have given mathematical model for unsteady MHD flow and heat transfer of dusty fluid between two cylinders with variable physical properties.

Frenet frames are a central construction in modern differential geometry, in which structure is described with respect to an object of interest rather than with respect to external coordinate systems. Some researchers like Kanwal [12], Truesdell [17], Indrasena [11], Purushotham [14], Bagewadi and Gireesha [2], [3] have applied differential geometry techniques to study the fluid flow. Further, the authors [2], [3] have studied two-dimensional dusty fluid flow in Frenet frame field system, which is one of the moving frame. Recently the authors [8], [9] have studied the flow of unsteady dusty fluid in different regions under varying time dependent pressure gradients. The present work deals with the study of flow of an unsteady dusty fluid through a channel having triangular cross-section under the influence of time dependent pressure gradient in Frenet frame field system. Initially it is assumed that the fluid and dust particles are to be at rest. The analytical expressions are obtained for velocities of both fluid and dust particles using Laplace transform technique and variable separable method. Further the skin friction at the boundary is calculated. The velocity profiles of both fluid and dust phase are shown graphically for different time $t$ and number density $N$.

2. Equations of Motion

The equations of motion of unsteady viscous incompressible fluid with uniform distribution of dust particles are given by [16]:

For fluid phase

\[ \nabla \cdot \mathbf{u} = 0, \quad \text{(Continuity)} \]  \hspace{1cm} (1)

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} \]  \hspace{1cm} (2)

54
\[ + \frac{kN}{\rho} (\mathbf{v} - \mathbf{u}) \]  
(Linear Momentum)

**For dust phase**
\[ \nabla \cdot \mathbf{v} = 0, \]  
(Continuity)  
\[ (3) \]

\[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{k}{m} (\mathbf{v} - \mathbf{v}^* \cdot \mathbf{v}) \]  
(Linear Momentum)  
\[ (4) \]

We have following nomenclature: We have following nomenclature:
\( \mathbf{v} \) —velocity of the fluid phase, \( \mathbf{v}^* \) —velocity of dust phase, \( \rho \) —density of the gas, \( p \) —pressure of the fluid, \( N \) —number density of dust particles, \( \nu \) —kinematic viscosity, \( k = 6\pi \sigma \mu \) —Stoke’s resistance (drag coefficient), \( \alpha \) —spherical radius of dust particle, \( m \) —mass of the dust particle, \( \mu \) —the co-efficient of viscosity of fluid particles, \( t \) —time.

3. FRENET FRAME FIELD SYSTEM

Let \( \mathbf{e}, \mathbf{n}, \mathbf{b} \) be triply orthogonal unit vectors tangent, principal normal, binormal respectively to the spatial curves of congruences formed by fluid phase velocity and dusty phase velocity lines respectively as shown in the figure-1.

![Figure-1: Frenet Frame Field System](image)

Geometrical relations are given by Frenet formulae [4]

\[ i) \frac{\partial \mathbf{e}}{\partial s} = k_s \mathbf{n}, \quad \frac{\partial \mathbf{n}}{\partial s} = \tau_s \mathbf{b} - k_s \mathbf{e}, \quad \frac{\partial \mathbf{b}}{\partial s} = -\tau_s \mathbf{n} \]

\[ ii) \frac{\partial \mathbf{e}}{\partial n} = k_n \mathbf{b}, \quad \frac{\partial \mathbf{b}}{\partial n} = -\sigma_n \mathbf{e} \quad \frac{\partial \mathbf{e}}{\partial n} = \sigma_n \mathbf{b} - k_n \mathbf{e} \]

\[ iii) \frac{\partial \mathbf{b}}{\partial n} = k_b \mathbf{e}, \quad \frac{\partial \mathbf{e}}{\partial b} = -\sigma_b \mathbf{n}, \quad \frac{\partial \mathbf{b}}{\partial b} = \sigma_b \mathbf{n} - k_b \mathbf{b} \]

\[ iv) \nabla \cdot \mathbf{e} = \theta_{na} + \theta_{bs}; \quad \nabla \cdot \mathbf{n} = \theta_{bn} - \theta_s; \quad \nabla \cdot \mathbf{b} = \theta_{nb} \]  
\[ (5) \]

where \( \partial/\partial s, \partial/\partial n \) and \( \partial/\partial b \) are the intrinsic differential operators along fluid phase velocity (or dust phase velocity) lines, tangential, principal normal and binormal.
The functions \((k_s, k'_s, k''_s)\) and \((\tau_s, \sigma'_s, \sigma''_s)\) are the curvatures and torsions of the above curves and \(\theta_n\) and \(\theta_b\) are normal deformations of these spatial curves along their principal normal and binormal respectively.

4. FORMULATION AND SOLUTION OF THE PROBLEM

The present discussion considers a laminar flow of viscous incompressible, dusty fluid through a rectangular channel. The flow is due to the influence of constant pressure gradient varying with time. Both the fluid and the dust particle clouds are supposed to be static at the beginning. The dust particles are assumed to be spherical in shape and uniform in size. The number density of the dust particles is taken as a constant throughout the flow. As Figure-2 shows, the axis of the channel is along binormal direction and the velocity components of both fluid and dust particles are respectively given by:

\[
\bar{u'} = u_b(s, n, t)\bar{b}, \quad \bar{v'} = v_b(s, n, t)\bar{b}
\] (6)

where \((u_s, u_n, u_b)\) and \((v_s, v_n, v_b)\) are velocity components of fluid and dust particles respectively.

![Figure 2: The Geometry of the Problem.](image)

By virtue of system of equations (5) the intrinsic decomposition of equations (2) and (4) using equation (6) one can get,

\[
0 = \frac{1}{\rho} \frac{\partial p}{\partial s} + \nu \left( \tau_s k_s u_b - 2 \sigma'_n \frac{\partial u_b}{\partial n} \right)
\]

\[
0 = \frac{1}{\rho} \frac{\partial p}{\partial n} + \nu \left( \sigma'_n k'_n u_b - 2 \tau_s \frac{\partial u_b}{\partial s} \right)
\]

\[
\frac{\partial u_b}{\partial t} = \frac{1}{\rho} \frac{\partial p}{\partial b} + \nu \left[ \frac{\partial^2 u_b}{\partial s^2} + \frac{\partial^2 u_b}{\partial n^2} - C_r u_b \right] + \frac{kN}{p} (v_b - u_b)
\] (7)

\[
\frac{\partial v_b}{\partial t} = \frac{k}{m} (u_b - v_b)
\] (8)

\[
v_b^2 k' = 0
\] (9)
where \( C_r = (\tau_s^2 + \sigma^2 + k''_b) \) is called curvature number [3].

From equation (9) we see that \( \psi_b k''_b = 0 \) which implies either \( \psi_b = 0 \) or \( k''_b = 0 \). The choice \( \psi_b = 0 \) is impossible, since if it happens then \( u_b = 0 \), which shows that the flow doesn’t exist. Hence \( k''_b = 0 \), it suggests that the curvature of the streamline along binormal direction is zero. Thus no radial flow exists.

Since we have assumed that the constant pressure gradient to be imposed on the system for \( t > 0 \), we can write

\[
-\frac{1}{\rho} \frac{\partial p}{\partial b} = c_0 \quad (\text{a constant})
\]  

(10)

Equation (7) and (8) are to be solved subject to the initial and boundary conditions:

\[
\begin{align*}
\text{Initial condition:} & \quad \text{at } t = 0; \quad u_b = 0, \quad v_b = 0 \\
\text{Boundary condition:} & \quad \text{for } t > 0; \quad u_b = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} \text{ at } s = \pm a, \\
& \quad u_b = 0 \text{ at } n = a \text{ \& } n = -a
\end{align*}
\]  

(11)

Let \( U_b \) and \( V_b \) are given by

\[
U_b = \int_0^\infty e^{-\tau t} u_b dt \quad \text{and} \quad V_b = \int_0^\infty e^{-\tau t} v_b dt
\]  

(12)

denote the Laplace transforms of \( u_b \) and \( v_b \) respectively.

Using relations (10) and (12) in equations (7), (8) and (11) one can obtain the following:

\[
xU_b = \frac{c_0}{x} + \nu \left( \frac{\partial^2 U_b}{\partial s^2} + \frac{\partial^2 U_b}{\partial n^2} - C_r U_b \right) + \frac{1}{\tau} (V_b - U_b)
\]  

(13)

\[
V_b = \frac{U_b}{(1 + \tau \lambda)}
\]  

(14)

\[
U_b = \frac{a_1}{x - i\omega_1} + \frac{a_2}{x - i\omega_2} \quad \text{at } s = a \text{ \& } s = -a
\]  

(15)

\[
\text{and } U_b = 0 \text{ at } n = a \text{ \& } n = -a,
\]

where \( \lambda = \frac{mN}{\rho} \) and \( \tau = \frac{m}{\kappa} \).

From equations (13) and (14) we obtain, the following equation

\[
\frac{\partial^2 U_b}{\partial s^2} + \frac{\partial^2 U_b}{\partial n^2} - Q^2 U_b + R = 0
\]  

(16)

where

\[
Q^2 = \left( C_r + \frac{x}{\nu} + \frac{x l}{\nu (1 + \tau \lambda)} \right) \quad \text{and} \quad R = \frac{c_0}{\nu x}
\]
To solve equation (13) we assume the solution in the following form [18]

\[ U_b(s, n) = w_1(s, n) + w_2(s) \]  

(17)

Substitution of \( U_b(s, n) \) in equation (13) yields

\[ \frac{\partial^2 w_1}{\partial s^2} + \frac{\partial^2 w_2}{\partial n^2} - R^2 (w_1 + w_2) = 0 \]

so that if \( w_2 \) satisfies

\[ \frac{\partial^2 w_2}{\partial s^2} - R^2 w_2 = 0 \]

then

\[ \frac{\partial^2 w_1}{\partial s^2} + \frac{\partial^2 w_1}{\partial n^2} - Q^2 w_1 = 0 \]  

(18)

In similar manner if \( U_b(s, n) \) is inserted in no slip boundary conditions, one can obtain

\[
\begin{cases}
U_b(a, n) = w_1(a, n) + w_2(a) = \frac{a_1}{x - iw_1} + \frac{a_2}{x + iw_2}, \\
U_b(-a, n) = w_1(-a, n) + w_2(-a) = \frac{a_1}{x - iw_1} + \frac{a_2}{x + iw_2}, \\
U_b(s, a) = w_1(s, a) + w_2(s) = 0, \\
U_b(s, -a) = w_1(s, -a) + w_2(s) = 0
\end{cases}
\]

By solving the problem

\[ \frac{\partial^2 w_2}{\partial s^2} - Q^2 w_2 + R = 0, \quad w_2(a) = \frac{a_1}{x - iw_1} + \frac{a_2}{x + iw_2}, \quad w_2(-a) = \frac{a_1}{x - iw_1} + \frac{a_2}{x + iw_2} \]

we obtain the solution in the form

\[ w_2(s) = \frac{\cosh(Qs)}{\cosh(Qa)} \left( \frac{a_1}{x - iw_1} + \frac{a_2}{x + iw_2} \right) - \frac{R}{Q^2} \left( \frac{\cosh(Qs) - \cosh(Qa)}{\cosh(Qa)} \right) \]  

(19)

Using variable separable method, the solution of the problem (18) with the conditions

\[ w_1(a, n) = 0, \quad w_1(-a, n) = 0, \quad w_1(s, a) = -w_2(s), \quad w_1(s, -a) = -w_2(s) \]

is obtained in the form

\[ w_1(s, n) = -\frac{2\pi}{a^2} \sum_{\tau_1=0}^{\infty} \sin \left( \frac{\tau_1 \pi s}{a} \right) \left( \frac{a_1}{x - iw_1} + \frac{a_2}{x + iw_2} \right) \]

\[ \times \frac{\cosh(An) \left[ 1 - (-1)^\tau \cosh(Qa) \right]}{A^2 \cosh(Qa) \cosh(Aa)} \]

\[ + \frac{2\pi}{a^2} \sum_{\tau_1=0}^{\infty} \sin \left( \frac{\tau_1 \pi s}{a} \right) \frac{R}{Q^2} \frac{1 - (-1)^r \cosh(Qa)}{\cosh(Qa) \cosh(Aa)} \cosh(An) \]

\[ - \frac{2}{\pi} \sum_{\tau_1=0}^{\infty} \frac{1}{\tau_1} \sin \left( \frac{\tau_1 \pi s}{a} \right) \frac{R}{Q^2} \frac{1 - (-1)^r \cosh(An)}{\cosh(Aa)} \]  

(20)
where \( A = \sqrt{Q^2 a_1^2 + x^2} \)

Now by substituting (19) and (20) in (17) we have

\[
U_b(s, n) = -\frac{2\pi}{a^2} \sum_{r_1=0}^{\infty} \sin \left( \frac{r_1 \pi}{a} s \right) \left( \frac{a_1}{x - iw_1} + \frac{a_2}{x + iw_2} \right) \times \frac{\cosh(An) [1 - (-1)^r_1 \cosh(Qa)]}{A^2 \cosh(Qa) \cosh(Aa)}
\]

\[
+ \frac{2\pi}{a^2} \sum_{r_1=0}^{\infty} \sin \left( \frac{r_1 \pi}{a} s \right) \frac{R}{Q^2} \frac{A^2 \cosh(Qa) \cosh(Aa)}{\cosh(An) [1 - (-1)^r_1 \cosh(Qa)]}
\]

\[
- \frac{2\pi}{a^2} \sum_{r_1=0}^{\infty} \frac{1}{r_1} \sin \left( \frac{r_1 \pi}{a} s \right) \frac{R}{Q^2} \frac{1}{\cosh(Qa)(1 + x\tau)} \frac{\cosh(Qa)(1 + x\tau)}{\cosh(Qa)}
\]

\[
+ \frac{2\pi}{a^2} \sum_{r_1=0}^{\infty} \frac{1}{r_1} \sin \left( \frac{r_1 \pi}{a} s \right) \frac{R}{Q^2} \frac{1}{\cosh(Aa)(1 + x\tau)} \frac{\cosh(Qa)(1 + x\tau)}{\cosh(Qa)}
\]

\[
- \frac{R}{Q^2} \frac{\cosh(Qa) - \cosh(Qa)}{\cosh(Qa)}
\]

Using \( U_b \) in equation (14) one can see that

\[
V_b(s, n) = -\frac{2\pi}{a^2} \sum_{r_1=0}^{\infty} \sin \left( \frac{r_1 \pi}{a} s \right) \left( \frac{a_1}{x - iw_1} + \frac{a_2}{x + iw_2} \right) \times \frac{[1 - (-1)^r_1 \cosh(Qa)] \cosh(An)}{A^2 \cosh(Qa) \cosh(Aa) (1 + x\tau)}
\]

\[
+ \frac{2\pi}{a^2} \sum_{r_1=0}^{\infty} \sin \left( \frac{r_1 \pi}{a} s \right) \frac{R}{Q^2} \frac{A^2 \cosh(Qa) \cosh(Aa)}{\cosh(An) [1 - (-1)^r_1 \cosh(Qa)]} \frac{1}{(1 + x\tau)}
\]

\[
- \frac{2\pi}{a^2} \sum_{r_1=0}^{\infty} \frac{1}{r_1} \sin \left( \frac{r_1 \pi}{a} s \right) \frac{R}{Q^2} \frac{1}{\cosh(Aa)(1 + x\tau)} \frac{\cosh(Qa)(1 + x\tau)}{\cosh(Qa)}
\]

\[
+ \frac{2\pi}{a^2} \sum_{r_1=0}^{\infty} \frac{1}{r_1} \sin \left( \frac{r_1 \pi}{a} s \right) \frac{R}{Q^2} \frac{1}{\cosh(Qa)(1 + x\tau)} \frac{\cosh(Qa)(1 + x\tau)}{\cosh(Qa)}
\]

By taking inverse Laplace transformation to \( U_b \) and \( V_b \), we obtain \( u_b \) and \( v_b \) as follows:

\[
u_b(s, n, t) = -\frac{2\pi}{a^2} \sum_{r_1=0}^{\infty} r_1 \sin \left( \frac{r_1 \pi}{a} s \right) \left\{ a_1 \frac{(\phi_1 \cos(w_1 t) - \phi_2 \sin(w_1 t))}{(\gamma^2 + \beta^2)(c^2 + d^2)(e^2 + f^2)} \right\}
\]
\[
\begin{align*}
\frac{i(\phi_1 \sin(w_1 t) + \phi_2 \cos(w_1 t))}{(y_3^2 + z_3^2)(c_1^2 + C_1^2)(c_3^2 + C_3^2)} & - a_1(-1)^r_1 \left( \frac{(\phi_3 \cos(w_1 t) - \phi_4 \sin(w_1 t))}{(y_3^2 + z_3^2)(c_1^2 + C_1^2)} + a_1[1 - (-1)^r_1 \cos(Q_1a)] \right) \\
\frac{i(\phi_3 \sin(w_1 t) + \phi_4 \cos(w_1 t))}{(y_3^2 + z_3^2)(c_1^2 + C_1^2)} & + a_1[1 - (-1)^r_1 \cos(Q_1a)] \cos(Q_1a)(x_1 - x_2) \\
\times \left[ e^{x t}(x_1 + i w_1) - e^{x t}(x_2 + i w_1) \right] \\
& \left[ \frac{a_1 \nu \pi}{a^2} \sum_{r_2=0}^{\infty} \left( -1 \right)^{r_2} \frac{\cosh(A_1 n)}{A_1^2 \cosh(A_1 a)} \cdot \frac{x_1 + w_1}{x_2 + w_2} \right]
\end{align*}
\]
\[ \times \left[ \frac{e^{x_1 t} - e^{x_2 t}}{x_1 - x_2} \right] - 4c_0 \sum_{r_2=0}^{\infty} \frac{(-1)^{r_2}}{(2r_2 + 1) A_1} \cosh(A_1a) \left[ \frac{e^{x_3 t}(1 + x_3 \tau)^2}{x_3[l + (1 + x_3 \tau)^2]} \right] \\
+ \frac{e^{x_4 t}(1 + x_4 \tau)^2}{x_4[l + (1 + x_4 \tau)^2]} - 4c_0 \sum_{r_2=0}^{\infty} \frac{(-1)^{r_2}}{(2r_2 + 1) Q_2} \cosh(Q_2a) \left[ \frac{e^{x_5 t}(1 + x_5 \tau)^2}{x_5[l + (1 + x_5 \tau)^2]} + \frac{e^{x_6 t}(1 + x_6 \tau)^2}{x_6[l + (1 + x_6 \tau)^2]} \right] \right) \\
\times \cos \left[ \frac{(2r_2 + 1)\pi n}{2a} \right] \left[ \frac{e^{x_7 t}(1 + x_7 \tau)^2}{x_7[l + (1 + x_7 \tau)^2]} + \frac{e^{x_8 t}(1 + x_8 \tau)^2}{x_8[l + (1 + x_8 \tau)^2]} \right] \right) \\
- \frac{2}{\pi} \sum_{r_1=0}^{\infty} \frac{1}{r_1 \pi s} \sin \left[ \frac{r_1 \pi s}{a} \right] \left\{ c_0 \cosh(Yn) \right\} \left\{ \nu X^2 \cosh(Ya) \right\} \\
\times \cosh(Q_1a) \left[ \frac{e^{x_9 t}(1 + x_9 \tau)^2}{x_9[l + (1 + x_9 \tau)^2]} + \frac{e^{x_10 t}(1 + x_{10} \tau)^2}{x_{10}[l + (1 + x_{10} \tau)^2]} \right] \\
+ \frac{c_0}{\nu} \cosh(Q_1a) \left[ \frac{e^{x_11 t}(1 + x_{11} \tau)^2}{x_{11}[l + (1 + x_{11} \tau)^2]} + \frac{e^{x_12 t}(1 + x_{12} \tau)^2}{x_{12}[l + (1 + x_{12} \tau)^2]} \right] \\
+ \frac{a_1 \nu \pi}{d^2} \sum_{r_2=0}^{\infty} \frac{(-1)^{r_2}}{(2r_2 + 1) \cosh} \left[ \frac{(2r_2 + 1)\pi s}{2a} \right] \\
\times \left[ \frac{e^{x_13 t}(x_3 + iw_1)(1 + x_3 \tau)^2}{(x_3^2 + w_1^2)[(l + (x_3 \tau + 1)^2)]} + \frac{e^{x_14 t}(x_4 + iw_1)(1 + x_4 \tau)^2}{(x_4^2 + w_1^2)[(l + (x_4 \tau + 1)^2)]} \right] \\
+ \frac{a_2}{\nu} \cosh(Xs) - \cosh(Xa) \left[ \frac{e^{x_15 t}(x_3 - iw_2)(1 + x_3 \tau)^2}{(x_3^2 + w_2^2)[(l + (x_3 \tau + 1)^2)]} + \frac{e^{x_16 t}(x_4 - iw_2)(1 + x_4 \tau)^2}{(x_4^2 + w_2^2)[(l + (x_4 \tau + 1)^2)]} \right] \\
- \frac{c_0}{\nu} \cosh(Xs) - \cosh(Xa) \left[ \frac{e^{x_17 t}(x_3 - iw_2)(1 + x_3 \tau)^2}{(x_3^2 + w_2^2)[(l + (x_3 \tau + 1)^2)]} + \frac{e^{x_18 t}(x_4 - iw_2)(1 + x_4 \tau)^2}{(x_4^2 + w_2^2)[(l + (x_4 \tau + 1)^2)]} \right] \\
+ \frac{4c_0}{\pi} \sum_{r_2=0}^{\infty} \frac{(-1)^{r_2}}{(2r_2 + 1) \cosh} \left[ \frac{(2r_2 + 1)\pi s}{2a} \right] \left[ \frac{e^{x_19 t}(x_3 - iw_2)(1 + x_3 \tau)^2}{(x_3^2 + w_2^2)[(l + (x_3 \tau + 1)^2)]} + \frac{e^{x_20 t}(x_4 - iw_2)(1 + x_4 \tau)^2}{(x_4^2 + w_2^2)[(l + (x_4 \tau + 1)^2)]} \right] \\
+ \frac{e^{x_21 t}(x_3 - iw_2)(1 + x_3 \tau)^2}{(x_3^2 + w_2^2)[(l + (x_3 \tau + 1)^2)]} \]
\[ v_b(s, n, t) = \frac{-2\pi}{a^2} \sum_{r_1} \sin \left[ \frac{r_1 \pi s}{a} \right] \left\{ a_1 \left[ \frac{(\phi_{13} \cos(w_1t) - \phi_{14} \sin(w_1t))}{(1 + w_1^2 \tau^2)(y_1^2 + x_1^2)(c_1^2 + c_2^2)(c_3^2 + c_4^2)} \right] \\
+ \frac{i(\phi_{14} \cos w_1t + \phi_{13} \sin w_1t)}{(1 + w_1^2 \tau^2)(y_1^2 + x_1^2)(c_1^2 + c_2^2)(c_3^2 + c_4^2)} \right\} \\
- a_1(-1)^{r_1} \left[ \frac{(\phi_{15} \cos(w_1t) - \phi_{16} \sin(w_1t)) + i(\phi_{15} \sin(w_1t) + \phi_{16} \cos(w_1t))}{(1 + w_1^2 \tau^2)(y_1^2 + x_1^2)(c_1^2 + c_2^2)(c_3^2 + c_4^2)} \right] \\
+ a_1 \left[ \frac{1 - (-1)^{r_1} \cos(Q_1a)}{\cos(Q_1a)(x_1 - x_2)} \right] \left[ \frac{e^{x_1t}(x_1 + iw_1)}{(x_1^2 + w_1^2)(1 + x_1^2) - (1 + x_2^2)(x_2^2 + w_2^2)} \right] \\
+ \frac{e^{x_1t}(x_4 + iw_1)(1 + x_4^2)}{(x_4^2 + w_4^2)(1 + x_4^2 + (x_3^2 + x_4^2)(x_5^2 + x_6^2))} \right] \\
+ \frac{4\nu a_1}{\pi} \sum_{r_2=0}^{\infty} \left[ \frac{(-1)^{r_2} + \nu \phi_{17} \cos w_2t + \nu \phi_{18} \sin w_2t}{(1 + w_2^2 \tau^2)(y_2^2 + x_2^2)(c_1^2 + c_2^2)(c_3^2 + c_4^2)} \right] \\
- \frac{1 - (-1)^{r_1} \cos(Q_2a)}{\cos(Q_2a)} \right] \left[ \frac{e^{x_2t}(x_3 + iw_1)(1 + x_3^2)}{(x_3^2 + w_3^2)(1 + x_3^2 + (x_3^2 + x_4^2)(x_5^2 + x_6^2))} \right] \\
+ \frac{e^{x_2t}(x_4 + iw_1)(1 + x_4^2)}{(x_4^2 + w_4^2)(1 + x_4^2 + (x_3^2 + x_4^2)(x_5^2 + x_6^2))} \right] \\
+ a_2 \left[ \frac{(\phi_{17} \cos w_2t + \phi_{18} \sin w_2t) + i(\phi_{18} \cos w_2t - \phi_{17} \sin w_2t)}{(1 + w_2^2 \tau^2)(y_2^2 + x_2^2)(c_1^2 + c_2^2)(c_3^2 + c_4^2)} \right] \\
- a_2(-1)^{r_1} \left[ \frac{(\phi_{19} \cos w_2t + \phi_{20} \sin w_2t) + i(\phi_{20} \cos w_2t - \phi_{19} \sin w_2t)}{(1 + w_2^2 \tau^2)(y_2^2 + x_2^2)(c_1^2 + c_2^2)(c_3^2 + c_4^2)} \right] \\
+ \frac{a_2 \nu \pi}{a^2} \sum_{r_2=0}^{\infty} (-1)^{r_2}(2r_2 + 1) \left[ \frac{e^{x_2t}(x_3 - iw_2)(1 + x_3^2)}{(x_3^2 + w_3^2)(1 + x_3^2 + (x_3^2 + x_4^2)(x_5^2 + x_6^2))} \right] \\
+ \frac{4\nu a_2}{\pi} \sum_{r_2=0}^{\infty} \left[ \frac{(-1)^{r_2} + \nu \phi_{17} \cos w_2t + \nu \phi_{18} \sin w_2t}{(1 + w_2^2 \tau^2)(y_2^2 + x_2^2)(c_1^2 + c_2^2)(c_3^2 + c_4^2)} \right] \\
+ \frac{e^{x_2t}(x_4 - iw_2)(1 + x_4^2)}{(x_4^2 + w_4^2)(1 + x_4^2 + (x_3^2 + x_4^2)(x_5^2 + x_6^2))} \right] \\
+ \frac{e^{x_2t}(x_5 - iw_2)(1 + x_5^2)}{(x_5^2 + w_5^2)(1 + x_5^2 + (x_3^2 + x_4^2)(x_5^2 + x_6^2))} \right] \\
+ \frac{\cos(Q_2a)}{\cos(Q_2a)} \right] \left[ \frac{e^{x_3t}(x_3 - iw_2)(1 + x_3^2)}{(x_3^2 + w_3^2)(1 + x_3^2 + (x_3^2 + x_4^2)(x_5^2 + x_6^2))} \right] \\
+ \frac{e^{x_3t}(x_4 - iw_2)(1 + x_4^2)}{(x_4^2 + w_4^2)(1 + x_4^2 + (x_3^2 + x_4^2)(x_5^2 + x_6^2))} \right] \\
+ \frac{e^{x_3t}(x_5 - iw_2)(1 + x_5^2)}{(x_5^2 + w_5^2)(1 + x_5^2 + (x_3^2 + x_4^2)(x_5^2 + x_6^2))} \right] \\
+ \frac{2\pi}{a^2} \sum_{r_1=0}^{\infty} \left[ \frac{r_1 \pi s}{a} \right] \\
+ \frac{c_0 \left[ 1 - (-1)^{r_1} \cos(Q_1a) \right] \cos(Y_1)}{\nu X^2 Y^2 \cosh(Q_1a) \cos(Y_1) + \nu X^2 Y^2 \cosh(Q_1a) \cos(Y_1)} + \frac{c_0 \left[ 1 - (-1)^{r_1} \right] \cosh(Q_1n)}{\nu X^2 Y^2 \cosh(Q_1a) \cos(Y_1) + \nu X^2 Y^2 \cosh(Q_1a) \cos(Y_1)} \right\} \]
\[
\times \left[ \frac{e^{xt}}{x_1(1 + x_1^2)} - \frac{e^{xt}}{x_2(1 + x_2^2)} \right] + \frac{c_0}{\nu} \left[ 1 - (-1)^{\tau} \cos(Q_1 \tau) \right] \left[ \cosh(Q_1 \tau) (x_1 - x_2) \right]
\]

\[
\times \left[ \frac{e^{xt}}{x_1(1 + x_1^2)} - \frac{e^{xt}}{x_2(1 + x_2^2)} \right] - \frac{4c_0}{\pi} \sum_{\tau_r=0}^{(2\tau_2 + 1)} \frac{(-1)^{\tau_2}}{A_1} \cos(A_1 \tau) \cosh(A_1 \tau)
\]

\[
\times \left[ \frac{e^{xt}(1 + x_3^2)}{x_3[l + (1 + x_3^2)^2]} + \frac{e^{xt}(1 + x_4^2)}{x_4[l + (1 + x_4^2)^2]} \right] - \frac{4c_0}{\pi} \sum_{\tau_r=0}^{(2\tau_2 + 1)} \frac{(-1)^{\tau_2}}{A_1} \cos(A_1 \tau) \cosh(A_1 \tau)
\]

\[
\times \left[ \frac{1 - (-1)^{\tau} \cos(Q_2 \tau)}{Q_2} \cos(Q_2 \tau) \right] \left[ \begin{array}{c} \frac{e^{xt}(1 + x_5^2)}{x_5[l + (1 + x_5^2)^2]} \\ \frac{e^{xt}(1 + x_6^2)}{x_6[l + (1 + x_6^2)^2]} \end{array} \right] \left[ - \frac{2}{\pi} \sum_{r_1=0}^{\infty} \frac{[1 - (-1)^r]}{r_1} \sin \left[ \frac{r_1 \pi s}{2} \right] \right]
\]

\[
\times \left[ \frac{c_0 \cosh(Yn)}{\nu X^2 \cosh(Ya)} + \frac{c_0 \cosh(Q_1 n)}{\nu \cosh(Q_1 a)(x_7 - x_8)} \right] \left[ \begin{array}{c} e^{xt}(1 + x_7^2) \\ e^{xt}(1 + x_8^2) \end{array} \right] \left[ \begin{array}{c} \frac{2\tau_r + 1}{2a} \cos \left[ \frac{(2\tau_r + 1)\pi s}{2a} \right] \\ \frac{2\tau_r + 1}{2a} \cos \left[ \frac{(2\tau_r + 1)\pi s}{2a} \right] \end{array} \right]
\]

\[
+ \frac{a_1}{a^2} \sum_{r_2=0}^{\infty} (-1)^{r_2} (2\tau_2 + 1) \cosh \left[ \frac{(2\tau_r + 1)\pi s}{2a} \right]
\]

\[
+ \frac{a_2}{a^2} \sum_{r_2=0}^{\infty} (-1)^{r_2} (2\tau_2 + 1) \cosh \left[ \frac{(2\tau_r + 1)\pi s}{2a} \right]
\]

\[
\times \left[ \frac{e^{xt}(x_3 + iw_1)(1 + x_3^2)}{(x_3^2 + w_1^2)(l + (x_3^2 + 1)^2)} + \frac{e^{xt}(x_4 + iw_1)(1 + x_4^2)}{(x_4^2 + w_1^2)(l + (x_4^2 + 1)^2)} \right]
\]

\[
+ \frac{a_1}{a^2} \sum_{r_2=0}^{\infty} (-1)^{r_2} (2\tau_2 + 1) \cosh \left[ \frac{(2\tau_r + 1)\pi s}{2a} \right]
\]

\[
+ \frac{a_2}{a^2} \sum_{r_2=0}^{\infty} (-1)^{r_2} (2\tau_2 + 1) \cosh \left[ \frac{(2\tau_r + 1)\pi s}{2a} \right]
\]

\[
\times \left[ \frac{e^{xt}(x_3 - iw_2)(1 + x_3^2)}{(x_3^2 + w_2^2)(l + (x_3^2 + 1)^2)} + \frac{e^{xt}(x_4 - iw_2)(1 + x_4^2)}{(x_4^2 + w_2^2)(l + (x_4^2 + 1)^2)} \right]
\]

\[
- \frac{c_0}{\nu X^2} \left[ \cosh(Xs) - \cosh(Xa) \right] - \frac{4c_0}{\pi} \sum_{r_2=0}^{(2\tau_2 + 1)} (-1)^{r_2} \cos \left[ \frac{(2\tau_r + 1)\pi s}{2a} \right]
\]

\[
\times \left[ \frac{e^{xt}(x_3 - iw_2)(1 + x_3^2)}{(x_3^2 + w_2^2)(l + (x_3^2 + 1)^2)} + \frac{e^{xt}(x_4 - iw_2)(1 + x_4^2)}{(x_4^2 + w_2^2)(l + (x_4^2 + 1)^2)} \right]
\]

\[
+ \frac{a_1}{a^2} \sum_{r_2=0}^{\infty} (-1)^{r_2} (2\tau_2 + 1) \cosh \left[ \frac{(2\tau_r + 1)\pi s}{2a} \right]
\]

\[
+ \frac{a_2}{a^2} \sum_{r_2=0}^{\infty} (-1)^{r_2} (2\tau_2 + 1) \cosh \left[ \frac{(2\tau_r + 1)\pi s}{2a} \right]
\]

\[
\times \left[ \frac{e^{xt}(x_3 + iw_2)(1 + x_3^2)}{(x_3^2 + w_2^2)(l + (x_3^2 + 1)^2)} + \frac{e^{xt}(x_4 + iw_2)(1 + x_4^2)}{(x_4^2 + w_2^2)(l + (x_4^2 + 1)^2)} \right]
\]
Shearing Stress (Skin Friction):
The Shear stress at the boundaries \( s = a, \ s = -a \) and \( n = a, \ n = -a \) are given by

\[
D_{an} = \frac{2\pi^2 \mu}{a^3} \sum_{r_1=0}^{\infty} r_1^2 \left\{ a_1 \left[ \frac{(\phi_1 \cos(w_1 t) - \phi_2 \sin(w_1 t)) + i(\phi_1 \sin(w_1 t) + \phi_2 \cos(w_1 t))}{(y_2^2 + z_2^2)(c_1^2 + d_1^2)(c_2^2 + d_2^2)} \right] - a_1 (-1)^{r_1} \left[ \frac{(\phi_3 \cos(w_1 t) - \phi_4 \sin(w_1 t)) + i(\phi_3 \sin(w_1 t) + \phi_4 \cos(w_1 t))}{(y_3^2 + z_3^2)(c_1^2 + d_1^2)} \right] \right.
\]

\[
+ a_1 \left[ \frac{1 - (-1)^{r_1} \cos(Q_1 a)}{\cos(Q_1 a)(x_1 - x_2)} \right] \left[ \frac{e^{\gamma t}(x_1 + iw_1)}{x_1^2 + w_1^2} - \frac{e^{\gamma t}(x_2 + iw_1)}{x_2^2 + w_1^2} \right] \right.
\]

\[
+ \frac{a_1 \nu \pi}{a^2} \sum_{r_2=0}^{\infty} (-1)^{r_2} \frac{(2r_2 + 1) \cosh(A_1 n)}{A_1 \cosh(A_1 a)} \left[ \frac{e^{\gamma t}(x_3 + iw_1)(1 + x_3 r)^2}{(x_3^2 + w_1^2)[(l + (x_3 r + 1)^2)]} \right]
\]

\[
+ \frac{e^{\gamma t}(x_4 + iw_1)(1 + x_4 r)^2}{(x_4^2 + w_1^2)[(l + (x_4 r + 1)^2)]} - \frac{4\nu a_2}{a^2} \sum_{r_3=0}^{\infty} (-1)^{r_3} \frac{(1 - (-1)^{r_3} \cos(Q_2 a))}{\cos(Q_2 a)}
\]

\[
\times \cos \left[ \frac{(2r_2 + 1) an}{2a} \right] \left[ \frac{e^{\gamma t}(x_5 + iw_1)(1 + x_5 r)^2}{(x_5^2 + w_1^2)[(l + (x_5 r + 1)^2)]} \right]
\]

\[
+ \frac{e^{\gamma t}(x_6 + iw_1)(1 + x_6 r)^2}{(x_6^2 + w_1^2)[(l + (x_6 r + 1)^2)]} \right\}
\]

\[
- a_2 \left[ \frac{(\phi_6 \cos(w_2 t) - \phi_7 \sin(w_2 t)) + i(\phi_6 \sin(w_2 t) - \phi_7 \cos(w_2 t))}{(y_6^2 + z_6^2)(c_1^2 + d_1^2)(c_2^2 + d_2^2)} \right]
\]

\[
- a_2 \left[ (-1)^{r_1} \frac{(\phi_7 \cos(w_2 t) - \phi_6 \sin(w_2 t)) + i(\phi_7 \sin(w_2 t) - \phi_6 \cos(w_2 t))}{(y_7^2 + z_7^2)(c_1^2 + d_1^2)} \right]
\]

\[
+ a_2 \left[ \frac{1 - (-1)^{r_1} \cos(Q_1 a)}{\cos(Q_1 a)(x_1 - x_2)} \right] \left[ \frac{e^{\gamma t}(x_7 - iw_2)}{x_7^2 + w_2^2} - \frac{e^{\gamma t}(x_8 - iw_2)}{x_8^2 + w_2^2} \right] \right.
\]

\[
+ \frac{a_2 \nu \pi}{a^2} \sum_{r_2=0}^{\infty} (-1)^{r_2} \frac{(2r_2 + 1) \cosh(A_1 n)}{A_1 \cosh(A_1 a)} \left[ \frac{e^{\gamma t}(x_9 - iw_2)(1 + x_9 r)^2}{(x_9^2 + w_2^2)[(l + (x_9 r + 1)^2)]} \right]
\]

\[
+ \frac{e^{\gamma t}(x_{10} - iw_2)(1 + x_{10} r)^2}{(x_{10}^2 + w_2^2)[(l + (x_{10} r + 1)^2)]} - \frac{4\nu a_2}{a^2} \sum_{r_3=0}^{\infty} (-1)^{r_3} \frac{(1 - (-1)^{r_3} \cos(Q_2 a))}{\cos(Q_2 a)}
\]

\[
\times \cos \left[ \frac{(2r_2 + 1) an}{2a} \right] \left[ \frac{e^{\gamma t}(x_{11} - iw_2)(1 + x_{11} r)^2}{(x_{11}^2 + w_2^2)[(l + (x_{11} r + 1)^2)]} \right]
\]

\[
+ \frac{e^{\gamma t}(x_{12} - iw_2)(1 + x_{12} r)^2}{(x_{12}^2 + w_2^2)[(l + (x_{12} r + 1)^2)]} \right\} - \frac{2\pi^2 \mu}{a^3} \sum_{r_1}^{\infty} (-1)^{r_1} r_1^2
\]
\[
\times \left\{ \frac{c_0}{\nu x_7} [1 - (1)^r \cosh(xa)] \cosh(yn) + \frac{c_0}{\nu \cosh(Q_1 a)} [1 - (1)^r \cosh(Q_1 n) y^2 \cosh(xa) \cosh(yna)] \right. \\
\times \left[ \frac{e^{xt}t - e^{xt}}{x_7} + \frac{c_0}{\nu} \left[ 1 - (1)^r \cos(Q_1 a) \cos(Q_1 n) \right] \frac{e^{xt}t - e^{xt}}{x_7} \right] \\
\left\{ \frac{c_0}{\nu Q_1^2} \cosh(Q_1 a) (x_7 - x_8) + \frac{1}{Q_1^2} \cosh(Q_1 a) (x_7 - x_8) \right\} \\
- \frac{4c_0}{\pi} \sum_{r_2=0}^{\infty} \frac{(-1)^r^2 \cosh(A_1 n)}{(2r_2 + 1) A_1^2 \cosh(A_1 a)} \left[ \frac{e^{xt}t (1 + x_3 \tau)^2}{x_3 \left[ 1 + (1 + x_3 \tau)^2 \right]} + \frac{e^{xt}t (1 + x_4 \tau)^2}{x_4 \left[ 1 + (1 + x_4 \tau)^2 \right]} \right] \\
= \frac{4c_0}{\pi} \sum_{r_2=0}^{\infty} \frac{(-1)^r^2 \cosh(A_1 n)}{(2r_2 + 1) Q_2^2 \cosh(Q_1 a)} \left[ \frac{e^{xt}t (1 + x_3 \tau)^2}{x_3 \left[ 1 + (1 + x_3 \tau)^2 \right]} + \frac{e^{xt}t (1 + x_4 \tau)^2}{x_4 \left[ 1 + (1 + x_4 \tau)^2 \right]} \right] \\
+ \frac{a_1 \pi^2 \nu}{2a^3} \sum_{r_2=0}^{\infty} \frac{(-1)^r^2 (2r_2 + 1) Q_2^2}{(2r_2 + 1) Q_2^2} \left[ \frac{e^{xt}t (1 + x_3 \tau)^2}{x_3 \left[ 1 + (1 + x_3 \tau)^2 \right]} + \frac{e^{xt}t (1 + x_4 \tau)^2}{x_4 \left[ 1 + (1 + x_4 \tau)^2 \right]} \right] \\
+ \alpha_a \mu \sum_{r_2=0}^{\infty} \frac{e^{xt}t}{(x_3 + w_1) \left[ 1 + (x_3 + w_1) \right]} \left[ \frac{e^{xt}t}{(x_4 + w_1) \left[ 1 + (x_4 + w_1) \right]} \right] + \alpha_a \mu \left[ \frac{e^{xt}t}{(x_3 + w_1) \left[ 1 + (x_3 + w_1) \right]} \right] \\
= \alpha_a \mu \sum_{r_2=0}^{\infty} \frac{e^{xt}t}{(x_3 + w_1) \left[ 1 + (x_3 + w_1) \right]} \left[ \frac{e^{xt}t}{(x_4 + w_1) \left[ 1 + (x_4 + w_1) \right]} \right] \\
+ \alpha_a \mu \left[ \frac{e^{xt}t}{(x_3 + w_1) \left[ 1 + (x_3 + w_1) \right]} \right] \\
- \sin w_2 \left( m_3 \cos(xa) + m_2 \sin(xa) \right) \\
- \cos w_2 \left( m_3 \sin(xa) - m_2 \cos(xa) \right) \\
+ \frac{a_2 \nu}{2a^3} \sum_{r_2=0}^{\infty} \frac{e^{xt}t (1 + x_3 \tau)^2 (x_3 + i w_1)}{x_3 \left[ 1 + (1 + x_3 \tau)^2 \right]} \\
+ \frac{e^{xt}t (1 + x_4 \tau)^2 (x_4 + i w_1)}{x_4 \left[ 1 + (1 + x_4 \tau)^2 \right]} + \frac{c_0}{\nu} \sin(xa) \\
+ \frac{c_0}{\nu} \cos(xa) \cosh(xa).\]
\[ D_{-on} = -D_{on} \]
\[ D_{sa} = \frac{-2\pi\mu}{a^2} \sum_{r_1=0}^{\infty} r_1 \sin \left[ \frac{r_1\pi s}{a} \right] \left[ \frac{a_1}{k_1} \left( \cos w_1 t (m_1 H_{11} - m_2 G_{11} - m_3 G_{11}) - m_4 H_{11}) p_1 (m_1 G_{11} + m_2 H_{11} + m_3 H_{11} - m_4 G_{11}) p_2 \right) \right. \]
\[ - \sin w_1 t \left( (m_1 H_{11} - m_2 G_{11} - m_3 G_{11} - m_4 H_{11}) p_2 - (m_1 G_{11} + m_2 H_{11}) \right) \]
\[ + m_3 H_{11} - m_4 G_{11}) p_1 \] \[ + i \sin w_1 t \left( (m_1 H_{11} - m_2 G_{11} - m_3 G_{11} - m_4 H_{11}) p_1 \right) \]
\[ + (m_1 G_{11} + m_2 H_{11} + m_3 H_{11} - m_4 G_{11}) p_2 \] \[ + i \cos w_1 t \left( (m_1 H_{11} - m_2 G_{11} - m_3 G_{11} - m_4 H_{11}) p_1 \right) \]
\[ - m_3 G_{11} - m_4 H_{11}) p_2 - (m_1 G_{11} + m_2 H_{11} + m_3 H_{11} - m_4 G_{11}) p_1 \] \[ - \frac{a_1(-1)^{r_1}}{k_2} \left[ \cos w_1 t (m_5 H_{11} - m_6 G_{11} - m_7 G_{11} - m_8 G_{11} + m_9 G_{11}) \right. \]
\[ + m_{10} H_{11} - m_{12} G_{11}) - \sin w_1 t (m_9 H_{11} - m_{10} G_{11} - m_{11} G_{11} - m_{12} H_{11}) \]
\[ - m_5 G_{11} - m_6 H_{11} - m_7 H_{11} - m_8 G_{11}) \]
\[ + i \sin w_1 t (m_5 H_{11} - m_6 G_{11} - m_7 G_{11} - m_8 G_{11}) \]
\[ + m_7 G_{11} - m_8 G_{11} + m_9 G_{11} + m_{10} H_{11} - m_{12} G_{11}) \]
\[ + i \cos w_1 t (m_9 H_{11} \right) \]
\[ - m_{10} G_{11} - m_{11} G_{11} - m_{12} G_{11} - m_{13} G_{11} - m_{14} H_{11} - m_{15} H_{11} - m_{16} G_{11}) \]
\[ + \frac{a_1 \nu \pi}{a} \sum_{r_2=0}^{\infty} (-1)^{r_2}(2r_2 + 1) \frac{\sinh(A_1 a)}{A_1 \cosh(A_1 a)} \]
\[ \times \left[ \frac{e^{x_1 t}(x_3 + i w_1)(1 + x_3^2 t)^2}{(x_3 + i w_1)[(1 + x_3^2 t + 1)^2]} \right] \]
\[ + \frac{2v a_3}{a} \sum_{r_2=0}^{\infty} \cos(Q_2 a) \left[ \frac{e^{x_1 t}(1 + x_3^2 t)^2}{x_5[1 + (1 + x_3^2 t)^2]} \right] \]
\[ + \frac{a_3}{k_3} \left[ \cos w_1 t (m_{13} H_{12} - m_{14} G_{12} - m_{15} G_{12} - m_{16} H_{12}) p_3 + (m_{13} G_{12}) \right. \]
\[ + m_{14} H_{12} + m_{15} H_{12} - m_{16} G_{12}) p_4 \] \[ + \sin w_2 t [(m_{13} G_{12} + m_{14} H_{12} + m_{15} H_{12}) \)
\[ - m_{16} G_{12}) p_3 - (m_{13} H_{12} - m_{14} G_{12} - m_{15} G_{12} - m_{16} H_{12}) p_4 \]
\[ + i \cos w_2 t \left( (m_{13} G_{12} + m_{14} H_{12} + m_{15} H_{12} - m_{16} G_{12}) p_3 - (m_{13} H_{12} - m_{14} G_{12}) \right] \]
\[\begin{align*}
&- m_{15}G_{12} - m_{16}H_{12})p_4 - i \sin \omega t \left[ (m_{13}H_{12} - m_{14}G_{12} - m_{15}G_{12} + m_{16}H_{12})p_4 \right] \\
&+ m_{16}H_{12})p_4 + (m_{13}G_{12} + m_{14}H_{12} + m_{15}H_{12} - m_{16}G_{12})p_4) \\
&- a_2(-1)^{r_1} \left[ \cos \omega t \left[ (m_{17}H_{12} - m_{18}G_{12} - m_{19}G_{12} \\
&- m_{20}H_{12} - m_{21}G_{12} - m_{22}H_{12} + m_{23}H_{12} - m_{24}G_{12} \\
&+ \sin \omega t \left[ (m_{17}G_{12} - m_{18}H_{12} - m_{19}H_{12} - m_{20}G_{12} - m_{21}H_{12} - m_{22}G_{12} \\
&+ m_{23}G_{12} - m_{24}H_{12}) + \cos \omega t \left[ (m_{17}G_{12} - m_{18}H_{12} - m_{19}H_{12} - m_{20}G_{12} \\
&- m_{21}H_{12} - m_{22}G_{12} + m_{23}G_{12} - m_{24}H_{12} \\
&- i \sin \omega t \left[ (m_{17}H_{12} - m_{18}G_{12} - m_{19}G_{12} - m_{20}H_{12} - m_{21}G_{12} - m_{22}H_{12} \\
&+ m_{23}H_{12} - m_{24}G_{12} \right) + \frac{a_2\nu \pi}{a^2} \sum_{n=0}^{\infty} \left( (-1)^{n+2}(2m_2 + 1) \cosh(A_1) \right) \\
&\times \left[ \frac{e^{\pi t}(x_3 + i\omega_1)(1 + x_3\tau)^2}{(x_3^2 + \omega_1^2)((1 + (x_3\tau + 1)^2)} + \frac{e^{\pi t}(x_4 + i\omega_1)(1 + x_4\tau)^2}{(x_4^2 + \omega_1^2)((1 + (x_4\tau + 1)^2)} \right] \\
&+ \frac{2\pi \mu}{a^2} \sum_{l=0}^{\infty} \left( (-1)^{r_1} \cos(Q_2a) \right) \\
&\times \sinh(Q_1a) \left[ \frac{\cos(Q_1a(x_7 - x_8)}{x_7 - x_8} - \frac{4\nu}{\nu X^2} \sum_{n=0}^{\infty} \left( (-1)^{n+1} \cosh(A_1a) \right) \\
&\times \left[ \frac{e^{\pi t}(1 + x_3\tau)^2}{x_3(1 + (1 + x_3\tau)^2)} + \frac{e^{\pi t}(1 + x_4\tau)^2}{x_4(1 + (1 + x_4\tau)^2)} \right] \\
&- \frac{2\nu}{a^2} \sum_{n=0}^{\infty} \left( (-1)^{r_1} \sinh(Q_2a) \right) \left[ \frac{e^{\pi t}(1 + x_5\tau)^2}{x_5(1 + (1 + x_5\tau)^2)} \right] \\
&+ \frac{e^{\pi t}(1 + x_6\tau)^2}{x_6(1 + (1 + x_6\tau)^2)} \right] \right) \\
&- \frac{2\mu}{a^2} \sum_{l=0}^{\infty} \left( (-1)^{r_1} \sinh(Q_2a) \right) \left[ \frac{e^{\pi t}(1 + x_5\tau)^2}{x_5(1 + (1 + x_5\tau)^2)} \right] \\
&+ \frac{e^{\pi t}(1 + x_6\tau)^2}{x_6(1 + (1 + x_6\tau)^2)} \right] \\
&\times \left\{ \frac{c_0Y \sinh(Ya)}{\nu X^2 \cosh(Ya)} + \frac{c_0Q_1 \sinh(Q_1a(x_7 - x_8)}{\nu X^2 \cosh(Q_1a(x_7 - x_8)} \left[ \frac{e^{\pi t}(1 + x_3\tau)^2}{x_3(1 + (1 + x_3\tau)^2)} + \frac{e^{\pi t}(1 + x_4\tau)^2}{x_4(1 + (1 + x_4\tau)^2)} \right] \\
&- \frac{c_0\pi^2}{a^2} \sum_{l=0}^{\infty} \left( (2r_2 + 1)^2 \right) \left[ \frac{e^{\pi t}(1 + x_5\tau)^2}{x_5(1 + (1 + x_5\tau)^2)} + \frac{e^{\pi t}(1 + x_6\tau)^2}{x_6(1 + (1 + x_6\tau)^2)} \right] \right\} \right] \end{align*}\]
\[
+ \frac{2\pi^2\mu}{a^3} \sum_{r_1=0}^{\infty} r_1^3 \cos\left(\frac{r_1 \pi s}{a}\right) \\
\times \left\{ \left[ \left( G_1 \cos(w_1 t) - H_1 \sin(w_1 t) + i(G_1 \sin(w_1 t) + H_1 \cos(w_1 t)) \right) \right] a_1 \\
- a_1 (-1)^{r_1} \left[ \left( G_2 \cos(w_1 t) - H_2 \sin(w_1 t) + i(G_2 \sin(w_1 t) + H_2 \cos(w_1 t)) \right) \right] a_1 \\
+ \frac{a_1 \nu \pi}{a^2} \sum_{r_2=0}^{\infty} (-1)^{r_2} (2r_2 + 1) \cosh(A_{1a}) A_1^2 \cosh(A_{1a}) \left[ \frac{e^{x_1 t}(x_1 - i w_1)}{x_1^2 + w_1^2} - \frac{e^{x_2 t}(x_2 + i w_1)}{x_2^2 + w_1^2} \right] \\
+ \frac{a_2 \nu \pi}{a^2} \sum_{r_2=0}^{\infty} (-1)^{r_2} (2r_2 + 1) \cosh(A_{1a}) A_1^2 \cosh(A_{1a}) \left[ \frac{e^{x_3 t}(x_3 + i w_1)}{x_3^2 + w_1^2}(1 + x_3 \tau)^2 \right] \\
+ \frac{a_2 \nu \pi}{a^2} \sum_{r_2=0}^{\infty} (-1)^{r_2} (2r_2 + 1) \cosh(A_{1a}) A_1^2 \cosh(A_{1a}) \left[ \frac{e^{x_4 t}(x_4 - i w_1)}{x_4^2 + w_1^2}(1 + x_4 \tau)^2 \right] \\
- \frac{2\pi^2\mu}{a^3} \sum_{r_1=0}^{\infty} r_1^3 \cos\left(\frac{r_1 \pi s}{a}\right) \\
\times \left\{ c_0 \left[ 1 - (-1)^{r_1} \cosh(Xa) \right] \cosh(Ya) + c_0 \left[ 1 - (-1)^{r_1} \cosh(Q_{1a}) \right] \cosh(Q_{1a}) \cos(x_7 - x_8) \right\} \\
\times \left\{ \frac{e^{x_1 t}}{x_7} - \frac{e^{x_2 t}}{x_8} + c_0 \left[ 1 - (-1)^{r_1} \cosh(Q_{1a}) \right] \cosh(Q_{1a}) \left[ \frac{e^{x_1 t}}{x_1} - \frac{e^{x_2 t}}{x_2} \right] \\
- \frac{4c_0}{\pi} \sum_{r_2=0}^{\infty} (-1)^{r_2} \cosh(A_{1a}) \left[ \frac{e^{x_3 t}(1 + x_3 \tau)^2}{x_3 \tau + (1 + x_3 \tau)^2} \right] \\
+ \frac{e^{x_4 t}(1 + x_4 \tau)^2}{x_4 \tau + (1 + x_4 \tau)^2} \right\} \\
\times \left\{ \frac{c_0}{\nu x^2} + \frac{c_0}{\nu} \frac{1}{x_7 - x_8} \left[ \frac{e^{x_1 t}}{x_7} - \frac{e^{x_2 t}}{x_8} \right] - \frac{a_1 \mu}{c_1^2 + c_2^2} \left[ G_3 \cos w_1 t - H_3 \sin w_1 t \right] \right\} \\
\right.
\]
\[ + \left( H_5 \cos w_1 t + G_5 \sin w_1 t \right) + \frac{a_1 \pi^2 \nu \mu}{2a^3} \sum_{r_2=0}^\infty (2r_2 + 1)^2 \\
\times \left[ \frac{e^{x_3 t} (x_3 + iw_1) (1 + x_3 r)^2}{(x_3^2 + w_1^2) [(l + (x_3 r + 1)^2)]} + \frac{e^{x_4 t} (x_4 + iw_1) (1 + x_4 r)^2}{(x_4^2 + w_1^2) [(l + (x_4 r + 1)^2)]} \right] \\
+ \frac{a_2 \mu}{c_8^2 + c_6^2} \left[ G_6 \cos w_2 t + H_6 \sin w_2 t + i (H_6 \cos w_2 t - G_6 \sin w_2 t) \right] \\
+ \frac{a_2 \nu^2 \mu}{2a^3} \sum_{r_2=0}^\infty (2r_2 + 1)^2 \left[ \frac{e^{x_3 t} (1 + x_3 r)^2 (x_3 + iw_1)}{x_3 [(l + (1 + x_3 r)^2)]} \\
- \frac{c_{20} \mu \sinh(Xa)}{\nu \cosh(Xa)} - \frac{2c_{20} \mu}{a} \left[ \frac{e^{x_3 t} (1 + x_3 r)^2}{x_3 [(l + (1 + x_3 r)^2)]} \right] \right] \\
+ \frac{e^{x_4 t} (1 + x_4 r)^2}{x_4 [(l + (1 + x_4 r)^2)]} \right] \\
D_{-sa} = -D(sa) \]

where

\[ y_1 = \frac{c_1 \nu + c_2 \nu^2 + w_1^2 l + w_1^2 t}{\nu (1 + w_1^2 r^2)}, \quad z_1 = \frac{w_1 + w_1 l + w_1^2 r^2}{\nu (1 + w_1^2 r^2)}, \quad y_2 = \frac{y_1 + \sqrt{y_1^2 + z_1^2}}{2}, \]

\[ y_3 = \frac{c_1 \nu + c_2 \nu^2 + c_3 w_1^2 + c_4 \nu^2 l + w_1^2 l + w_1^2 t + r_1^2 \nu^2 w_1^2 + w_1^3 \nu^2 + r_1^2 \nu^2 w_1^3}{\nu a^2 (1 + w_1^2 r^2)}, \]

\[ y_4 = \frac{y_3 + \sqrt{y_3^2 + z_3^2}}{2}, \quad z_4 = \frac{-y_3 + \sqrt{y_3^2 + z_3^2}}{2}, \quad c_1 = \cosh(y_2 a) \cos(z_2 a), \]

\[ c_2 = \sinh(y_2 a) \sin(z_2 a), \quad c_3 = \cosh(y_4 a) \cos(z_4 a), \quad c_4 = \sinh(y_4 a) \sin(z_4 a), \]

\[ d_1 = \cosh(y_4 n) \cos(z_4 n), \quad d_2 = \sin(y_4 n) \sin(z_4 n), \quad e_1 = d_1 y_3 - d_2 z_3, \]

\[ e_2 = d_2 y_3 + d_1 z_3, \quad e_3 = c_1 c_3 - c_2 c_4, \quad e_4 = c_2 c_3 + c_1 c_4, \quad \phi_1 = e_1 c_3 + e_2 c_4, \]

\[ \phi_2 = e_1 c_4 - e_2 c_3, \quad \phi_3 = e_1 c_3 + e_2 c_4, \quad \phi_4 = e_1 c_4 - e_2 c_3, \quad b_1 = a^2 r, \]

\[ b_2 = c_r a^2 \nu r + a^2 + l a^2 + r_1^2 \nu^2 r, \quad b_3 = c_r a^2 \nu + r_1^2 \nu^2, \quad x_1 = \frac{-b_2 + \sqrt{b_2^2 - 4b_1 b_3}}{2b_1}, \]

\[ x_2 = \frac{-b_2 - \sqrt{b_2^2 - 4b_1 b_3}}{2b_1}, \quad Q_1^2 = \frac{r_1^2 \pi^2}{a^2}, \quad Q_1 = \frac{r_1 \pi}{a}, \quad b_4 = 4a^2 r, \]
The text contains mathematical equations and expressions. It appears to be a page from a research paper or a scientific document, possibly related to fluid dynamics or a similar field of study. The equations are complex and involve trigonometric functions, hyperbolic functions, and other mathematical operations.
5. Conclusion

Figures 3 and 5 show the velocity profiles for the fluid and dust particles, which are paraboloid in nature. From these it is observed that the flow of fluid particles is parallel to that of dust. Also the velocity of both fluid and dust particles, which are nearer to the axis of flow, move with the greater velocity. One can observe that the appreciable effect of Number density of the dust particles of both fluid and dust i.e., as \( N \) increases velocities of both the phases decreases. Further observation shows that if the dust is very fine i.e., mass of the dust particles is negligibly small then the relaxation time of dust particle decreases and ultimately as \( \tau \to 0 \) the velocities of fluid and dust particles will be the same. Also it is evident from the graphs that as time \( t \) increases the velocity of both the phases decreases ultimately for large time \( t \) the velocity becomes zero.
Figure-3: Variation of fluid velocity with $s$ and $n$ (for $t = 0.1$ & $t = 0.2$)

Figure-4: Variation of dust velocity with $s$ and $n$ (for $t = 0.1$ & $t = 0.2$)

Figure-5: Variation of fluid velocity with $s$ and $n$ (for $N = 0.25$ & $N = 0.5$)
Mahesha, B.J.Gireesha, G.K.Ramesh and C.S.Bagewadi - Unsteady Flow

Figure-6: Variation of dust velocity with $s$ and $n$ (for $N = 0.25 \& N = 0.5$)

REFERENCES


Mahesha
Department of Mathematics
University BDT Engineering College
Davanagere, Karnataka, INDIA.
email: maheshubdt@gmail.com

Bijjanal Jayanna Gireesha
Department of Mathematics
Kuvempu University
Shankaraghatta-577 451
Shimoga, Karnataka, INDIA.
email: bijgireesu@rediffmail.com

Gosikere Kenchappa Ramesha
Department of Mathematics
Kuvempu University
Shankaraghatta-577 451
Shimoga, Karnataka, INDIA.
email: gkrmaths@gmail.com
Channabasappa Shanthappa Bagewadi
Department of Mathematics
Kuvempu University
Shankaraghatta-577 451
Shimoga, Karnataka, INDIA.
email: prof_bagewadi@yahoo.co.in
UNSTEADY FLOW OF A CONDUCTING DUSTY FLUID BETWEEN TWO CIRCULAR CYLINDERS

G. K. RAMESH, MAHESHA, B. J. GIREESHA AND C. S. BAGEWADI

ABSTRACT. The present analysis deals with the study of laminar flow of a conducting dusty fluid with uniform distribution of dust particles between two circular cylinders. Initially the fluid and dust particles are at rest. The flow is due to the influence of time dependent pressure gradient and the differential rotations of the circular cylinders. The exact solutions for both fluid and dust velocities are obtained using Variable Separable method. Further the skin friction at the boundaries is calculated. Finally the changes in the velocity profiles with \( R \) are shown graphically.

1. INTRODUCTION

The influence of dust particles on viscous flows has great importance in petroleum industry and in the purification of crude oil. Other important applications of dust particles in a boundary layer include soil erosion by natural winds and dust entrainment in a cloud during nuclear explosion. Also such flows occur in a wide range of areas of technical importance like fluidization, flow in rocket tubes, combustion, paint spraying and more recently blood flows in capillaries.


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Yang Lei and Bakhtier Farouk [17] investigated three-dimensional mixed convection flows in a horizontal annulus with a heated rotating inner circular cylinder. Colette Calmelet-Eluhu and Philip Crooke [4] studied unsteady conducting dusty gas flow through a circular pipe in the presence of an applied and induced magnetic field. The authors Bagewadi and Gireesha [2], [3] studied two-dimensional dusty fluid flow in Frenet frame field system and recently the authors [7], [8] obtained solutions for the flow of unsteady dusty fluid under varying time dependent pressure gradients through different regions like parallel plates, rectangular channel and open rectangular channel.

The present investigation deals with the study of unsteady flow of a conducting dusty fluid between two circular cylinders. Here the flow is due to the influence of time dependent pressure gradient and differential rotations of the cylinders. The fluid and dust particles are assumed to be at rest initially. The analytical expressions are obtained for velocities of fluid and dust particles. Further the skin friction at the boundaries is calculated and graphical representation of the velocity profiles versus $R$ is given.

2. EQUATIONS OF MOTION

The equations of motion of conducting unsteady viscous incompressible fluid with uniform distribution of dust particles are given by [14]:

For fluid phase

\begin{align}
\nabla \cdot \vec{u} & = 0 \quad \text{(Continuity)} \\
\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} & = -\rho^{-1} \nabla p + \nu \nabla^2 \vec{u} + \frac{kN}{\rho} (\vec{v} - \vec{u}) + \frac{1}{\rho} (\vec{J} \times \vec{B}) \quad \text{(Linear Momentum)}
\end{align}

For dust phase

\begin{align}
\nabla \cdot \vec{v} & = 0 \quad \text{(Continuity)} \\
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} & = \frac{k}{m} (\vec{u} - \vec{v}) \quad \text{(Linear Momentum)}
\end{align}

We have following nomenclature: $\vec{u}$—velocity of the fluid phase, $\vec{v}$—velocity of dust phase, $\rho$—density of the gas, $p$—pressure of the fluid, $N$—number density of dust particles, $\nu$—kinematic viscosity, $k = 6\pi a \mu$—Stoke’s resistance (drag coefficient), $a$—spherical radius of dust particle, $m$—mass of the dust particle, $\mu$—the coefficient of viscosity of fluid particles, $t$—time and $\vec{J}$ and $\vec{B}$ given by Maxwell’s equations and Ohm’s law, namely,

\begin{align}
\nabla \times \vec{H} & = 4\pi \vec{J}, \quad \nabla \times \vec{B} = 0, \quad \nabla \times \vec{E} = 0, \quad \vec{J} = \sigma [\vec{E} + \vec{u} \times \vec{B}]
\end{align}

Here $\vec{H}$—magnetic field, $\vec{J}$—current density, $\vec{B}$—magnetic flux, $\vec{E}$—electric field and $\sigma$—the electrical conductivity of the fluid.
It is assumed that the effect of induced magnetic fields produced by the motion of the electrically conducting gas is negligible and no external electric field is applied. With those assumptions the magnetic field \( J \times B \) of the body force in (2.2) reduces simply to \(-\sigma B_0^2 \mathbf{u}\), where \( B_0 \) is the intensity of the imposed transverse magnetic field.

### 3. Formulation of the Problem

Consider a flow of viscous incompressible, conducting dusty fluid between two circular cylinders. The inner cylinder is of unit radius and outer cylinder is of radius \( b \). The flow is due to the influence of time dependent pressure gradient and differential rotations of the cylinders. It is assumed that the inner and outer cylinders rotate with different angular velocities. Both the fluid and the dust particle clouds are supposed to be static at the beginning. The dust particles are assumed to be spherical in shape and uniform in size. The number density of the dust particles is taken as a constant throughout the flow. As Figure 1 shows, the axis of the channel is along z-axis and the velocity components of both fluid and dust particles are respectively given by:

\[
\begin{align*}
  u_r &= 0; \quad u_\theta = 0; \quad u_z = u_z(r,t); \\
  v_r &= 0; \quad v_\theta = 0; \quad v_z = v_z(r,t)
\end{align*}
\]

where \((u_r, u_\theta, u_z)\) and \((v_r, v_\theta, v_z)\) are velocity components of fluid and dust particles, respectively.

![Figure 1. Schematic diagram of the flow.](image)

By virtue of equation (3.1) the intrinsic decomposition of equations (2.1) to (2.4) in cylindrical polar coordinates give the following forms:

\[
\begin{align*}
  (3.2) \quad &-\frac{1}{\rho} \frac{\partial p}{\partial r} = 0, \\
  (3.3) \quad &\frac{\partial u_z}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left[ \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} \right] + \frac{kN}{\rho} (v_z - u_z) - \frac{\sigma B_0^2}{\rho} u_z, \\
  (3.4) \quad &\frac{\partial v_z}{\partial t} = \frac{k}{m} (u_z - v_z),
\end{align*}
\]
Let us introduce the following non-dimensional quantities:

\[ R = \frac{r}{a}, \quad \bar{z} = \frac{z}{a}, \quad \bar{p} = \frac{pa^2}{\rho \nu^2}, \quad T = \frac{\nu}{a^2}, \quad u = \frac{u_s a}{\nu}, \quad v = \frac{u_s a}{\nu}, \]

\[ \beta = \frac{l}{\gamma} = \frac{Nk a^2}{\rho \nu}, \quad l = \frac{Nm}{\rho}, \quad \gamma = \frac{\nu m}{k a^2}. \]

Transform the equations (3.2)-(3.4) to the non-dimensional forms as

\[ \frac{\partial^2 u}{\partial T^2} + (l + 1 + M^2 \gamma) \frac{\partial u}{\partial T} - \gamma \frac{\partial}{\partial T} \left[ \frac{\partial^2 u}{\partial R^2} + \frac{1}{R} \frac{\partial u}{\partial R} \right] = \beta (v - u) - M^2 u, \]

\[ \frac{\partial v}{\partial T} = (u - v), \]

where \( M = \frac{Bo a \sqrt{\sigma/\mu}}{\nu} \) is Hartmann number.

Since we have assumed that the time dependent pressure gradient is impressed on the system for \( t > 0 \), so we can write

\[ \frac{1}{\rho} \frac{\partial p}{\partial T} = c + d e^{i \omega t}, \]

where \( c, d \) and \( \alpha \) are reals.

Eliminating \( v \) from (3.7) and (3.8) and then substituting the expression for pressure gradient, one can get

\[ \frac{\partial^2 u}{\partial T^2} + (l + 1 + M^2 \gamma) \frac{\partial u}{\partial T} - \gamma \frac{\partial}{\partial T} \left[ \frac{\partial^2 u}{\partial R^2} + \frac{1}{R} \frac{\partial u}{\partial R} \right] = c + d e^{i \omega t} + \left[ \frac{\partial^2 u}{\partial R^2} + \frac{1}{R} \frac{\partial u}{\partial R} \right] - M^2 u. \]

4. Solution Part

Let the solution of the equation (3.9) be written in the form [16], [11]

\[ u = U(R) + V(R, T), \]

where \( U \) is the steady part and \( V \) is the unsteady part of the fluid velocity.

Separating the steady part from the unsteady part of the equation (3.9), we get

\[ \frac{\partial^2 U}{\partial R^2} + \frac{1}{R} \frac{\partial U}{\partial R} - M^2 U = -c, \]

\[ \frac{\partial^2 V}{\partial R^2} + (l + 1 + M^2 \gamma) \frac{\partial V}{\partial T} - \gamma \frac{\partial}{\partial T} \left[ \frac{\partial^2 V}{\partial R^2} + \frac{1}{R} \frac{\partial V}{\partial R} \right] = d e^{i \omega t} + \left[ \frac{\partial^2 V}{\partial R^2} + \frac{1}{R} \frac{\partial V}{\partial R} \right] - M^2 V. \]
Case 1. Periodic Motion.

Consider the boundary conditions

\[ u = u_1 \sin(\alpha T), \quad \text{at} \quad R = 1, \]
\[ u = u_2 \sin(\alpha T), \quad \text{at} \quad R = b, \]

where \( u_1 \) and \( u_2 \) are uniform angular velocities.

Since \( u = U(R) + V(R, T) \), one can see that the boundary conditions become as follows:

\[
\begin{align*}
U &= 0 \quad \text{and} \quad V = u_1 \sin(\alpha T) \quad \text{at} \quad R = 1, \\
U &= 0 \quad \text{and} \quad V = u_2 \sin(\alpha T) \quad \text{at} \quad R = b.
\end{align*}
\]

Now, by solving equation (4.2) using the boundary conditions (4.4), one can get

\[
U = \frac{c}{M^2} \left( \frac{T_1 J_0(MR) + T_2 K_0(MR)}{T_0} - 1 \right),
\]

where \( J_0 \) and \( K_0 \) are Bessel’s functions of the first and the second kind, respectively, of order zero.

Assume the solution of the equation (4.3) is in the form

\[
V = g(R)e^{\lambda T},
\]

where \( g(R) \) is an unknown function to be determined.

Using equation (4.6) in (4.3), one can obtain

\[
\frac{\partial^2 g}{\partial R^2} + \frac{1}{R}\frac{\partial g}{\partial R} - \lambda_1^2 g = -\lambda_2,
\]

where \( \lambda_1 = \frac{(M^2 - \alpha^2) + \alpha(1 + \alpha)}{(1 + \alpha)^2} \) and \( \lambda_2 = \frac{d}{(1 + \alpha)^2} \).

Using the boundary conditions (4.4) one can obtain the solution of (4.7)

\[
g(R) = \frac{\lambda_2}{\lambda_1^2} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right]
+ \frac{\sin(\alpha T)}{e^{\lambda T}} \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right],
\]

Using this in (4.6), we get

\[
V = \frac{\lambda_2 e^{\lambda T}}{\lambda_1^2} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right]
+ \sin(\alpha T) \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right].
\]
Now, using equations (4.9) and (4.5) in (4.1), we obtain the fluid velocity \( u \) in the form

\[
u = \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) + T_2 K_0(MR)}{T_0} - 1 \right] + \frac{\lambda_2 e^{i\alpha t}}{\lambda_1^2} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] + \sin(\alpha T) \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right],
\]

Also, the dust phase velocity is obtained from equation (3.8) as

\[
v = A e^{-\frac{1}{2} \alpha T} + \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) + T_2 K_0(MR)}{T_0} - 1 \right] + \frac{\lambda_2 e^{i\alpha t}}{\lambda_1^2(1 + i\alpha \gamma)} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] + \frac{1}{1 + \alpha^2 \gamma^2} \left[ \sin \alpha T - \alpha \gamma \cos \alpha T \right] \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right],
\]

where

\[
A = \frac{\alpha \gamma}{1 + \alpha^2 \gamma^2} \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right] - \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) + T_2 K_0(MR)}{T_0} - 1 \right] - \frac{\lambda_2 (1 - \alpha \gamma)}{\lambda_1^2(1 + \alpha^2 \gamma^2)} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right].
\]

Figure 2. Variation of fluid and dust velocities with \( R \) for Case 1.
Shearing Stress (Skin Friction).

The Shear stress at the boundaries $R = 1$ and $R = b$, respectively, is given by

\[
D_1 = \frac{\mu c}{M} \left[ \frac{T_1 J_0'(M)}{T_0} + \frac{T_2 K_0'(M)}{T_0} \right] + \frac{\mu \lambda_2 e^{i\alpha t}}{\lambda_1} \left[ \frac{Q_1 J_0'(\lambda_1) + Q_3 K_0'(\lambda_1)}{Q_0} \right] + \mu \lambda_1 \sin(\alpha T) \left[ \frac{Q_2 J_0'(\lambda_1) + Q_4 K_0'(\lambda_1)}{Q_0} \right]
\]

\[
D_b = \frac{\mu c}{M} \left[ \frac{T_1 J_0'(Mb)}{T_0} + \frac{T_2 K_0'(Mb)}{T_0} \right] + \frac{\mu \lambda_2 e^{i\alpha t}}{\lambda_1} \left[ \frac{Q_1 J_0'(\lambda_1 b) + Q_3 K_0'(\lambda_1 b)}{Q_0} \right] + \mu \lambda_1 \sin(\alpha T) \left[ \frac{Q_2 J_0'(\lambda_1 b) + Q_4 K_0'(\lambda_1 b)}{Q_0} \right]
\]

Case 2. Impulsive Motion.

In impulsive motion, we consider the boundary conditions

\[
\begin{align*}
\text{at } R = 1, \quad u &= u_1 \delta(T), \\
\text{at } R = b, \quad u &= u_b \delta(T),
\end{align*}
\]

where $\delta(T)$ is the Dirac delta function.

Using these boundary conditions, one can see that the solution for velocities of fluid and dust phases is obtained as

\[
u = \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) + T_2 K_0(MR)}{T_0} - 1 \right] + \frac{\lambda_2 e^{i\alpha t}}{\lambda_1} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] + \delta(T) \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right],
\]
Figure 4. Variation of fluid and dust velocities with $R$ for Case 2.

Figure 5. Variation of fluid and dust velocities with $R$ for Case 2.

and

$$v = \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) + T_2 K_0(MR)}{T_0} - 1 \right]$$

$$+ \frac{\lambda_2 e^{i\alpha t}}{(1 + i\alpha \gamma) \lambda_1^2} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right]$$

$$+ \frac{e^{-\gamma T}}{\gamma} \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right] + A_1 e^{-\gamma T},$$

(4.13)

where

$$A_1 = -\frac{1}{\gamma} \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right]$$

$$- \frac{\lambda_2}{\lambda_1^2 (1 + i\alpha \gamma)} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right]$$

$$- \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) + T_2 K_0(MR)}{T_0} - 1 \right].$$
Shearing Stress (Skin Friction).

The Shear stress, i.e., the skin friction at \( R = 1 \) and \( R = b \), respectively, is given by

\[
D_1 = \frac{\mu c}{M} \left[ \frac{T_1 J_0(M) + T_2 K_0(M)}{T_0} \right] + \frac{\mu \lambda_2 e^{i\alpha t}}{\lambda_1} \left[ \frac{Q_1 J_0(\lambda_1) + Q_3 K_0(\lambda_1)}{Q_0} \right] + \mu \delta(T) \lambda_1 \left[ \frac{Q_2 J_0(\lambda_1) + Q_4 K_0(\lambda_1)}{Q_0} \right],
\]

\[
D_2 = \frac{\mu c}{M} \left[ \frac{T_1 J_0(Mb) + T_2 K_0(Mb)}{T_0} \right] + \frac{\mu \lambda_2 \mu}{\lambda_1} \left[ \frac{Q_1 J_0(\lambda_1 b) + Q_3 K_0(\lambda_1 b)}{Q_0} \right] + \mu \delta(T) \lambda_1 \left[ \frac{Q_2 J_0(\lambda_1 b) + Q_4 K_0(\lambda_1 b)}{Q_0} \right].
\]

Case 3. Transition Motion.

For transition motion, we consider the boundary conditions

\[ u = u_1 H(T) e^{\alpha t}, \quad \text{at} \quad R = 1, \]

\[ u = u_2 H(T) e^{\alpha t}, \quad \text{at} \quad R = b, \]

where \( H(T) \) is the Heaviside's unit step function.

Using these boundary conditions, the solution for velocities of fluid and dust phases can be written as

\[
u = \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) + T_2 K_0(MR)}{T_0} - 1 \right] + \frac{\lambda_2 e^{\alpha t}}{\lambda_1^2} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] + H(T) e^{\alpha t} \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right]
\]

and

\[
v = A_2 e^{-\frac{\alpha}{2} T} + \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) + T_2 K_0(MR)}{T_0} - 1 \right] + \frac{\lambda_2 e^{\alpha t}}{\lambda_1^2(1+i\alpha \gamma)} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right]
\]

\[
e^{-\frac{i\alpha}{2} T} e^{\left(\frac{i}{2} + \alpha \gamma\right) T} \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right] \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right],
\]

where

\[
A_2 = -\frac{\lambda_2}{\lambda_1^2(1+i\alpha \gamma)} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] - \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) + T_2 K_0(MR)}{T_0} - 1 \right].
\]
Shearing Stress (Skin Friction).

The skin friction at $R = 1$ and $R = b$, respectively, is given by

$$D_1 = \frac{\mu c}{M} \left[ \frac{T_1 J_0(M)}{T_0} + T_2 K_0'(M) \right] + \frac{\mu \lambda_2 e^{\alpha t}}{\lambda_1} \left[ \frac{Q_1 J_0' \lambda_1 + Q_3 K_0'(\lambda_1)}{Q_0} \right]$$

$$+ \mu \lambda_1 H(T)e^{\alpha T} \left[ \frac{Q_2 J_0'(\lambda_1) + Q_4 K_0'(\lambda_1)}{Q_0} \right],$$

$$D_0 = \frac{\mu c}{M} \left[ \frac{T_1 J_0(Mb) + T_2 K_0'(Mb)}{T_0} \right] + \frac{\mu \lambda_2 e^{\alpha t}}{\lambda_1} \left[ \frac{Q_1 J_0'(\lambda_1 b) + Q_3 K_0'(\lambda_1 b)}{Q_0} \right]$$

$$+ \mu \lambda_1 H(T)e^{\alpha T} \left[ \frac{Q_2 J_0'(\lambda_1 b) + Q_4 K_0'(\lambda_1 b)}{Q_0} \right].$$
**Case 4. Motion for a Finite Time.**

For this case, we consider the boundary conditions

\[
\begin{align*}
  u &= u_1[H(T) - H(T - t)], \quad \text{at} \quad R = 1, \\
  u &= u_2[H(T) - H(T - t)], \quad \text{at} \quad R = b,
\end{align*}
\]

where \( H(T) \) is the Heaviside step function.

Using these boundary conditions, we found the solution for velocities of fluid and dust phases as follows:

\[
\begin{align*}
  u &= \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) + T_2 K_0(MR)}{T_0} - 1 \right] \\
  &\quad + \frac{\lambda_2 e^{i\alpha t}}{\lambda_1^2} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] \\
  &\quad + [H(T) - H(T - t)] \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right]
\end{align*}
\]

(4.16)

and

\[
\begin{align*}
  v &= A_3 e^{-\alpha T} + \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) + T_2 K_0(MR)}{T_0} - 1 \right] \\
  &\quad + \frac{\lambda_2 e^{i\alpha t}}{\lambda_1^2(1 + i\alpha \gamma)} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] \\
  &\quad + \frac{e^{-\alpha T} e^{1+\alpha \gamma} - 1}{(1 + \alpha \gamma)} [H(T) - H(T - t)] \left[ \frac{Q_2 J_0(\lambda_1 R) + Q_4 K_0(\lambda_1 R)}{Q_0} \right],
\end{align*}
\]

(4.17)

where

\[
A_3 = -\frac{\lambda_2}{\lambda_1^2 (1 + i\alpha \gamma)} \left[ \frac{Q_1 J_0(\lambda_1 R) + Q_3 K_0(\lambda_1 R)}{Q_0} - 1 \right] - \frac{c}{M^2} \left[ \frac{T_1 J_0(MR) + T_2 K_0(MR)}{T_0} - 1 \right].
\]
Shearing Stress (Skin Friction).

The Shear stress, i.e. the skin friction at the boundaries $R = 1$ and $R = b$, respectively, is given by

$$D_1 = \frac{\mu c}{M} \left[ \frac{T_1 J_0(M) + T_2 K_0'(M)}{T_0} \right] + \frac{\mu \lambda_2 e^{i\omega t}}{\lambda_1} \left[ \frac{Q_1 J_0'(\lambda_1) + Q_3 K_0'(\lambda_1)}{Q_0} \right] + \mu \lambda_1 [H(T) - H(T - t)] \left[ \frac{Q_2 J_0'(\lambda_1) + Q_4 K_0'(\lambda_1)}{Q_0} \right],$$

$$D_b = \frac{\mu c}{M} \left[ \frac{T_1 J_0(Mb) + T_2 K_0'(Mb)}{T_0} \right] + \frac{\mu \lambda_2 e^{i\omega t}}{\lambda_1} \left[ \frac{Q_1 J_0'(\lambda_1 b) + Q_3 K_0'(\lambda_1 b)}{Q_0} \right] + \mu \lambda_1 [H(T) - H(T - t)] \left[ \frac{Q_2 J_0'(\lambda_1 b) + Q_4 K_0'(\lambda_1 b)}{Q_0} \right],$$

where

$$T_0 = J_0(M)K_0(Mb) - J_0(Mb)K_0(M), \quad T_1 = K_0(Mb) - K_0(M),$$

$$T_2 = J_0(M) - J_0(Mb),$$

$$Q_0 = J_0(\lambda_1)K_0(\lambda_1 b) - J_0(\lambda_1 b)K_0(\lambda_1), \quad Q_1 = K_0(\lambda_1 b) - K_0(\lambda_1),$$

$$Q_2 = u_1 K_0(\lambda_1 b) - u_2 K_0(\lambda_1), \quad Q_3 = J_0(\lambda_1) - J_0(\lambda_1 b),$$

$$Q_4 = u_2 J_0(\lambda_1) - u_1 J_0(\lambda_1 b).$$

5. Conclusion

In the present paper, we have studied the laminar flow of a conducting dusty fluid between two circular cylinders. The four different cases based on the time dependent pressure gradient are discussed. Variable separable and Eigen expansion methods are employed to solve the governing equations. The graphs for velocity profiles are shown as in Figures from 2 to 9 for different values of parameters like Hartmann number $(M)$ and Time $(T)$ showing that they are parabolic in nature. From Figures 2, 4, 6 and 8 one can observed the appreciable effect of Hartmann
number on the flow of both fluid and dust phases, i.e. the magnetic field has retarding influence. Also, it is evident from the Figures 3, 5, 7 and 9 that as time increases the velocities of both phases decrease, which is desirable in physical situations. Further, we can see that if $\gamma \to 0$, i.e. if the dust is very fine, then the velocities of both fluid and dust particles will be the same.

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REFERENCES

3. , A study of two dimensional steady dusty fluid flow under varying pressure gradient, Tensor, N.S., 64 (2003), 232–240.
Unsteady Flow and Heat Transfer of a Dusty Fluid Between Two Infinite Parallel Plates

B.J.GIREESH, C.S.BAGEWADI AND *MAHESHA
Department of Mathematics, Kuvempu University, Shankaraghatta-577451, Shimoga, Karnataka, India.
*Department of Mathematics, University BDT Engineering College, Davanagere.
e-mail: prof_bagewadi@rediffmail.com and bjgireesu@rediffmail.com

ABSTRACT:
The geometry of laminar flow and heat transfer of an unsteady viscous liquid with uniform distribution of dust particles between two infinite stationary parallel plates under the influence of exponential pressure gradient has been considered. The plates are maintained at a constant temperature. Solutions are obtained for fluid and dust phase velocity and temperature distributions using differential geometry techniques. The results are discussed with the help of graphs.

Key Words: Frenet frame field system; parallel plates, laminar flow, dusty fluid; velocity of dust phase and fluid phase, heat transfer, temperature.

AMS Subject Classification (2000): 76T10, 76T15;

1. Introduction:

The study of dusty fluid flow has importance in many areas like environmental pollution, smoke emission from vehicles, emission of effluents from industries like cement, the cooling effects of air conditioners, flying ash produced from thermal reactors and formation of raindrops etc. Also it is useful in lunar ash flow which explains many features of lunar soil. The study of dusty fluids under different physical conditions have been carried out by Several authors like Marble [10], Samba Siva Rao [15], Michael and Miller
Gireesha et al.


The study of fluid flow using differential geometry techniques i.e., Frenet frame field system were traced in many papers: Kanwal [9], Trusdell [17], Indrasena [8], Purushotham [13], Bagewadi, Shantharajappa and Gireesha [1, 2, 3]. Further, recently the authors [2, 3] have studied two-dimensional dusty fluid flow in Frenet frame field system under varying physical parameters temperature and pressure. The present paper deals with the study of geometry of laminar flow and heat transfer of an unsteady dusty fluid between two infinite stationary parallel plates under the influence of exponential pressure gradient has been considered. The plates are maintained at a constant temperature. The analytical solutions are obtained for fluid and dust phase velocity and temperature distributions using Frenet frame field system. The graphs are plotted for fluid and dust phase velocity and for temperatures.

2. Equations of Motion:

The equations of motion of unsteady viscous incompressible fluid with uniform distribution of dust particles are given by [14]:

For fluid phase

(2.1) \( \nabla \cdot \bar{u} = 0 \) (Continuity)

(2.2) \( \frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} = -\rho \nabla p + \nu \nabla^2 \bar{u} + \frac{f}{\tau_e} (\bar{v} - \bar{u}) \) (Linear Momentum)

(2.3) \( \rho \left( \frac{\partial E}{\partial t} + (\bar{u} \cdot \nabla) E \right) = Q + (\bar{v} - \bar{u}) \cdot F + k \nabla \cdot (\nabla T) \) (Energy)

For dust phase
(2.4) \( \nabla \cdot \mathbf{v} = 0 \) (Continuity)

(2.5) \( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\tau_v} (\mathbf{u} - \mathbf{v}) \) (Linear Momentum)

(2.6) \( N \left( \frac{\partial E}{\partial t} + (\mathbf{u} \cdot \nabla) E \right) = -Q \) (Energy)

We have following nomenclature:

\[ E = c_p T, \quad E_p = c_m T_p; \quad Q = N c_p (T - T_p) / \tau \] is the thermal interaction between fluid and dust particle phase, \( F = N (\mathbf{u} - \mathbf{v}) \) is the velocity interaction force between the fluid and dust particle phase, \( \tau_v = m / 6 \pi \mu = m / K \) is the velocity relaxation time of the dust particles, \( \tau_T = m C_p / 4 \pi \alpha k \) is the thermal relaxation time of the dust particles, \( k \nabla (\nabla T) \) is the rate of heat added to the fluid by conduction in unit volume; \( \mathbf{u}, \rho, p, v, T, c_p, \) & \( k \) are respectively, the velocity vector, density, pressure, kinematic viscosity, temperature, specific heat & thermal conductivity of the fluid; \( \mathbf{v}, N, T_p, c_m, m \) are respectively, the velocity vector, number density, temperature, specific heat, mass concentration. \( K = 6 \pi \mu \) - Stoke’s resistance coefficient and \( t \) - time.

Let \( s, n, b \) be triply orthogonal unit vectors tangent, principal normal, binormal respectively to the spatial curves of congruences formed by fluid phase velocity & dusty phase velocity lines respectively as shown in the figure 1.
Geometrical relations are given by Frenet formulae\[4\]

\( i) \ \frac{\partial \vec{S}}{\partial s} = K \ n, \quad \frac{\partial \vec{n}}{\partial s} = \vec{\tau}, \quad \frac{\partial \vec{b}}{\partial s} = -\vec{\tau}, \ n \)

\( ii) \ \frac{\partial \vec{n}}{\partial n} = K \ s, \quad \frac{\partial \vec{b}}{\partial n} = -\sigma \ s, \quad \frac{\partial \vec{b}}{\partial s} = \sigma \ b - K \ n \)

\( iii) \ \frac{\partial \vec{b}}{\partial b} = k \ s, \quad \frac{\partial \vec{n}}{\partial b} = -\sigma \ s, \quad \frac{\partial \vec{n}}{\partial s} = \sigma \ n - k \ b \)

\( iv) \ \nabla \cdot s = \theta_m + \theta_b; \quad \nabla \cdot n = \theta_{mb} - k_s; \quad \nabla \cdot b = \theta_{nb} \)

where \( \partial / \partial s, \partial / \partial n \) & \( \partial / \partial b \) are the intrinsic differential operators along fluid phase velocity (or dust phase velocity) lines, principal normal and binormal. The functions \( (k, k', k'_b) \) & \( (\tau, \sigma, \sigma'_b) \) are the curvatures and torsion of the above curves and \( \theta_m \) & \( \theta_b \) are normal deformations of these spatial curves along their principal normal and binormal respectively.

3. Formulation of the Problem:

Consider the dusty fluid flowing between two infinite stationary horizontal parallel plates separated by a distance \( 2h \) in the absence of body force. The dusty particles are assumed to be uniformly distributed and spherical in shape. Two plates are kept at constant temperatures \( T_i \) for the lower plate and \( T_j \) for the upper plate with \( T_j > T_i \). An exponential pressure gradient with respect to time is influenced on the flow.

As in figure 2, the flow be in tangential direction and perpendicular to binormal direction. Then the velocities of fluid and dust are of the form

\[
\begin{align*}
\vec{u} &= u_s \ n, \\
\vec{v} &= v_s \ n
\end{align*}
\]

\[ (3.1) \]

i.e \( u_n = u_b = 0 \) & \( v_n = v_b = 0 \). where \( (u_s, u_n, u_b) \) & \( (v_s, v_n, v_b) \) denote the velocity components of fluid & dust respectively.
Since the fluid flow is in between two infinite stationary parallel plates, we can assume the velocity of both fluid & dust particles do not vary along tangential direction. Suppose the fluid extends to infinity in the principal normal direction, then the velocities of both may be neglected in this direction.

\[ b = h \quad u_s = v_s = 0 \quad T = T_2 \]

\[ b = -h \quad u_s = v_s = 0 \quad T = T_1 \]

**Figure-2: Schematic diagram of the dusty fluid flow.**

Since the fluid flow is in between two infinite stationary parallel plates, we can assume the velocity of both fluid and dust particles are functions of \( b \) and \( t \), i.e., \( u, = u, \,(b,t) \) & \( v, = v, \,(b,t) \).

4. Solutions of the Problem:

By virtue of system of equations (2.7) the intrinsic decomposition of equations (2.2), (2.3), (2.5) and (2.6) give the following forms:

\[
\frac{\partial u_s}{\partial t} = - \frac{1}{\rho} \frac{\partial p}{\partial b} + v \left[ \frac{\partial^2 u_s}{\partial b^2} - C_s, u_s \right] + \frac{f}{\tau_s} (v, - u,)
\]

(4.1)

\[
2u_s^2 k, = - \frac{1}{\rho} \frac{\partial p}{\partial b} + v \left[ 2\sigma_s^* \frac{\partial u_s}{\partial b} - u_s k_s \right]
\]

(4.2)

\[
0 = - \frac{1}{\rho} \frac{\partial p}{\partial b} + v \left[ u, k, \tau, + 2k_s \frac{\partial u_s}{\partial b} \right]
\]

(4.3)

\[
\frac{\rho c_v}{k} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial b^2} + \theta_k \frac{\partial T}{\partial b} + \frac{N c_p}{\tau} (\tau - T) + \frac{N}{\tau} (v, - u,)^2
\]

(4.4)
\[
\frac{\partial v_s}{\partial t} = \frac{k}{m} (u_s - v_s)
\]

(4.6) \[2v_s^2 k_s = 0\]

(4.7) \[c_m \frac{\partial T_p}{\partial t} = c_p \frac{T - T_p}{\tau_f}\]

where \(C_s = \left( \sigma_s^{x^2} + k_s^{y^2} + k_s^{y^2} + \sigma_s^{z^2} \right)\) is called curvature number [3].

The boundary conditions of the problem are given by

\[
\begin{align*}
&u_s = 0, \quad v_s = 0, \quad T = T_1, \quad \text{at} \quad b = -h \\
&u_s = 0, \quad v_s = 0, \quad T = T_2, \quad \text{at} \quad b = h
\end{align*}
\]

where \(T_1\) and \(T_2\) are constant temperatures at lower and upper plates respectively.

From equation (4.6) we see that \(v_s^2 k_s = 0\) which implies either \(v_s = 0\) or \(k_s = 0\). The choice \(v_s = 0\) is impossible, since if it happens then \(u_s = 0\), which shows that the flow doesn't exist. Hence \(k_s = 0\), it suggests that the curvature of the streamline along tangential direction is zero. Thus no radial flow exists.

Since we have assumed the pressure gradient is exponentially varying with respect to time, we can write

\[
\frac{-1}{\rho} \frac{\partial P}{\partial s} = \alpha e^{-\beta s}
\]

where \(\alpha\) and \(\beta\) are real constants.

In view of (4.8) we can express \(u_s, v_s, T\ & T_p\) as

(4.9) \[u_s = \omega_1 (b) e^{-\beta s}; \quad v_s = \omega_2 (b) e^{-\beta s};\]

(4.10) \[T = \theta_1 (b) e^{-2\beta s}; \quad T_p = \theta_2 (b) e^{-2\beta s};\]
Using (4.8) and (4.9) in (4.1), (4.5) and (4.4), (4.7) one obtains respectively

\begin{equation}
- \beta \cdot \omega_1 = \alpha + \nu \left[ \frac{\partial \omega_1}{\partial b^2} - C_a \omega_1 \right] + \frac{f}{\tau_v} (\omega_2 - \omega_1)
\end{equation}

\begin{equation}
\omega_2 = \frac{\omega_1}{1 - \tau_v \beta^2}.
\end{equation}

\begin{equation}
\frac{\partial^2 \theta}{\partial b^2} + \theta_{,m} \frac{\partial \theta}{\partial b} + \frac{2 \beta^2 \rho_c}{k} \theta + \frac{N \epsilon_c}{\tau_T} (\theta - \theta_T) + \frac{N}{\tau_T} (\omega_2 - \omega_1)^2 = 0
\end{equation}

\begin{equation}
\theta_P = \left( \frac{c_p \theta}{c_p - 2 \beta^2 c_w \tau_T} \right)
\end{equation}

Eliminating \( \omega_2 \) from (4.11) and (4.12) we obtain the following equation

\begin{equation}
\frac{d^2 \omega_1}{db^2} - Q^2 \omega_1 = -\frac{\alpha}{\nu}
\end{equation}

where

\[ Q^2 = \left( C_v - \frac{\beta^2}{\nu} - \frac{f \beta^2}{\nu (1 - \tau_v \beta^2)} \right) \quad \text{and} \quad f = \frac{mN}{\rho} \]

The velocities of fluid and dust particle are obtained by solving the equation (4.15) subjected to the boundary conditions

\( \omega_1 = 0, \quad \text{at} \quad b = -h \) and

\( \omega_1 = 0, \quad \text{at} \quad b = h \)

Now from (4.15) and using above boundary conditions, we get

\[ \omega_1 = \frac{\alpha}{\nu Q^2} \left( \frac{1 - \cosh (Qb \tau_T)}{\cosh (Qb \tau_T)} \right) \]

By virtue of \( \omega_1 \) in (4.12), we obtained as follows
Eliminating $\theta_{\mu}$ from (4.13) and (4.14) we obtain the following equation

\begin{equation}
\frac{d^2 \theta}{db^2} + a_1 \frac{d \theta}{db} + a_2 \theta = -\frac{N}{\tau_k} (\omega_2 - \omega_1)^2
\end{equation}

where $a_1 = \theta_{ab}$ and $a_2 = \frac{2\beta^2 c_p}{\kappa} \left( \rho + \frac{N c_m}{c_p - 2\beta^2 c_m \tau_k} \right)$.

Consider the boundary conditions for temperature as

\[ \theta = T_1 e^{2\beta_1} \text{ at } b = -h \]
\[ \theta = T_2 e^{2\beta_1} \text{ at } b = h \]

The temperatures of fluid and dust particles are obtained by using $\omega_1, \omega_2$, and above boundary conditions in equation (4.16) as

\[ \theta = \frac{b_1}{2a_2 \sinh (r_2 - r_1) h} \left( 2 + \frac{1}{\cosh^2 Q h} \right) e^{\omega_2 \sinh r_1 h} - e^{\omega_2 \sinh r_2 h} \]
\[ - \frac{b_3}{2a_2 \cosh^2 Q h \sinh (r_2 - r_1) h} \left( \frac{Ae^{\omega_1} - Be^{\omega_2}}{(1 + 4b_1 Q^2)^2 - 4b_2^2 Q^2} \right) \]
\[ + \frac{b_1}{a_2 \cosh Q h \sinh (r_2 - r_1) h} \left( \frac{Ce^{\omega_1} - De^{\omega_2}}{(1 + b_1 Q^2)^2 - b_2^2 Q^2} \right) \]
\[ + \frac{e^{2\beta_1}}{2 \sinh (r_2 - r_1) h} \left( (T_1 e^{\omega_1 h} - T_2 e^{-\omega_1 h}) e^{\omega_2 h} - (T_1 e^{\omega_1 h} - T_2 e^{-\omega_1 h}) e^{\omega_2 h} \right) \]
\[ + \frac{b_1}{a_2} + \frac{b_3}{4a_2 \cosh^2 Q h} \left( 2 + \frac{e^{2\omega_1}}{1 + 2b_1 Q + 4b_1^2 Q^2} + \frac{e^{-2\omega_1}}{1 - 2b_1 Q + 4b_1^2 Q^2} \right) \]
\[ - \frac{b_1}{2a_2 \cosh Q h} \left( \frac{e^{\omega_1 h}}{1 + b_1 Q + b_2 Q^2} + \frac{e^{-\omega_1 h}}{1 - b_1 Q + b_2 Q^2} \right) \]
\[
\theta_p = \frac{c_p}{c_p - 2\beta^2 e_{w} \tau_f} \left\{ \frac{b_3}{2a_3 \sinh (r_2 - r_1) h} \left( 2 + \frac{b_3}{\cosh^2 Qh} \right) e^{\beta h} \sinh r_1 h - e^{\beta h} \sinh r_2 h \right\}
\]

\[- \frac{1}{2a_3 \cosh^2 Qh \sinh (r_2 - r_1) h} \left( \frac{1 + 4b_2 Q^2}{1 - 4b_2 Q^2} \right) \left( A e^{\beta h} - B e^{-\beta h} \right) \]

\[+ \frac{b_3}{a_3 \cosh Qh \sinh (r_2 - r_1) h} \left( \frac{C e^{\beta h} - D e^{-\beta h}}{1 + b_2 Q^2 - b_2^2 Q^2} \right) \]

\[+ \frac{e^{2\beta h}}{2 \sinh (r_2 - r_1) h} \left( T_1 e^{\beta h} - T_2 e^{-\beta h} \right) \left( T_1 e^{\beta h} - T_2 e^{-\beta h} \right) \]

\[+ \frac{b_3}{a_3} \left( 2 + \frac{e^{Qh}}{1 + 2b_2 Q + 4b_2 Q^2} + \frac{e^{-Qh}}{1 - 2b_2 Q + 4b_2 Q^2} \right) \]

\[= \frac{b_3}{2a_3 \cosh Qh \left( 1 + b_2 Q + b_2 Q^2 \right)} \left( 1 - b_2 Q + b_2 Q^2 \right) \]

By substituting \( \omega_1, \omega_2, \theta \) and \( \theta_p \) in equations (4.9) and (4.10) we get the required relations for \( u, v, T \) and \( T_p \).

\[u = \frac{\alpha e^{-\beta_2}}{v Q^2} \left( 1 - \frac{\cosh (Qb)}{\cosh (Qh)} \right) \]

\[v = \frac{\alpha e^{-\beta_2}}{v Q^2 (1 - \tau_1, \beta_2)} \left( 1 - \frac{\cosh (Qb)}{\cosh (Qh)} \right) \]
\[ T = e^{s/2} \left[ \frac{b_3}{2a_3 \sinh(r_2 - r_1) \cosh Q^2} \left( 2 + \frac{1}{\cosh^2 Q^2} \right) e^{s \sinh r_1 h} - e^{s \sinh r_2 h} \right] \\
+ \frac{b_3}{2a_3 \cosh Q^2} \sinh(r_2 - r_1) \cosh Q^2 \left( \frac{A e^{s \sinh r_1 h} - e^{s \sinh r_2 h}}{(1 + b_1 Q^2)^2 - b_1^2 Q^2} \right) \]
\[ + \frac{e^{s/2}}{2 \sinh(r_2 - r_1) \cosh Q^2} \left( T_1 e^{s \sinh r_1 h} - T_1 e^{s \sinh r_2 h} - T_1 e^{s \sinh r_1 h} - T_1 e^{s \sinh r_2 h} \right) \]
\[ + \frac{1}{4a_3 \cosh^2 Q^2} \left( 2 + \frac{1}{2h_Q + 4b_Q^2 + 1 - 2h_Q + 4b_Q^2} \right) \]
\[ + \frac{a_3}{2a_3 \cosh Q^2} \left( \frac{e^{s/2}}{1 + b_1 Q + b_1 Q^2 + b_1 Q - b_1 Q} \right) \]
\[ (4.19) \]

\[ T_r = e^{s/2} \left[ \frac{b_1}{2a_2 \sinh(r_2 - r_1) \cosh Q^2} \left( 2 + \frac{1}{\cosh^2 Q^2} \right) e^{s \sinh r_1 h} - e^{s \sinh r_2 h} \right] \\
+ \frac{b_1}{2a_2 \cosh Q^2} \sinh(r_2 - r_1) \cosh Q^2 \left( \frac{A e^{s \sinh r_1 h} - e^{s \sinh r_2 h}}{(1 + b_1 Q^2)^2 - b_1^2 Q^2} \right) \]
\[ + \frac{e^{s/2}}{2 \sinh(r_2 - r_1) \cosh Q^2} \left( T_1 e^{s \sinh r_1 h} - T_1 e^{s \sinh r_2 h} - T_1 e^{s \sinh r_1 h} - T_1 e^{s \sinh r_2 h} \right) \]
\[ + \frac{1}{4a_2 \cosh^2 Q^2} \left( 2 + \frac{1}{2h_Q + 4b_Q^2 + 1 - 2h_Q + 4b_Q^2} \right) \]
\[ + \frac{a_2}{2a_2 \cosh Q^2} \left( \frac{e^{s/2}}{1 + b_1 Q + b_1 Q^2 + b_1 Q - b_1 Q} \right) \]
\[ (4.20) \]

where
\[ A = (1 + 4b_Q^2) \cosh 2Q^2 \sinh r_1 h + 2b_Q \sinh 2Q^2 \cosh r_1 h \]
\[ B = (1 + 4b_Q^2) \cosh 2Q^2 \sinh r_1 h + 2b_Q \sinh 2Q^2 \cosh r_1 h \]
\[ C = (1 + b_Q^2) \cosh Q^2 \sinh r_1 h + b_Q \sinh Q^2 \cosh r_1 h \]
\[ D = (1 + b_Q^2) \cosh Q^2 \sinh r_1 h + b_Q \sinh Q^2 \cosh r_1 h \]
\[ a_1 = \frac{N \tau \beta^2 \alpha^2}{\sqrt{\kappa}}, \quad b_3 = \frac{a_3}{a_2}, \quad b_1 = \frac{1}{a_3}, \quad b_1 = a_3 = \frac{N \tau \beta^2 \alpha^2}{k(1 - \tau \beta^2) \sqrt{\kappa} Q^2} \]
\[ r_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}, \quad r_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} \]

**Conclusion:**

- The velocity profiles for the fluid and dust particles are drawn as in figures 3, which are parabolic in nature. From these it is observed that velocity of fluid particles is parallel to that of dust. Also the velocity of both fluid and dust particles, which are nearer to the axis of flow, move with the greater velocity. Further one can observe that if the dust is very fine i.e., mass of the dust particles is negligibly small then the relaxation time of dust particle decreases and ultimately as \( \tau_r \to 0 \) the velocities of fluid and dust particles will be the same (from equations (4.17) & (4.18)).

- This means that the flow of dust particles merge with fluid flow or vice-versa and the flow becomes single phase instead of two phase. Hence the velocity interaction force \( F \) between the dust and fluid phases does not exist.

- This occurs when color powder is mixed with liquid, for example, injecting of medicine powder in distilled water and preparation of different colors by the mixture of water and color powder.

- If the mass of the dust particles are big then by definition of \( \tau_r \) it will be fairly large and from the equations (4.17) & (4.18) dust velocity is greater than fluid velocity. This type of flow is found in industries i.e., the flow of smoke soon after its emission from chimney.

- Figure 4 shows that variation of Temperature of both fluid and dust particles at different timings, which are also parabolic in nature. From this figure it is observed that the temperature of both fluid and dust particles increase suddenly near the lower plate but whereas at upper plate they increase slowly.

- If the dust is very fine i.e., the mass of the dust particles is negligibly small then \( \tau_r \) decreases and as \( \tau_r \to 0 \) then from (4.19) & (4.20) the temperatures of fluid and dust particles will be same i.e., the two phase flow will reduces to single phase, so it is not possible. Hence no heat transfer exist from fluid phase to dust or dust to fluid, only when the specific heat of dust particles reduces to zero.
Figure 3: Variation of Fluid and Dust Phase Velocity with $h$

Figure 4: Variation of Fluid and Dust particle Temperature with $h$
References:


