CHAPTER 5
CHAPTER 5

THE STUDY OF BERWALD SPACE WITH

$(\alpha, \beta)$-METRIC

Publications based on this Chapter

- The study of Berwald connection in a Finsler space with $(\alpha, \beta)$-metric, Communicated.

- Mean Berwald curvature of homogeneous $(\alpha, \beta)$-metrics, Communicated.

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Chapter 5

The study of Berwald space with $(\alpha, \beta)$-metric

5.1 Introduction

A Berwald space is an affinely connected space defined by Berwald [36] which is a Finsler space such that the coefficients $G^i_{jk}$ of the Berwald connection $\nabla$ depend on position only. Y. Ichijyo [31] proved that on a Berwald space, the parallel translation along any geodesic preserves the Minkowski functionals. Thus, Berwald spaces can be viewed as Finsler spaces modeled on a single Minkowski space.

L. Berwald introduced a connection and two curvature tensors. The Berwald connection is torsion free but is not necessarily metric-compatible. The Berwald connection is convenient when dealing with Finsler spaces of constant flag curvature. It is most directly related to the non-linear connection coefficients and most amenable to the study of the geometry of paths. In general, the Berwald connection is non-linear and linear co-ordinates $(y^1, ..., y^n)$ defined on a fixed tangent space $T_p(M)$ are transported along curves $c(t)$ to proper non-linear co-ordinates $(\bar{y}^1(t), ..., \bar{y}^n(t))$ of the other tangent spaces $T_{c(t)}(M)$, where $M$ is an $n$-dimensional manifold.

T. Okada [52] studied on the Minkowskian product of Finsler spaces and Berwald
connection. M. Matsumoto [40] established the relation on the Berwald connection of a Finsler space with an \((\alpha, \beta)\)-metric. H. S. Park, H. Y. Park and B. D. Kim [27] studied on the Berwald connection of a Finsler space with a special \((\alpha, \beta)\)-metric. Nabil L. Youssef, S. H. Abed and A. Suleiman [45] investigated Cartan and Berwald connections in the pullback formalism. B. Bidabad and A. Tayebi [14] established the different properties of generalized Berwald connections. T. Mestdag and W. Sarlet [43] studied the Berwald connection for time-dependent second-order differential equations and its applications in theoretical mechanics. Shaoqiang Deng and Xiaoyang Wang [64] gave the formula for \(S\)-curvature of homogeneous \((\alpha, \beta)\)-metrics and they used that formula to deduce a formula for the mean Berwald curvature \(E_{ij}\) of Randers metrics. Throughout this chapter, the terminology and notation are referred to Matsumoto’s monograph [39].

5.2 Berwald space with special \((\alpha, \beta)\)-metric

Let \(F^n = (M, L(\alpha, \beta))\) be an \(n\)-dimensional Finsler space with an \((\alpha, \beta)\)-metric

\[
L(\alpha, \beta) = c_1 \alpha + c_2 \beta + \frac{\alpha^2}{\beta}, \quad c_1 \neq 0, \quad c_2
\]

where \(\alpha\) is a Riemannian metric and \(\beta\) is a differential 1-form. The Riemannian space \(R^n = (M, \alpha)\) is called the associated Riemannian space with \(F^n\) and the Christoffel symbols of \(R^n = (M, \alpha)\) are indicated by \(\gamma^i_{jk}\). Then the Riemannian connection \(\gamma^i_{jk}\) gives rise to the linear Finsler connection \(F_\Gamma = (\gamma^i_{jk}, \gamma^i_0, 0)\), where the subscript 0 means a contraction by \(y^j\).

Now, we shall find the Berwald connection \(B\Gamma\) in \(F^n\). Putting

\[
2G^i = \gamma^i_0 + 2B^i, \quad (5.2.2)
\]
from \((B2), (B3)\) and \((B4)\) of definition 1.3.6., we have
\[
G^i_j = \partial_j G^i = \gamma^i_{0_{j}} + B^i_{j},
\]
\[
G^i_{j_k} = \partial_j G^i_k = \gamma^i_{j_k} + B^i_{j_k},
\]
(5.2.3)
where \(B^i_j = \partial_j B^i\) and \(B^i_{j_k} = \partial_k B^i_j\).

The axiom \((B1)\) of definition 1.3.6: \(L_{,i} = \partial_i L - G^r_i \partial_r L = 0\) is written as
\[
L_{,i} B^k_j y^j y^i_k + \alpha L_{,i} (B^r_j b_r - b_{j|i}) y^j = 0,
\]
(5.2.4)
where \(y_k = a_{k|i} y^i\).

Now, we establish the following theorem.

**Theorem 5.2.1.** Let \(F^n\) be the Finsler space with a special \((\alpha, \beta)\)-metric \(L(\alpha, \beta) = c_1 \alpha + c_2 \beta + \frac{\alpha^2}{\beta}, c_1 \neq 0,\) with the Berwald connection \(B\Gamma = (G^i_{j_k}, G^i_{0_{j}}, 0)\). Then we have the following:

(i) If \((c_2 \beta^2 - \alpha^2) \neq 0,\) then \(F^n\) is a Berwald space if and only if \(b_{j|i} = 0\) and the Berwald connection is \((\gamma^i_{j_k}, \gamma^i_{0_{j}}, 0)\).

(ii) If \((c_2 \beta^2 - \alpha^2) = 0,\) then \(F^n\) is a Berwald space if and only if \(B^k_j = 0\) and the Berwald connection is \((\gamma^i_{j_k}, \gamma^i_{0_{j}}, 0)\).

**Proof.** We find the condition for \(F^n\) to be a Berwald space by applying the Matsumoto's method of [40]. Since the metric function \(L\) is given by (5.2.1), we get
\[
L_{,i} = c_1 + \frac{2 \alpha}{\beta},
\]
\[
L_{,i} = c_2 - \frac{\alpha^2}{\beta^2}.
\]
(5.2.5)
Substituting (5.2.5) in (5.2.4), we have
\[
\alpha \{2 \beta B^k_j y^j y^i_k + (c_2 \beta^2 - \alpha^2)(B^r_j b_r - b_{j|i}) y^j \} + c_1 \beta^2 B^k_j y^j y^i_k = 0.
\]
(5.2.6)
Now, we assume that the Finsler space $F^n$ with $(\alpha, \beta)$-metric given by (5.2.1) is a Berwald space, that is, $G_{j^i k}$ is a function of the position alone. Then we have $B^k_{j^i} = B^k_{j^i}(x)$, so that the second term is rational and $\alpha$ is an irrational polynomial in $(y^j)$. Thus, we have

$$2\beta B^k_{j^i} y^j y^k + (c_2 \beta^2 - \alpha^2) (B^k_{j^i} b_t - b_{j^i} y^j) y^k = 0.$$

Then we have

$$\beta^2 B^k_{j^i} y^j y^k = 0. \quad (5.2.7)$$

From the above two equations, we have

$$(c_2 \beta^2 - \alpha^2) (B^k_{j^i} b_k - b_{j^i} y^j) y^k = 0. \quad (5.2.8)$$

Next, we proceed with different cases:

Case (i): Suppose $(c_2 \beta^2 - \alpha^2) \neq 0$. Then (5.2.8) yields

$$(B^k_{j^i} b_k - b_{j^i} y^j) y^k = 0, \quad (5.2.9)$$

which implies

$$B^k_{j^i} b_k - b_{j^i} y^k = 0. \quad (5.2.10)$$

From (5.2.7), we have $B^k_{j^i} y^k y^k = 0$, which implies

$$B^k_{j^i} y^j y^k + B^k_{h^i} y^k y^h = 0, \quad \Rightarrow B^k_{j^i} y^j a_{kh} y^h + B^k_{h^i} y^h a_{kj} y^j = 0. \quad (5.2.11)$$

Contracting (5.2.11) with $b_j b_h$, we have

$$(B^k_{j^i} a_{kh} + B^k_{h^i} a_{kj}) \beta^2 = 0, \quad (5.2.12)$$

which gives

$$B^k_{j^i} a_{kh} + B^k_{h^i} a_{kj} = 0. \quad (5.2.13)$$

From (5.2.13), we have $B^k_{j^i} = 0$ and from (5.2.10), we have $b_{j^i} = 0$. 

Conversely, according to [22], if \( b_{ijk} = 0 \), then the Finsler space \( F^n \) with the \((\alpha, \beta)\)-metric is a Berwald space.

Case (ii): Suppose \((c_2\beta^2 - \alpha^2) = 0\), which implies \( c_2 = 0 \). In this case, (5.2.1) reduces to \( L = c_1\alpha + \frac{\alpha^2}{\beta} \), \( c_1 \neq 0 \), the special \((\alpha, \beta)\)-metric. From (5.2.7), we can have

\[
B_j^k y^j y_k = 0 \tag{5.2.14}
\]

and from which, we have

\[
B_j^k a_{kh} + B_h^k a_{kj} = 0,
\]

which gives \( B_j^k = 0 \).

On the other hand, again by [22], \( F^n \) with the mentioned special \((\alpha, \beta)\)-metric is a Berwald space. This completes the proof.

### 5.3 Berwald connection with respect to the special \((\alpha, \beta)\)-metric

In this section, we find the concrete form of Berwald connection in the Finsler space with an \((\alpha, \beta)\)-metric given by (5.2.1). The Berwald connection is determined by \( B_j^k \) in the equation (5.2.4) uniquely. From (5.2.4) and (5.2.5), we get

\[
(c_1\beta^2 + 2\alpha\beta)B_j^k y^j y_k + \alpha(c_2\beta^2 - \alpha^2)(B_j^k b_k - b_{jik}) y^j = 0. \tag{5.3.1}
\]

Equation (5.3.1) can be rewritten as

\[
(c_2\beta^2 - \alpha^2)(b_{jik}) y^j = \{(c_1\beta^2 + 2\alpha\beta)e_k + (c_2\beta^2 - \alpha^2)b_k\} B^k_i, \tag{5.3.2}
\]

where \( e_k = \frac{\mu_k}{\alpha} \). We put

\[
r_{ij} = \frac{(b_{ij} + b_{jii})}{2}, \quad s_{ij} = \frac{(b_{ij} - b_{jii})}{2}.
\]
Transvecting (5.3.2) by $y^i$ and by using the homogeneity, we have

$$(c_2\beta^2 - \alpha^2)\tau_{i0} = 2\{(c_1\beta^2 + 2\alpha\beta)e_k + (c_2\beta^2 - \alpha^2)b_k\}B_k^i. \quad (5.3.3)$$

Conversely, differentiating (5.3.3) by $y^i$ and by the virtue of $\hat{\partial}_i\alpha = e_i$, $\hat{\partial}_i\varepsilon_k = \frac{\varepsilon_i - \varepsilon_k}{\alpha}$, we have

$$(c_2\beta^2 - \alpha^2)\tau_{i0} + (c_2\beta^2 - e_i\alpha)\tau_{i0} = \{(c_1\beta^2 + 2\alpha\beta)e_k + (c_2\beta^2 - \alpha^2)b_k\}B_k^i +\{(c_1\beta^2 + 2\alpha\beta)(\frac{\alpha k - \varepsilon_k}{\alpha})\}
+ (c_1\beta^2 + b_i\alpha + e_i\beta)2e_k
+ (c_2\beta^2 - e_i\alpha)2b_k\}B_k^i. \quad (5.3.4)$$

From (5.3.2), (5.3.3) and (5.3.4), we have

$$a_{ki}\left\{\frac{c_1\beta^2 + 2\alpha\beta}{\alpha}\right\}B^k = (c_2\beta^2 - e_i\alpha)\tau_{i0} + (c_2\beta^2 - \alpha^2)s_{i0} + \left\{\frac{c_1\beta^2 + 2\alpha\beta}{\alpha}\right\}e_i e_k B^k - (c_1\beta^2 + b_i\alpha + e_i\beta) e_k B^k - (c_2\beta^2 - e_i\alpha) b_k B^k. \quad (5.3.5)$$

Put $e_k B^k = E$ and $b_k B^k = D$ and divide (5.3.5) by $\frac{c_1\beta^2 + 2\alpha\beta}{\alpha}$, we get

$$a_{ki}B^k = \left\{E + \frac{\alpha [\alpha D - (\alpha r_{00} + \beta E)]}{c_1\beta^2 + 2\alpha\beta}\right\}e_i + \frac{\alpha (c_2\beta^2 - \alpha^2)}{c_1\beta^2 + 2\alpha\beta}s_{i0} + \frac{\alpha \{c_2\beta r_{00} - [(c_1\beta + \alpha)E + c_2\beta D]\}}{c_1\beta^2 + 2\alpha\beta}b_i. \quad (5.3.5)$$

Contract the above equation with $a^{ij}$, we obtain

$$B^i = P_1 e^i + P_2 s_0^i + P_3 b^i, \quad (5.3.6)$$

where

$$P_1 = E + \frac{\alpha [\alpha D - (\alpha r_{00} + \beta E)]}{c_1\beta^2 + 2\alpha\beta}, \quad P_2 = \frac{\alpha (c_2\beta^2 - \alpha^2)}{c_1\beta^2 + 2\alpha\beta}, \quad P_3 = \frac{\alpha \{c_2\beta r_{00} - [(c_1\beta + \alpha)E + c_2\beta D]\}}{c_1\beta^2 + 2\alpha\beta}. \quad (5.3.7)$$
From (5.3.3), we get

\[(c_2\beta^2 - \alpha^2)r_{00} = 2((c_1\beta^2 + 2\alpha\beta)E + 2(c_2\beta^2 - \alpha^2)D).\]  

(5.3.8)

Transvecting (5.3.6) by \(b_i\) and by virtue of \(b_i e^i = \frac{\partial}{\partial},\) we have

\[
\alpha^2\beta(c_2b^2 - 1)r_{00} + \alpha^2(c_2\beta^2 - \alpha^2)s_0 = [(c_1\beta + \alpha)(b^2\alpha^2 - \beta^2)] E + [\alpha\beta(c_1\beta + \alpha + c_2b^2)] D.
\]

(5.3.9)

By solving (5.3.8) and (5.3.9), we get \(D\) and \(E.\) Thus, we have

**Theorem 5.3.1.** Let \(F^n\) be a Finsler space with special \((\alpha, \beta)\)-metric \(L(\alpha, \beta) = c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta},\ c_1 \neq 0.\) Then the vector field \(B^i(x, y)\) in the Berwald connection is given by (5.3.6).

**Example:**

In an \((\alpha, \beta)\)-metric given by (5.2.1), if \(c_1 = c_2 = 1,\) then the metric

\[L(\alpha, \beta) = \alpha + \beta + \frac{\alpha^2}{\beta}\]

(5.3.10)

is a special \((\alpha, \beta)\)-metric. For the Finsler space with the special \((\alpha, \beta)\)-metric (5.3.10), from (5.3.8) and (5.3.9), we determine the quantities \(D\) and \(E\) by the following two equations.

\[r_{00} = \frac{2\beta(\beta + 2\alpha)}{\beta^2 - \alpha^2}E + 2D,\]

\[
\alpha^2(\beta^2 - \alpha^2)s_0 + \alpha^2\beta(b^2 - 1)r_{00} = (\alpha + \beta)(b^2\alpha^2 - \beta^2)E + \alpha\beta(\alpha + \beta + b^2)D.
\]

(5.3.11)

From the above two equations, we get

\[D = \frac{K_1r_{00} + K_2s_0}{C},\]

\[E = \frac{\alpha(\beta^2 - \alpha^2)(K_3r_{00} - K_4s_0)}{C},\]

(5.3.12)
where

\[ K_1 = 2\alpha^2 \beta^2 (2\alpha + \beta)(b^2 - 1) - (\alpha + \beta)(\beta^2 - \alpha^2)(b^2\alpha^2 - \beta^2), \]

\[ K_2 = 2\alpha^2 \beta (2\alpha + \beta)(\beta^2 - \alpha^2). \]

\[ K_3 = \beta [3\alpha + \beta + (1 - 2\alpha)b^2], \]

\[ K_4 = 2\alpha(\beta^2 - \alpha^2), \]

\[ C = 2\{\alpha\beta^2 (2\alpha + \beta)(\alpha + \beta + b^2) - (\alpha + \beta)(\beta^2 - \alpha^2)(b^2\alpha^2 - \beta^2)\}. \] (5.3.13)

From (5.3.7), (5.3.8) and (5.3.12), we get

\[ P_1 = \frac{A_1 r_{00} + 2\alpha \beta(\beta^2 - \alpha^2)B_1 s_0}{\beta(\beta + 2\alpha)C}, \]

\[ P_2 = \frac{\alpha(\beta^2 - \alpha^2)}{\beta(\beta + 2\alpha)}, \]

\[ P_3 = \frac{A_2 \alpha \beta r_{00} - 2\alpha^2(\beta^2 - \alpha^2)(\alpha^2 + \alpha \beta + \beta^2)s_0}{\beta(\beta + 2\alpha)C}, \]

where

\[ A_1 = (\alpha + \beta)(\beta^2 - \alpha^2) \left\{ \beta^2 (2\alpha + \beta) + b^2 [\alpha^3 + \beta^2 (1 - 2\alpha)] \right\} + 2\alpha^2 \beta^2 (2\alpha + \beta) \left[ b^2 (\alpha - 1) - (2\alpha + \beta) \right], \]

\[ B_1 = (\alpha^2 + \alpha \beta + \beta^2) - (\beta + 2\alpha)(\beta^2 - \alpha^2), \]

\[ A_2 = 2\alpha \beta^2 (2\alpha + \beta) [(2\alpha + \beta) - b^2 (\alpha - 1)], \]

\[ - (\alpha + \beta)(\beta^2 - \alpha^2) [\alpha \beta + b^2 \alpha (1 - \alpha) + (3\alpha^2 - \beta^2)]. \] (5.3.14)

Thus, in a Finsler space with the special \((\alpha, \beta)\)-metric (5.2.1), the vector field \(B^i(x, y)\) in (5.2.2) is given as:

\[ B^i = \frac{\alpha \{A_1 r_{00} + 2\alpha \beta (\beta^2 - \alpha^2)B_1 s_0\}}{\beta(2\alpha + \beta)C} e^i + \frac{\alpha(\beta^2 - \alpha^2)}{\beta(2\alpha + \beta)} s^i_0 e^i + \alpha \left\{ \frac{A_2 \beta r_{00} - 2\alpha^2 (\beta^2 - \alpha^2)(\alpha^2 + \alpha \beta + \beta^2)s_0}{\beta(2\alpha + \beta)C} \right\} b^i, \]

where \(A_1, B_1, A_2\) and \(C\) are given in (5.3.13) and (5.3.14).
5.4 Homogeneous spaces

A family of spaces which has many applications in Physics is homogeneous spaces (in particular, Lie groups) equipped with invariant metrics. The study of homogeneous spaces (Lie groups) with invariant Riemannian metrics has been a very interesting field in recent decades ([15],[33],[44],[51]).

There are several interesting curvatures in Finsler Geometry.

Now, we recall [64] the definition of S-curvature. Let $V$ be an $n$-dimensional real vector space and $L$ be a Minkowski norm on $V$. For a basis $\{e_i\}$ of $V$, let

$$\sigma_F = \frac{Vol(B^n)}{Vol \{ (y) \in R^n \mid L(ye_i) < 1 \}},$$

where Vol means the volume of a subset in the standard Euclidean space $R^n$ and $B^n$ is the open ball of radius 1. This quantity is generally dependent on the choice of the basis $\{e_i\}$. But it is easily seen that

$$\tau(y) = ln \frac{\sqrt{det (g_{ij}(y))}}{\sigma_F}, \quad y \in V \setminus \{0\}$$

is independent of the choice of the basis. $\tau = \tau(y)$ is called the distortion of $(V, L)$.

Now let $(M, L)$ be a Finsler space. Let $\tau(x, y)$ be the distortion of the Minkowski norm $L_x$ on $T_xM$. For $y \in T_xM \setminus \{0\}$, let $\tau(t)$ be the geodesic with $\tau(0) = x$ and $\tau_0 = y$.

Then the quantity

$$S(x, y) = \frac{d}{dt} [\tau(\sigma(t), \dot{\sigma}(t))]_{t=0}$$

is called the S-curvature of the Finsler space $(M, L)$. A Finsler space $(M, L)$ is said to have almost isotropic S-curvature if there exists a smooth function $c(x)$ on $M$ and a closed 1-form $\eta$ such that:

$$S(x, y) = (n + 1) (c(x)L(y) + \eta(y)), \quad x \in M, \quad y \in T_xM.$$
In the above equation, if $\eta = 0$, then $(M, L)$ is said to have isotropic $S$-curvature. If $\eta = 0$ and $c(x)$ is a constant, then $(M, L)$ is said to have constant $S$-curvature.

The mean Berwald curvature ($E$-curvature) is an important non-Riemannian quantity defined by [16]

\[ E_{ij} = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} \left( \frac{\partial G^m}{\partial y^m} \right), \]

where $G^m = G^m(x, y)$ are the spray coefficients.

The group $I(M, L)$ of isometries of a Finsler space $(M, L)$ is a Lie transformation group of $M$ [9]. If $I(M, L)$ acts transitively on $M$, then $(M, L)$ is called homogeneous. Let $(G/H, L)$ be a homogeneous $(\alpha, \beta)$-metric of the form $L = \alpha \phi(s)$, where $s = \beta/\alpha$ with $\alpha$ a $G$-invariant Riemannian metric on the coset space $G/H$ of a Lie group $G$ and $\beta$ a $G$-invariant vector field on $G/H$. Also $G/H$ is a reductive homogeneous manifold in the sense of Nomizu [51], that is, the Lie algebra of $G$ has a decomposition:

\[ g = h \oplus m, \quad (\text{direct sum of subspaces}) \]

such that $Ad(h) \subset m$, $\forall h \in H$.

Let $\langle , \rangle$ be the corresponding inner product on $m$. Let $u_1, u_2, \ldots, u_m$ be an orthonormal basis of $(m, \langle , \rangle)$ such that $u_m = \frac{u}{|u|}$.

We use the following theorem [64].

**Theorem 5.4.1.** Let $L = \alpha \phi(s)$ be a $G$-invariant $(\alpha, \beta)$-metric on the reductive homogeneous manifold $G/H$ with a decomposition of the Lie algebra

\[ g = h \oplus m. \]

Then the $S$-curvature of $L$ has the form

\[ S(\alpha, y) = -\frac{1}{\alpha(y)} \frac{\Phi}{2\Delta^2} \left( -c([u, y]_m, y) - \alpha(y)Q([u, y]_m, u) \right), \quad y \in m, \]
where \( u \) is the vector in \( m \) corresponding to the 1-form \( \beta \) and we have identified \( m \) with the tangent space of \( G/H \) at the origin \( o = H \).

We know that [64]
\[
S = \frac{\partial G^m}{\partial y^m} - (\ln \sigma(x))_{x^k} y^k ,
\]
where \( \ln \sigma(x)_{x^k} \) is the function of \( x \) because \( \ln \sigma(x) \) is the function of \( x \). Hence
\[
0 = \frac{\partial^2}{\partial y^i \partial y^j} [ (\ln \sigma(x))_{x^k} y^k ] .
\]
This means that
\[
\frac{\partial^2 S}{\partial y^i \partial y^j} = \frac{\partial^2}{\partial y^i \partial y^j} \left[ \frac{\partial G^m}{\partial y^m} - (\ln \sigma(x))_{x^k} y^k \right]
= \frac{\partial^2}{\partial y^i \partial y^j} \left( \frac{\partial G^m}{\partial y^m} \right) = 2E_{ij} .
\]
Now, we compute
\[
\frac{\partial^2 S}{\partial y^i \partial y^j} (o,y) = \frac{\partial^2 S(o,y)}{\partial y^i \partial y^j} = 2E_{ij}(o,y) .
\]
By theorem 5.4.1, we have
\[
\frac{\partial^2 S(o,y)}{\partial y^i \partial y^j} = \frac{\partial^2}{\partial y^i \partial y^j} \left( \frac{c\Phi}{2\Delta^2 \alpha(y)} ([u,y]_m,y) \right) + \frac{\partial^2}{\partial y^i \partial y^j} \left( \frac{\Phi Q}{2\Delta^2 ([u,y]_m,u)} \right) .
\]
Note that
\[
\frac{\partial s}{\partial y} = \frac{1}{\alpha} \left( b_m - \frac{y_m}{\alpha} \right) , \quad \frac{\partial \alpha}{\partial y^m} = \frac{y_m}{\alpha} ,
\]
where \( y_m = a_{mj} y^j \). Since \( u_1, u_2, \ldots, u_m \) is an orthonormal basis, we have \( a_{mj}/\alpha = \delta^m_j \).

Therefore at the origin, we have \( y_m = y^m \).
5.5 Mean Berwald curvature of homogeneous Kropina metric

Consider Kropina metric $L = \frac{\alpha^2}{\beta}$, which gives $\phi = \frac{1}{s}$ and $\phi' = -\frac{1}{s^2}$.

Let $\psi = \phi - s\phi'$. Then, we have

$$Q = \frac{\phi'}{\psi} = -\frac{1}{2s},$$

$$Q' = \frac{1}{2s^2},$$

$$Q'' = -\frac{1}{s^3},$$

$$\Delta = 1 + sQ + (b^2 - s^2)Q' = \frac{b^2}{2s^2},$$

$$\Psi = \frac{Q'}{2\Delta} = \frac{1}{2b^2},$$

$$\Phi = -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q''$$

$$= \frac{b^2(n + 1)}{2s^3}.$$

Letting $c = 1$ in $S(o, y)$, we get

$$S(o, y) = -\frac{1}{\alpha(y)} \frac{\Phi}{2s^2} \left(-\langle [u, y]_m, y \rangle - \alpha(y)Q\langle [u, y]_m, u \rangle\right),$$

$$= \frac{(n + 1)s}{\alpha(y)b^2} \langle [u, y]_m, y \rangle - \frac{(n + 1)}{2b^2} \langle [u, y]_m, u \rangle.$$

Therefore,

$$2E_{ij}(o, y) = \frac{\partial^2 S(o, y)}{\partial y^i \partial y^j},$$

$$= \frac{\partial^2}{\partial y^i \partial y^j} \left( \frac{(n + 1)s}{\alpha(y)b^2} \langle [u, y]_m, y \rangle - \frac{(n + 1)}{2b^2} \langle [u, y]_m, u \rangle \right),$$

$$= \frac{\partial^2}{\partial y^i \partial y^j} \left( \frac{(n + 1)s}{\alpha(y)b^2} \langle [u, y]_m, y \rangle \right) - \frac{\partial^2}{\partial y^i \partial y^j} \left( \frac{(n + 1)}{2b^2} \langle [u, y]_m, u \rangle \right).$$

(5.5.1)
Now,
\[
\frac{\partial^2}{\partial y^i \partial y^j} \left( \frac{(n + 1)s}{\alpha(y) b^2} \langle [u, y]_{m, y} \rangle \right) = \frac{n + 1}{b^2} \frac{\partial^2}{\partial y^i \partial y^j} \left\{ \frac{s}{\alpha(y) b^2} \langle [u, y]_{m, y} \rangle \right\}
\]
\[
= \frac{n + 1}{b^2} \left\{ \frac{s}{\alpha(y)} \frac{\partial^2}{\partial y^i \partial y^j} \left( \frac{\langle [u, y]_{m, y} \rangle}{\alpha(y)} \right) + \frac{\partial \langle [u, y]_{m, y} \rangle}{\partial y^i} \frac{\partial}{\partial y^j} \left( \frac{s}{\alpha(y)} \right) + \frac{\partial}{\partial y^j} \left( \frac{s}{\alpha(y)} \right) \frac{\partial \langle [u, y]_{m, y} \rangle}{\partial y^i} \right\},
\]
where
\[
\frac{\partial \langle [u, y]_{m, y} \rangle}{\partial y^i} = \langle [u, u_i]_{m, y} \rangle + \langle [u, y]_{m, u_i} \rangle,
\]
\[
\frac{\partial^2 \langle [u, y]_{m, y} \rangle}{\partial y^i \partial y^j} = \langle [u, u_i]_{m, u_j} \rangle + \langle [u, u_j]_{m, u_i} \rangle
\]
and
\[
\frac{\partial}{\partial y^j} \left( \frac{s}{\alpha(y)} \right) = \frac{\alpha(y) \frac{\partial s}{\partial y^j} - s \frac{\partial \alpha(y)}{\partial y^j}}{|\alpha(y)|^2}
\]
\[
= b_j - 2s \frac{\alpha_y^j}{\alpha^2(y)}
\]
Hence
\[
\frac{\partial^2}{\partial y^i \partial y^j} \left( \frac{s}{\alpha(y)} \right) = \frac{\partial}{\partial y^j} \left[ \frac{\partial}{\partial y^i} \left( \frac{s}{\alpha(y)} \right) \right]
\]
\[
= \frac{\partial}{\partial y^i} \left[ \frac{b_j - 2s \alpha^j_y}{\alpha^2(y)} \right]
\]
\[
= -\frac{2s \delta_i^j}{\alpha^2(y)} - \frac{2}{\alpha^4(y)} (b_i y^j + b_j y^i) + \frac{8s y^j y^j}{\alpha^6(y)}.
\]
Therefore, we have

\[
\frac{\partial^2}{\partial y^i \partial y^j} \left( \frac{(n + 1)s}{\alpha(y)b^2} \langle [u, y]_m, y \rangle \right) = \frac{(n + 1)}{b^2} \left\{ \frac{s}{\alpha(y)} \left[ \langle [u, u_i]_m, u_j \rangle + \langle [u, u_j]_m, u_i \rangle \right] + \langle [u, y]_m, y \rangle \left[ \frac{2s\delta_i^j}{\alpha^3(y)} - \frac{2}{\alpha^4(y)} (b_i y^j + b_j y^i) + \frac{8s y^j y^i}{\alpha^5(y)} \right] + \left[ \langle [u, u_j]_m, y \rangle + \langle [u, y]_m, u_j \rangle \right] \left( \frac{b_i - 2s y^i}{\alpha^2(y)} \right) \right\} . (5.5.2)
\]

Next, we have

\[
\frac{\partial^2}{\partial y^i \partial y^j} \left\{ \frac{(n + 1)}{2b^2} \langle [u, y]_m, u \rangle \right\} = \frac{(n + 1)}{2b^2} \frac{\partial^2}{\partial y^i \partial y^j} \langle [u, y]_m, u \rangle
\]

\[
= 0.
\]

Therefore, (5.5.1) gives

\[
2E_{ij}(o, y) = \frac{\partial^2 s(o, y)}{\partial y^i \partial y^j} = \frac{(n + 1)}{b^2} \left\{ \frac{s}{\alpha(y)} \left[ \langle [u, u_i]_m, u_j \rangle + \langle [u, u_j]_m, u_i \rangle \right] + \langle [u, y]_m, y \rangle \left[ \frac{2s\delta_i^j}{\alpha^3(y)} - \frac{2}{\alpha^4(y)} (b_i y^j + b_j y^i) + \frac{8s y^j y^i}{\alpha^5(y)} \right] + \left[ \langle [u, u_j]_m, y \rangle + \langle [u, y]_m, u_j \rangle \right] \left( \frac{b_i - 2s y^i}{\alpha^2(y)} \right) \right\} . (5.5.3)
\]

Thus, we can state

**Theorem 5.5.1.** Let \((G/H, L)\) be a homogeneous Kropina metric of the form \(L = \alpha \phi(s)\), where \(s = \frac{\beta}{\alpha}\) with \(\alpha\) a \(G\)-invariant Riemannian metric on the coset space \(G/H\) of a Lie group \(G\) and \(\beta\) a \(G\)-invariant vector field on \(G/H\). Then mean Berwald curvature of homogeneous Kropina metric is given as in (5.5.3).
5.6 Mean Berwald curvature of homogeneous generalised Kropina metric

Consider \( L = \frac{\alpha^{m+1}}{\beta^m} \), \((m \neq 0, -1)\) which gives
\[
\phi(s) = \frac{1}{s^m}, \quad \phi'(s) = \frac{-m}{s^{m+1}}.
\]
Let \( \psi = \phi - s\phi' \). Then we have
\[
Q = \frac{\phi'}{\psi} = -\frac{m}{(1+m)s},
\]
\[
Q' = \frac{m}{(1+m)s^2},
\]
\[
Q'' = -\frac{2m}{(1+m)s^3},
\]
\[
\Delta = 1 + sQ + (b^2 - s^2)Q' = \frac{(1-m)s^2 + b^2m}{(1+m)s^2},
\]
\[
\Psi = \frac{Q'}{2\Delta} = \frac{m}{2[(1-m)s^2 + b^2m]},
\]
\[
\Phi = -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q''
\]
\[
= \frac{2m\{n[(1-m)s^2 + b^2m] + b^2\}}{(1+m)^2 s^3}.
\]
Letting \( c = 1 \) in \( S(o, y) \), we get
\[
S(o, y) = -\frac{1}{\alpha(y) 2\Delta^2} \left(-\langle [u, y]_m, y \rangle - \alpha(y)Q \langle [u, y]_m, u \rangle \right)
\]
\[
= \frac{ms(nP + b^2)}{\alpha(y) P^2} \langle [u, y]_m, y \rangle - \frac{m^2(nP + b^2)}{(1+m)P^2} \langle [u, y]_m, u \rangle,
\]
where
\[
P = (1-m)s^2 + b^2m. \quad (5.6.1)
\]
Therefore
\[
2E_{ij}(o, y) = \frac{\partial^2 S(o, y)}{\partial y^i \partial y^j}
\]
\[
= \frac{\partial^2}{\partial y^i \partial y^j} \left\{ \frac{ms(nP + b^2)}{\alpha(y) P^2} \langle [u, y]_m, y \rangle \right\}
\]
\[
- \frac{\partial^2}{\partial y^i \partial y^j} \left\{ \frac{m^2(nP + b^2)}{(1+m)P^2} \langle [u, y]_m, u \rangle \right\}. \quad (5.6.2)
\]
Now

\[
\frac{\partial^2}{\partial y^i \partial y^j} \left\{ \frac{m s(nP + b^2)}{\alpha(y)P^2} ([u, y]_m, y) \right\} = m \left\{ \frac{s(nP + b^2)}{\alpha(y)P^2} \cdot \frac{\partial^2}{\partial y^i \partial y^j} ([u, y]_m, y) \right\} + \frac{\partial}{\partial y^i} ([u, y]_m, y) \cdot \frac{\partial}{\partial y^j} \left\{ \frac{s(nP + b^2)}{\alpha(y)P^2} \right\} + \frac{\partial}{\partial y^j} ([u, y]_m, y) \cdot \frac{\partial}{\partial y^i} \left\{ \frac{s(nP + b^2)}{\alpha(y)P^2} \right\},
\]

where

\[
\frac{\partial ([u, y]_m, y)}{\partial y^i} = ([u, u_i]_m, y) + ([u, y]_m, u_i),
\]

\[
\frac{\partial^2 ([u, y]_m, y)}{\partial y^i \partial y^j} = ([u, u_i]_m, u_j) + ([u, u_j]_m, u_i).
\]

Now

\[
\frac{\partial}{\partial y^j} \left\{ \frac{s(nP + b^2)}{\alpha(y)P^2} \right\} = \frac{1}{\alpha^2(y)p^3} \left\{ P_1 b_j - 2 A_1 s - \frac{y^j}{\alpha(y)} \right\},
\]

where

\[
P_1 = b^4 m(mn + 1) - 3 b^2 (1 - m) s^2 - (1 - m)^2 n s^4,
\]

\[
A_1 = b^3 \left[ b^2 m(mn + 1) - (1 - m)(1 - mn) s^2 \right].
\]

Hence

\[
\frac{\partial^2 \left\{ \frac{s(nP + b^2)}{\alpha(y)P^2} \right\}}{\partial y^i \partial y^j} = \frac{\partial}{\partial y^j} \left\{ \frac{\partial \left\{ \frac{s(nP + b^2)}{\alpha(y)P^2} \right\}}{\partial y^i} \right\} = \frac{-2 s A_1}{\alpha^3(y)p^3} \delta_i^j + \frac{A_2 y^i y^j}{\alpha^5(y)p^4} + \frac{A_3}{\alpha^4(y)p^4} (b_i y^j + b_j y^i) + \frac{A_4}{\alpha^3(y)p^4} b_i b_j.
\]

where

\[
A_2 = 8 s b^4 m [b^2 m(mn + 1) + (1 - m)(mn - 2) s^2],
\]
The study of Berwald space with \((\alpha, \beta)\)-metric

\[ A_3 = -2b^2b^4m^3(mn + 1) - 2b^2m(1 - m)s^2(mn + 4) + 3(1 - m)^2(1 - mn)s^4, \]

\[ A_4 = 2(1 - m)s[2b^2(1 - m)(3 - mn)s^2 - 3b^4m(mn + 2) + (1 - m)^2ns^4. \]

Therefore, we have

\[
\frac{\partial^2}{\partial y^i \partial y^j} \left( \frac{m^2(np + b^2)}{\alpha(y)p^2} \langle [u, y]_m, y \rangle \right) = m \left\{ \frac{s(np + b^2)}{\alpha(y)p^2} \left( \langle [u, u_i]_m, u_j \rangle + \langle [u, u_j]_m, u_i \rangle \right) \right. \\
+ \left. \left( \langle [u, u_j]_m, y \rangle + \langle [u, y]_m, u_i \rangle \right) \right. \\
\times \left[ \frac{1}{\alpha^2(y)p^3} \left( P_1b_i - 2A_1s\frac{y^i}{\alpha(y)} \right) \right] \\
+ \left[ \langle [u, y]_m, y \rangle \left( \frac{-2sA_1}{\alpha^3(y)p^3} \delta^i_j + \frac{A_2y^iy^j}{\alpha^3(y)p^4} \right) + \frac{A_3}{\alpha^4(y)p^4} (b_iy^j + b_jy^i) + \frac{A_4}{\alpha^5(y)p^4} b_ib_j \right] \\
+ \left( \langle [u, u_i]_m, y \rangle + \langle [u, y]_m, u_i \rangle \right) \\
\times \left[ \frac{1}{\alpha^2(y)p^3} \left( P_1b_j - 2A_1s\frac{y^j}{\alpha(y)} \right) \right] \right\}, \tag{5.6.6}
\]

where \(P, P_1, A_1, A_2, A_3\) and \(A_4\) are given in (5.6.1), (5.6.3) and (5.6.5).

Note that

\[
\frac{\partial \langle [u, y]_m, u \rangle}{\partial y^i} = \langle [u, u_j]_m, u \rangle, \\
\frac{\partial^2 \langle [u, y]_m, u \rangle}{\partial y^i \partial y^j} = \frac{\partial}{\partial y^i} \langle [u, u_j]_m, u \rangle = 0.
\]

Now

\[
\frac{\partial^2}{\partial y^i \partial y^j} \left\{ \frac{m^2(np + b^2)}{(m + 1)p^2} \langle [u, y]_m, u \rangle \right\} = \frac{m^2}{(m + 1)} \frac{\partial^2}{\partial y^i \partial y^j} \left\{ \frac{(np + b^2)}{P^2} \langle [u, y]_m, u \rangle \right\} \\
= \frac{m^2}{(m + 1)} \left\{ \frac{\partial}{\partial y^i} \left[ \frac{(np + b^2)}{P^2} \right] \cdot \frac{\partial}{\partial y^j} \langle [u, y]_m, u \rangle \right\} \\
+ \frac{\partial^2}{\partial y^i \partial y^j} \left[ \frac{(np + b^2)}{P^2} \right] \cdot \langle [u, y]_m, u \rangle \\
+ \frac{\partial}{\partial y^i} \left[ \frac{(np + b^2)}{P^2} \right] \cdot \frac{\partial}{\partial y^j} \langle [u, y]_m, u \rangle \right\}. \tag{5.6.7}
\]
Now

\[
\frac{\partial}{\partial y^j} \left\{ \frac{(nP + b^2)}{P^2} \right\} = \frac{2(1 - m)s \left( s_{y_j}^{\alpha(y)} - b_j \right)(nP + 2b^2)}{\alpha(y)P^3},
\]

\[
\frac{\partial^2}{\partial y^i \partial y^j} \left\{ \frac{(nP + b^2)}{P^2} \right\} = \frac{P_1}{\alpha^2(y)P^3} \delta_i^j + \frac{B_1 y^i y^j}{\alpha^4(y)P^4} + \frac{B_2}{\alpha^3(y)P^4} (b_i y^j + b_j y^i) + \frac{B_3}{\alpha^2(y)P^4} b_i b_j,
\]

(5.6.8)

where

\[
B_1 = 8b^2 (1 - m)s^2 \left\{ (1 - m)(1 - mn)s^2 - b^2 m(mn + 2) \right\},
\]

\[
B_2 = 4(1 - m)s \left\{ b^4 m(mn + 2) - (1 - m)s^2 \left[ (1 - m)ns^2 + 4b^2 \right] \right\},
\]

\[
B_3 = 2(1 - m) \left\{ (1 - m)s^2 \left[ 10b^2 + 3(1 - m)ns^2 + 2b^2 mn \right] - b^4 m(mn + 2) \right\}.
\]

(5.6.9)

Therefore (5.6.7) gives

\[
\frac{\partial^2}{\partial y^i \partial y^j} \left\{ \frac{m^2(nP + b^2)}{(1 + m)P^2} \left\langle u, y \right\rangle_m, u \right\} = \frac{m^2}{m + 1} \left\{ \frac{2(1 - m)s \left( s_{y_j}^{\alpha(y)} - b_j \right)(nP + 2b^2)}{\alpha(y)P^3} \right. \\
\times \left\langle \left[ u, u_j \right]_m, u \right\rangle + \left[ \frac{P_1}{\alpha^2(y)P^3} \delta_i^j + \frac{B_1 y^i y^j}{\alpha^4(y)P^4} \right. \\
+ \frac{B_2}{\alpha^3(y)P^4} (b_i y^j + b_j y^i) + \frac{B_3}{\alpha^2(y)P^4} b_i b_j \right. \\
\times \left\langle \left[ u, y \right]_m, u \right\rangle + \left\langle \left[ u, u_i \right]_m, u \right\rangle \\
\times \left. \frac{2(1 - m)s \left( s_{y_j}^{\alpha(y)} - b_j \right)(nP + 2b^2)}{\alpha(y)P^3} \right\} (5.6.10)
\]
Finally from (5.6.2), we have

\[
2E_{ij}(\alpha, \beta) = m \left\{ \frac{s(np + b^2)}{\alpha(y)p^2} \left( \langle [u, u_i]m, u_j \rangle + \langle [u, u_j]m, u_i \rangle \right) + \langle [u, u_i]m, y \rangle + \langle [u, y]m, u_i \rangle \right\} \\
+ \frac{1}{\alpha^2(y)p^3} \left( P_1 b_i - 2A_1 s \frac{y^i}{\alpha(y)} \right) \\
+ \frac{-2sA_1}{\alpha^2(y)p^3} \delta^i_j + \frac{A_2 y^i y^j}{\alpha^2(y)p^4} \frac{A_3}{\alpha^4(y)p^4} (b_i y^j + b_j y^i) \\
+ \frac{A_4}{\alpha^3(y)p^4} b_i b_j \left( \langle [u, u_i]m, y \rangle + \langle [u, y]m, u_i \rangle \right) \\
\times \left\{ \frac{1}{\alpha^2(y)p^3} \left( P_1 b_j - 2A_1 s \frac{y^j}{\alpha(y)} \right) \right\} \\
+ \frac{m^2}{m + 1} \left\{ \frac{2(1 - m)s}{\alpha(y)p^3} \left( s \frac{y^i}{\alpha(y)} - b_i \right) (nP + 2b^2) \\
+ \frac{P_1}{\alpha^2(y)p^3} \delta^i_j + \frac{B_1 y^i y^j}{\alpha^4(y)p^4} \frac{B_2}{\alpha^3(y)p^4} (b_i y^j + b_j y^i) \frac{B_3}{\alpha^2(y)p^4} b_i b_j \right\} \\
\times \left( \langle [u, u_i]m, u \rangle + \langle [u, u_i]m, u \rangle \right) \\
\times \frac{2(1 - m)s}{\alpha(y)p^3} \left( s \frac{y^i}{\alpha(y)} - b_i \right) (nP + 2b^2) \right\}. \tag{5.6.11}
\]

Thus, we can state

**Theorem 5.6.1.** Let \((G/H, L)\) be a homogeneous generalized Kropina metric of the form \(L = \alpha \phi(s)\), where \(s = \frac{\beta}{\alpha}\) with \(\alpha\) a \(G\)-invariant Riemannian metric on the coset space \(G/H\) of a Lie group \(G\) and \(\beta\) a \(G\)-invariant vector field on \(G/H\). Then mean Berwald curvature of homogeneous generalized Kropina metric is given as in (5.6.11).

### 5.7 Conclusion

L. Berwald introduced a connection and two curvature tensors in 1926. The Berwald connection is convenient when dealing with Finsler spaces of constant flag curvature. It is most directly related to the non-linear connection coefficients and most amenable to the
study of the geometry of paths. In this chapter, we found the Berwald connection in a Finsler space with special \((\alpha, \beta)\)-metric and concrete form of the Berwald connection with an example. Also we obtained the formulae for mean Berwald curvature of homogeneous Kropina and homogeneous generalized Kropina metrics.

The important findings of this chapter are as follows:

1. Let \( F^n \) be the Finsler space with a special \((\alpha, \beta)\)-metric \( L(\alpha, \beta) = c_1 \alpha + c_2 \beta + \frac{a^2}{\beta}, c_1 \neq 0 \) with the Berwald connection \( B \Gamma = (G^i_{jk}, G^i_0, 0) \). Then we have the following.

   (a) If \((c_2 \beta^2 - \alpha^2) \neq 0\), then \( F^n \) is a Berwald space if and only if \( b_{ji} = 0 \) and the Berwald connection is \((\gamma^i_{jk}, \gamma^i_0, 0)\).

   (b) If \((c_2 \beta^2 - \alpha^2) = 0\), then \( F^n \) is a Berwald space if and only if \( B_j^i = 0 \) and the Berwald connection is \((\gamma^i_{jk}, \gamma^i_0, 0)\).

2. Let \( F^n \) be a Finsler space with special \((\alpha, \beta)\)-metric \( L(\alpha, \beta) = c_1 \alpha + c_2 \beta + \frac{a^2}{\beta}, c_1 \neq 0 \).

   Then the vector field \( B^i(x, y) \) in the Berwald connection is given by
   \[
   B^i = P_1 e^i + P_2 s^i + P_3 \beta^i,
   \]
   where
   \[
   P_1 = E + \frac{\alpha(\alpha D - (\alpha r_{00} + \beta E))}{c_1 \beta^2 + 2\alpha \beta},
   \]
   \[
   P_2 = \frac{\alpha(c_2 \beta^2 - \alpha^2)}{c_1 \beta^2 + 2\alpha \beta},
   \]
   \[
   P_3 = \frac{\alpha \{c_2 \beta r_{00} - [(c_1 \beta + \alpha)E + c_2 \beta D]\}}{c_1 \beta^2 + 2\alpha \beta}.
   \]

3. Let \((G/H, L)\) be a homogeneous Kropina metric of the form \( L = \alpha \phi(s) \), where \( s = \frac{\beta}{\alpha} \) with \( \alpha \) a \( G \)-invariant Riemannian metric on the coset space \( G/H \) of a Lie group \( G \) and \( \beta \) a \( G \)-invariant vector field on \( G/H \). Then mean Berwald curvature of homogeneous Kropina metric is given as in (5.5.3).
4. Let \((G/H, L)\) be a homogeneous generalized Kropina metric of the form \(L = \alpha \phi(s)\), where \(s = \frac{\beta}{\alpha}\) with \(\alpha\) a \(G\)-invariant Riemannian metric on the coset space \(G/H\) of a Lie group \(G\) and \(\beta\) a \(G\)-invariant vector field on \(G/H\). Then mean Berwald curvature of homogeneous generalized Kropina metric is given as in (5.6.11).