CHAPTER 2

Publications based on this Chapter


• On Kenmotsu manifold with respect to quarter symmetric metric connection, (Communicated).

• On Generalized conharmonic $\phi$-recurrent Kenmotsu manifold, (Communicated).
Chapter 2
On Kenmotsu Manifolds

An almost Contact metric manifold which satisfies the following conditions

\[(\nabla_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi \quad \text{and} \quad (2.0.1)\]

\[\nabla_X \xi = X - \eta(X)\xi, \quad (2.0.2)\]

is called a Kenmotsu manifold.

In 1971, K. Kenmotsu [66] introduced a class of Contact Riemannian manifolds satisfying some conditions and this manifold is known as Kenmotsu manifold. The properties of Kenmotsu manifold have been studied by several authors such as Pittis [86], De [37], C.Ozgur [81], De and Pathak [45], De, Yildiz and Yaliniz [54]. Curvatures on Kenmotsu manifold also been studied by the authors B.B.Sinha and A.K.Srivastava [111], Cihan and U.C.De [32], Venkatesha and C.S. Bagewadi [26], N.Asghari and Taleshian [4], S.K.Chaubey and R.H.Ojha [34], U.C.De and Krishnendu De [43], D.G.Prakasha, Amit Prakash [91], Venkatesha and C.S.Bagewadi [23] and many others. Later generalized properties of Kenmotsu manifold was studied by the authors Ozgur [81], Basari and Murathan [17]. On the other hand, in 1975, S.Golab [56] defined and studied quarter symmetric connection in a differentiable manifold with affine connection. After Golab, S.C. Rastogi [97] continued the systematic study of quarter-symmetric connection. Also
authors like Mukhopadhyay, A.K.Roy and B.Barua [76], Biswas, S.C., and U.C.De [47], Basari and Murathan [17], U.C.De, Ozgur, and Sular [48], D.G.Prakasha, Venkatesha and C.S.Bagewadi [23] and many others studied quarter-symmetric metric connection.

2.1 Preliminaries

Let $M^{2n+1}$ $(\phi, \xi, \eta, g)$ be a Kenmotsu manifold with the structure $(\phi, \xi, \eta, g)$. Then the following relations hold [5]:

\[
\phi^2(X) = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad (2.1.1)
\]

\[
\eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \quad \eta(\phi X) = 0, \quad (2.1.2)
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.1.3)
\]

\[
R(\xi, X)Y = (\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \quad (2.1.4)
\]

\[
(a)\nabla_X \xi = X - \eta(X)\xi, \quad (b)(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.1.5)
\]

\[
R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (2.1.6)
\]

\[
R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.1.7)
\]

\[
\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \quad (2.1.8)
\]

\[
S(X, \xi) = -2n\eta(X), \quad (2.1.9)
\]

\[
S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y), \quad (2.1.10)
\]

for all vector fields $X, Y, Z$, where $\nabla$ denotes the operator of covariant differentiation with respect to $g$, $\phi$ is a $(1, 1)$ tensor field, $S$ is the Ricci tensor of type $(0, 2)$ and $R$ is the Riemannian curvature tensor of the manifold.
Definition 8. A Kenmotsu manifold is said to be locally $\phi$-symmetric if

$$\phi^2((\nabla_W R)(X, Y)Z) = 0,$$

(2.1.11)

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.

Definition 9. A Kenmotsu manifold is said to be locally pseudo-projective $\phi$-symmetric if

$$\phi^2((\nabla_W \overline{P})(X, Y)Z) = 0,$$

(2.1.12)

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.

Definition 10. A Kenmotsu manifold is said to be pseudo-projective $\phi$-recurrent manifold if there exists a non-zero 1-form $A$ such that

$$\phi^2((\nabla_W \overline{P})(X, Y)Z) = A(W)\overline{P}(X, Y)Z,$$

(2.1.13)

for arbitrary vector fields $X, Y, Z, W$, where $\overline{P}$ is a pseudo-projective curvature tensor given by [89]

$$\overline{P}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y]$$

$$- \frac{r}{(2n + 1)} \left[ \frac{a}{2n} + b \right] [g(Y, Z)X - g(X, Z)Y],$$

(2.1.14)

where $a$ and $b$ are constants such that $a, b \neq 0$, $R$ is the curvature tensor, $S$ is the Ricci tensor and $r$ is the scalar curvature.

If the 1-form $A$ vanishes, then the manifold reduces to a locally pseudo-projective $\phi$-symmetric manifold.

Definition 11. A Kenmotsu manifold is said to be generalized pseudo-projective $\phi$-recurrent if its curvature tensor $\overline{P}$ satisfies the condition

$$\phi^2((\nabla_W \overline{P})(X, Y)Z) = A(W)\overline{P}(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y],$$

(2.1.15)
where $A$ and $B$ are two 1-forms, $B$ is non-zero and these are defined by

$$A(W) = g(W, \rho_1) \quad \text{and} \quad B(W) = g(W, \rho_2),$$

where $\rho_1, \rho_2$ are vector fields associated with 1-forms $A$ and $B$ respectively.

### 2.2 Generalized Pseudo-projective $\phi$-recurrent Kenmotsu Manifold

Let us consider a the generalized pseudo-projective $\phi$-recurrent Kenmotsu manifold. Then by virtue of (2.1.1) and (2.1.15) we have

$$-((\nabla_W \bar{P})(X, Y)Z) + \eta((\nabla_W \bar{P})(X, Y)Z)\xi$$

$$= A(W)\bar{P}(X, Y)Z + B(W)\left[ g(Y, Z)X - g(X, Z)Y \right], \quad \text{(2.2.1)}$$

from which it follows that

$$-g((\nabla_W \bar{P})(X, Y)Z, U) + \eta((\nabla_W \bar{P})(X, Y)Z)\eta(U) = A(W)g(\bar{P}(X, Y)Z, U)$$

$$+ B(W)\left[ g(Y, Z)g(X, U) - g(X, Z)g(Y, U) \right]. \quad \text{(2.2.2)}$$

Let $\{e_i\}, i = 1, 2, \ldots, 2n + 1$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (2.2.2) and taking summation over $i, 1 \leq i \leq 2n + 1$, we get

$$-[a + (2n - 1)b]g(\nabla_W S)(Y, Z) - b(\nabla_W S)(\xi, Z)\eta(Y)$$

$$= (a + 2nb)A(W) \left\{ S(Y, Z) - \frac{r}{(2n + 1)} g(Y, Z) \right\}$$

$$+ 2nB(W)g(Y, Z). \quad \text{(2.2.3)}$$
Replacing $Z$ by $\xi$ in (2.2.3) and using (2.1.2) and (2.1.9), we get

$$-[a + (2n - 1)b][\nabla_{W}S(Y, \xi)] = (a + 2nb)A(W)\eta(Y) \left\{ -2n - \frac{r}{2n + 1} \right\} + 2nB(W)\eta(Y). \quad (2.2.4)$$

Now we have

$$(\nabla_{W}S)(Y, \xi) = \nabla_{W}S(Y, \xi) - S(\nabla_{W}Y, \xi) - S(Y, \nabla_{W}\xi). \quad (2.2.5)$$

Using (2.1.5) and (2.1.9) in the above relation, it follows that

$$(\nabla_{W}S)(Y, \xi) = -[S(Y, W) + 2ng(Y, W)]. \quad (2.2.6)$$

In view of (2.2.4) and (2.2.6), we have

$$-[a + (2n - 1)b][-S(Y, W) + 2ng(Y, W)] = (a + 2nb)A(W)\eta(Y) \left\{ -2n - \frac{r}{2n + 1} \right\} + 2nB(W)\eta(Y). \quad (2.2.7)$$

Replacing $Y$ and $W$ by $\phi Y$ and $\phi W$ respectively in (2.2.7) and using (2.1.3) and (2.1.10), we obtain

$$S(Y, W) = -2ng(Y, W). \quad (2.2.8)$$

This leads to the following theorem:

**Theorem 2.2.1.** A generalized pseudo-projective $\phi$-recurrent Kenmotsu manifold $(M^{2n+1}, g)$ is an Einstein manifold.

Putting $Y = Z = e_{i}$ in (2.2.2) and taking summation over $i, i = 1, 2, \ldots, 2n + 1$, we get

$$(-a + b)(\nabla_{W}S)(X, U) - b\nabla_{W}rg(X, U) + \frac{d\tau(W)}{2n + 1} \left[ \frac{a}{2n} + b \right] [2ng(X, U)] + (a - b)\nabla_{W}S(X, \xi)\eta(U) + b\nabla_{W}r\eta(X)\eta(U) - \frac{d\tau(W)}{2n + 1} \left[ \frac{a}{2n} + b \right] [2n\eta(X)\eta(U)] = A(W)[(a - b)S(X, U) + brg(X, U) - \frac{r}{2n + 1} \left[ \frac{a}{2n} + b \right] 2ng(X, U)] + 2nB(W)g(X, U) = 0. \quad (2.2.9)$$
Putting $U = \xi$ in (2.2.9) we have,

$$A(W)\eta(X) \left[ -2na + br - 2nb - \frac{r}{2n+1} \left( \frac{a}{2n} + b \right) 2n \right] + 2nB(W)\eta(X) = 0. \quad (2.2.10)$$

Putting $X = \xi$ in (2.2.10) we have,

$$B(W) = \left[ a - \frac{br}{2n} + b + \frac{r}{2n(2n+1)}(a + 2nb) \right] A(W). \quad (2.2.11)$$

This leads to the following result:

**Theorem 2.2.2.** In a generalized pseudo-projective $\phi$-recurrent Kenmotsu manifold $(M^{2n+1}, g)$, the 1-forms $A$ and $B$ are related as in (2.2.11).

Now from (2.2.1) we have

$$(\nabla_{\nu}F)(X, Y)Z = \eta((\nabla_{\nu}F)(X, Y)Z)\xi$$

$$-A(W)\overline{F}(X, Y)Z - B(W)[g(Y, Z)X - g(X, Z)Y]. \quad (2.2.12)$$

This implies,

$$a(\nabla_{\nu}R)(X, Y)Z = a\eta((\nabla_{\nu}R)(X, Y)Z)\xi - aA(W)R(X, Y)Z$$

$$+ b[(\nabla_{\nu}S)(Y, Z)\eta(X) - (\nabla_{\nu}S)(X, Z)\eta(Y)]\xi$$

$$- b[(\nabla_{\nu}S)(Y, Z)X - (\nabla_{\nu}S)(X, Z)Y]$$

$$- bA(W)[S(Y, Z)X - S(X, Z)Y]$$

$$+ \frac{r}{(2n+1)} \left( \frac{a}{2n} + b \right) A(W)[g(Y, Z)X - g(X, Z)Y]$$

$$- B(W)[g(Y, Z)X - g(X, Z)Y]. \quad (2.2.13)$$
From (2.2.13) and the Bianchi identity we get

\[ aA(W)\eta(R(X, Y)Z) + aA(X)\eta(R(Y, W)Z) + aA(Y)\eta(R(W, X)Z) \]
\[ = bA(W)[S(X, Z)\eta(Y) - S(Y, Z)\eta(X)] \]
\[ - \frac{r}{(2n + 1)} \left[ \frac{a}{2n} + b \right] A(W)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)] \]
\[ + bA(X)[S(Y, Z)\eta(W) - S(W, Z)\eta(Y)] \]
\[ - \frac{r}{(2n + 1)} \left[ \frac{a}{2n} + b \right] A(X)[g(Y, Z)\eta(W) - g(W, Z)\eta(Y)] \]
\[ + bA(Y)[S(W, Z)\eta(X) - S(X, Z)\eta(W)] \]
\[ - \frac{r}{(2n + 1)} \left[ \frac{a}{2n} + b \right] A(Y)[g(W, Z)\eta(X) - g(X, Z)\eta(W)] \]
\[ + B(W)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + B(X)[g(W, Z)\eta(Y) - g(Y, Z)\eta(W)] \]
\[ + B(Y)[g(X, Z)\eta(W) - g(W, Z)\eta(X)]. \] (2.2.14)

By virtue of (2.1.8), we obtain from (2.2.14) that

\[ aA(W)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)] + aA(X)[g(Y, Z)\eta(W) - g(W, Z)\eta(Y)] \]
\[ + aA(Y)[g(W, Z)\eta(X) - g(X, Z)\eta(W)] \]
\[ = bA(W)[S(X, Z)\eta(Y) - S(Y, Z)\eta(X)] - \frac{r}{(2n + 1)} \left[ \frac{a}{2n} + b \right] A(W)[g(X, Z)\eta(Y)] \]
\[ - g(Y, Z)\eta(X)] + bA(X)[S(Y, Z)\eta(W) - S(W, Z)\eta(Y)] \]
\[ - \frac{r}{(2n + 1)} \left[ \frac{a}{2n} + b \right] A(X)[g(Y, Z)\eta(W) - g(W, Z)\eta(Y)] + bA(Y)[S(W, Z)\eta(X)] \]
\[ - g(X, Z)\eta(W)] - \frac{r}{(2n + 1)} \left[ \frac{a}{2n} + b \right] A(Y)[g(W, Z)\eta(X) - g(X, Z)\eta(W)] \]
\[ + B(W)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + B(X)[g(W, Z)\eta(Y) - g(Y, Z)\eta(W)] \]
\[ + B(Y)[g(X, Z)\eta(W) - g(W, Z)\eta(X)]. \] (2.2.15)
Putting $Y = Z = e_i$ in (2.2.15) and taking summation over $i$, $1 \leq i \leq 2n + 1$, we get

\[(a)A(W)\eta(X) = A(X)\eta(W)\]

\[(b)B(W)\eta(X) = B(X)\eta(W),\]  \hspace{1cm} (2.2.16)

for all vector fields $X, W$.

Replacing $X$ by $\xi$ in (2.2.16), we get

\[(a)A(W) = \eta(W)\eta(\rho_1)\]

\[(b)B(W) = \eta(W)\eta(\rho_2).\]  \hspace{1cm} (2.2.17)

From (2.2.16) and (2.2.17) we can state the following theorem:

**Theorem 2.2.3.** In a generalized pseudo-projective $\phi$-recurrent Kenmotsu manifold, the characteristic vector field $\xi$ and the vector fields $\rho_1$ and $\rho_2$ associated to the 1-forms $A$ and $B$ respectively are co-directional and the 1-forms $A$ and $B$ are given by (2.2.17).

From (2.1.14), it follows that

\[(\nabla_W \bar{P})(X, Y)\xi = a(\nabla_W R)(X, Y)\xi + b[(\nabla_W S)(Y, \xi)X - (\nabla_W S)(X, \xi)Y].\]  \hspace{1cm} (2.2.18)

By making use of (2.1.5), (2.1.6) and (2.1.9), we obtain from the above equation that

\[(\nabla_W \bar{P})(X, Y)\xi = a[g(X, W) - \eta(X)\eta(W)]Y
\]  
\[ - a[g(Y, W) - \eta(Y)\eta(W)]X
\]  
\[ - aR(X, Y)(W - \eta(W)\xi).\]  \hspace{1cm} (2.2.19)
By virtue of (2.1.8), it follows from (2.2.19) that,

$$\eta((\nabla_W P)(X, Y)\xi) = 0.$$  \hspace{1cm} (2.2.20)

Using (2.2.19) and (2.2.20) in (2.2.1), we have

$$-a[g(X, W) - \eta(X)\eta(W)] + a[g(Y, W) - \eta(Y)\eta(W)] - aR(X, Y)(W - \eta(W)\xi)$$

$$= A(W) \left[ aR(X, Y)\xi - \left( \frac{ar}{2n(2n + 1)} + \left( 2n + \frac{r}{(2n + 1)} \right) b \right) [\eta(Y)X - \eta(X)Y] \right]$$

$$+ B(W)[\eta(Y)X - \eta(X)Y].$$ \hspace{1cm} (2.2.21)

Again using (2.1.6) and (2.2.17) in (2.2.21), we obtain

$$aR(X, Y)W = a[g(X, W)Y - g(Y, W)X]$$

$$- \left[ \frac{(r + 2n(2n + 1))(a + nb)}{2n(2n + 1)} \right] [\eta(X)Y - \eta(Y)X]$$

$$+ [\eta(Y)X - \eta(X)Y]\eta(W)\eta(\rho_2).$$ \hspace{1cm} (2.2.22)

If $X$ and $Y$ are orthogonal to $\xi$, then (2.2.22) reduces to

$$R(X, Y)W = g(X, W)Y - g(Y, W)X,$$ \hspace{1cm} (2.2.23)

for all vector fields $X, Y$ and $W$.

Hence we can state the following theorem:

**Theorem 2.2.4.** A generalized pseudo-projective $\phi$-recurrent Kenmotsu manifold $(M^{2n+1}, g)$, $n \geq 1$, is a space of constant curvature, provided that $X$ and $Y$ are orthogonal to $\xi$. 
2.3 3-dimensional Locally Generalized Pseudo-projective $\phi$-recurrent Kenmotsu Manifold

It is known that in a three dimensional Kenmotsu manifold, the curvature tensor and the Ricci tensor are defined respectively by [45]

\[ R(X, Y)Z = \left( \frac{r + 4}{2} \right) [g(Y, Z)X - g(X, Z)Y] - \left( \frac{r + 6}{2} \right) [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi - g(X, Z)\eta(Y)\eta(Z)Y], \tag{2.3.1} \]

and

\[ S(X, Y) = \frac{1}{2} [(r + 2)g(X, Y) - (r + 6)\eta(X)\eta(Y)]. \tag{2.3.2} \]

Thus from (2.3.1), (2.3.2) and (2.1.14) we get

\[ F(X, Y)Z = \left[ a \left( \frac{r + 4}{2} \right) + b \left( \frac{r + 6}{2} \right) \right] [g(Y, Z)X - g(X, Z)Y] - \left[ a \left( \frac{r + 4}{2} \right) + b \left( \frac{r + 6}{2} \right) \right] [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] - \left[ a \left( \frac{r + 4}{2} \right) + b \left( \frac{r + 6}{2} \right) \right] [\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \tag{2.3.3} \]

Differentiate (2.3.3) covariantly with respect to $W$ we get

\[ (\nabla_W \overline{P})(X, Y)Z = dr(W) \left[ \frac{10a}{21} + \frac{5b}{14} \right] [g(Y, Z)X - g(X, Z)Y] - \frac{adr(W)}{2} [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] - dr(W) \left( \frac{a + b}{2} \right) [\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \tag{2.3.4} \]

Noting that we may assume that all vector fields $X, Y, Z, W$ are orthogonal to $\xi$ in the above relation, we have

\[ (\nabla_W \overline{P})(X, Y)Z = dr(W) \left[ \frac{10a}{21} + \frac{5b}{14} \right] [g(Y, Z)X - g(X, Z)Y]. \tag{2.3.5} \]
Applying $\phi^2$ to the both sides of (2.3.5) and using (2.2.1) and (2.2.2) we get

$$\phi^2(\nabla_wP)(X,Y)Z = d\tau(W) \left[ \frac{10a}{21} + \frac{5b}{14} \right] [g(Y,Z)X - g(X,Z)Y]. \tag{2.3.6}$$

Using (2.2.1), the equation (2.3.6) reduces to

$$A(W)P(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y]$$

$$= d\tau(W) \left[ \frac{10a}{21} + \frac{5b}{14} \right] [g(Y,Z)X - g(X,Z)Y]. \tag{2.3.7}$$

Putting $W = \{e_i\}$, where $\{e_i\}$, $i = 1, 2, 3$, is an orthonormal basis of the tangent space at any point of the manifold and taking summation over $i$, $1 \leq i \leq 3$, we obtain

$$P(X,Y)Z = \lambda [g(Y,Z)X - g(X,Z)Y], \tag{2.3.8}$$

where $\lambda = \left[ \frac{d\tau(e_i)}{A(e_i)} \left( \frac{10a}{21} + \frac{5b}{14} \right) - \frac{B(e_i)}{A(e_i)} \right]$ is a scalar, since $A$ is a non-zero 1-form. Then by Schur’s theorem $\lambda$ will be a constant on the manifold. Therefore, $(M^3, g)$ is of constant curvature $\lambda$. Thus we get the following theorem:

**Theorem 2.3.1.** A 3-dimensional generalized pseudo-projective $\phi$-recurrent Kenmotsu manifold $M^3$ is of constant curvature.

### 2.4 Generalized Conharmonic $\phi$-recurrent Kenmotsu Manifold

**Definition 12.** A Kenmotsu manifold is said to be locally $\phi$-symmetric[15] if

$$\phi^2((\nabla_wR)(X,Y)Z) = 0, \tag{2.4.1}$$

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$. 

---

On Kenmotsu Manifolds

---

Applying $\phi^2$ to the both sides of (2.3.5) and using (2.2.1) and (2.2.2) we get

$$\phi^2(\nabla_wP)(X,Y)Z = d\tau(W) \left[ \frac{10a}{21} + \frac{5b}{14} \right] [g(Y,Z)X - g(X,Z)Y]. \tag{2.3.6}$$

Using (2.2.1), the equation (2.3.6) reduces to

$$A(W)P(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y]$$

$$= d\tau(W) \left[ \frac{10a}{21} + \frac{5b}{14} \right] [g(Y,Z)X - g(X,Z)Y]. \tag{2.3.7}$$

Putting $W = \{e_i\}$, where $\{e_i\}$, $i = 1, 2, 3$, is an orthonormal basis of the tangent space at any point of the manifold and taking summation over $i$, $1 \leq i \leq 3$, we obtain

$$P(X,Y)Z = \lambda [g(Y,Z)X - g(X,Z)Y], \tag{2.3.8}$$

where $\lambda = \left[ \frac{d\tau(e_i)}{A(e_i)} \left( \frac{10a}{21} + \frac{5b}{14} \right) - \frac{B(e_i)}{A(e_i)} \right]$ is a scalar, since $A$ is a non-zero 1-form. Then by Schur’s theorem $\lambda$ will be a constant on the manifold. Therefore, $(M^3, g)$ is of constant curvature $\lambda$. Thus we get the following theorem:

**Theorem 2.3.1.** A 3-dimensional generalized pseudo-projective $\phi$-recurrent Kenmotsu manifold $M^3$ is of constant curvature.

### 2.4 Generalized Conharmonic $\phi$-recurrent Kenmotsu Manifold

**Definition 12.** A Kenmotsu manifold is said to be locally $\phi$-symmetric[15] if

$$\phi^2((\nabla_wR)(X,Y)Z) = 0, \tag{2.4.1}$$

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$. 

---

On Kenmotsu Manifolds
Definition 13. A Kenmotsu manifold is said to be locally conharmonically $\phi$-symmetric if
\[
\phi^2((\nabla_W \overline{C})(X, Y)Z) = 0,
\] (2.4.2)
for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.

Definition 14. A Kenmotsu manifold is said to be conharmonically $\phi$-recurrent manifold if there exists a non-zero 1-form $A$ such that
\[
\phi^2((\nabla_W \overline{C})(X, Y)Z) = A(W)\overline{C}(X, Y)Z,
\] (2.4.3)
for arbitrary vector fields $X, Y, Z, W$, where $\overline{C}$ is a conharmonic curvature tensor given by[60]
\[
\overline{C}(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y
+ g(Y, Z)QX - g(X, Z)QY],
\] (2.4.4)
for any vector fields $X, Y, Z$ on $M$. The manifold is said to be conharmonically flat if $\overline{C}$ vanishes identically on $M$.

Definition 15. A Kenmotsu manifold $(M^n, g)$ is said to be generalized conharmonic $\phi$-recurrent if its curvature tensor $\overline{C}$ satisfies the condition
\[
\phi^2((\nabla_W \overline{C})(X, Y)Z) = A(W)\overline{C}(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y],
\] (2.4.5)
where $A$ and $B$ are two 1-forms, $B$ is non-zero and these are defined by
\[
A(W) = g(W, \rho_1) \text{ and } B(W) = g(W, \rho_2)
\]
where $\rho_1$ and $\rho_2$ are the vector fields associated with 1-forms $A$ and $B$ respectively.
Let us consider the generalized conharmonic $\phi$-recurrent Kenmotsu manifold. Then by virtue of (2.1.1) and (2.4.5) we have

$$-((\nabla_w \xi)(X, Y)Z) + \eta((\nabla_w \xi)(X, Y)Z)\xi = A(W)\xi(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y],$$

(2.4.6)

from which it follows that

$$-g((\nabla_w \xi)(X, Y)Z, U) + \eta((\nabla_w \xi)(X, Y)Z)\eta(U) = A(W)g(\xi(X, Y)Z, U) + B(W)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)].$$

(2.4.7)

Let $\{e_i\}, i = 1, 2, ..., n$, be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = U = e_i$ in (2.4.7) and taking summation over $i, 1 \leq i \leq n$, we get

$$-\frac{1}{n-2}\sum_{i=1}^{n-2}(\nabla_w S)(e_i, U)\eta(Y, e_i) - \sum_{i=1}^{n-2}(\nabla_w S)(e_i, \xi)g(Y, e_i, \xi)\eta(Y) = A(W)\eta(Y) + (n-1)g(Y, Z)B(W).$$

(2.4.8)

Substituting $Z$ by $\xi$ in (2.4.8) and using (2.1.1), (2.1.2) and (2.1.9), we have

$$-\frac{1}{n-2}\sum_{i=1}^{n-2}(\nabla_w S)(e_i, \xi)g(Y, e_i) - \sum_{i=1}^{n-2}(\nabla_w S)(e_i, \xi)g(Y, e_i, \xi)\eta(Y) = A(W)\eta(Y) + (n-1)\eta(Y)B(W).$$

(2.4.9)

Also we have,

$$(\nabla_w S)(Y, \xi) = \nabla_w S(Y, \xi) - S(\nabla_w Y, \xi) - S(Y, \nabla_w \xi).$$

(2.4.10)

Using (2.1.9) in the above relation, it follows that

$$(\nabla_w S)(Y, \xi) = -(n-1)g(Y, W) - S(Y, W).$$

(2.4.11)
In view of (2.4.9) and (2.4.11) we have
\[
\frac{n-1}{n-2} S(Y,W) + \frac{(n-1)^2}{n-2} g(Y,W) \\
= -\left(\frac{r}{n-2}\right) \eta(Y)A(W) + (n-1)\eta(Y)B(W). \tag{2.4.12}
\]

Replacing $Y$ by $\phi Y$ and $W$ by $\phi W$ in (2.4.12) and using (2.1.3) and (2.1.10) we get
\[
S(Y,W) = -(n-1)g(Y,W). \tag{2.4.13}
\]

This leads to the following theorem:

**Theorem 2.4.1.** A generalized conharmonic $\phi$-recurrent Kenmotsu manifold is an Einstein manifold.

Again from (2.4.6), we have
\[
((\nabla^\omega C)(X,Y)Z) = \eta((\nabla^\omega C)(X,Y)Z)\xi
- A(W)C(X,Y)Z - B(W)[g(Y,Z)X - g(X,Z)Y]. \tag{2.4.14}
\]

This implies,
\[
(\nabla^\omega R)(X,Y)Z = \eta((\nabla^\omega R)(X,Y)Z)\xi - A(W)R(X,Y)Z
+ \frac{1}{n-2}[(\nabla^\omega S)(Y,Z)X - (\nabla^\omega S)(X,Z)Y + g(Y,Z)(\nabla^\omega Q)(X)
- g(X,Z)(\nabla^\omega Q)(Y)] - \frac{1}{n-2}[(\nabla^\omega S)(Y,Z)\eta(X) - (\nabla^\omega S)(X,Z)\eta(Y)
+ g(Y,Z)(\nabla^\omega Q)(X) - g(X,Z)(\nabla^\omega Q)(Y)]\xi + \frac{A(W)}{n-2}[S(Y,Z)X
- S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - B(W)[g(Y,Z)X
- g(X,Z)Y] - B(X)[g(W,Z)Y - g(Y,Z)W]
- B(Y)[g(X,Z)W - g(W,Z)X]. \tag{2.4.15}
\]
From (2.4.15) and the Bianchi identity we get

\[
A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z)
\]
\[
= \frac{A(W)}{n-2} [S(Y,Z)\eta(X) - S(X,Z)\eta(Y) + g(Y,Z)\eta(QX) - g(X,Z)\eta(QY)]
\]
\[
+ \frac{A(X)}{n-2} [S(W,Z)\eta(Y) - S(Y,Z)\eta(W) + g(W,Z)\eta(QY) - g(Y,Z)\eta(QW)]
\]
\[
+ \frac{A(Y)}{n-2} [S(X,Z)\eta(W) - S(W,Z)\eta(X) + g(X,Z)\eta(QW) - g(W,Z)\eta(QX)]
\]
\[
-B(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] - B(X)[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)]
\]
\[
-B(Y)[g(X,Z)\eta(W) - g(W,Z)\eta(X)]
\] (2.4.16)

By virtue of (2.1.8) we obtain from (2.4.16) that

\[
A(W)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] + A(X)[g(Y,Z)\eta(W) - g(W,Z)\eta(Y)]
\]
\[
+ A(Y)[g(W,Z)\eta(X) - g(X,Z)\eta(W)]
\]
\[
= \frac{A(W)}{n-2} [S(Y,Z)\eta(X) - S(X,Z)\eta(Y) + g(Y,Z)\eta(QX) - g(X,Z)\eta(QY)]
\]
\[
+ \frac{A(X)}{n-2} [S(W,Z)\eta(Y) - S(Y,Z)\eta(W) + g(W,Z)\eta(QY) - g(Y,Z)\eta(QW)]
\]
\[
+ \frac{A(Y)}{n-2} [S(X,Z)\eta(W) - S(W,Z)\eta(X) + g(X,Z)\eta(QW) - g(W,Z)\eta(QX)]
\]
\[
-B(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] - B(X)[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)]
\]
\[
-B(Y)[g(X,Z)\eta(W) - g(W,Z)\eta(X)]
\] (2.4.17)

Putting \( Y = Z = e_i \) in (2.4.17) and taking summation over \( i, 1 \leq i \leq n \), we get

\[(a) A(W)\eta(X) = A(X)\eta(W), \]
\[(b) B(W)\eta(X) = B(X)\eta(W) \] (2.4.18)

for all vector fields \( X, W \).
Replacing $X$ by $\xi$ in (2.4.18) we get

\[ (a)A(W) = \eta(W)\eta(\rho_1) \]
\[ (b)B(W) = \eta(W)\eta(\rho_2), \]  
\[ (2.4.19) \]

for any vector field $W$, where $A(\xi) = g(\xi, \rho_1) = \eta(\rho_1)$ and $B(\xi) = g(\xi, \rho_2) = \eta(\rho_2)$, $\rho_1$ and $\rho_2$ being the vector fields associated to the 1-forms $A$ and $B$.

Thus from (2.4.18) and (2.4.19), we state the following theorem:

**Theorem 2.4.2.** In a generalized conharmonic $\phi$-recurrent Kenmotsu manifold $(M^n, g)$, $n \geq 1$ the characteristic vector field $\xi$ and the vector fields $\rho_1$ and $\rho_2$ associated to the 1-forms $A$ and $B$ respectively are co-directional and the 1-forms $A$ and $B$ are given by (2.4.19).

Again from (2.4.6), we have

\[ -((\nabla_w \overline{C})(X, Y)Z) + \eta((\nabla_w \overline{C})(X, Y)Z)\xi \]
\[ = A(W)\overline{C}(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y]. \]  
\[ (2.4.20) \]

Replacing $Z$ by $\xi$ in the above equation, we get

\[ -((\nabla_w \overline{C})(X, Y)\xi) + \eta((\nabla_w \overline{C})(X, Y)\xi) \]
\[ = A(W)\overline{C}(X, Y)\xi + B(W)[g(Y, \xi)X - g(X, \xi)Y]. \]  
\[ (2.4.21) \]
On the other hand we have

\[ ((\nabla_w \overline{\nabla})(X, Y)\xi) = (\nabla_w R)(X, Y)\xi - \frac{1}{n-2}[-(n-1)g(Y, W)X
\]
\[ - S(Y, W)X + (n-1)g(X, W)Y + S(X, W)Y], \quad (2.4.22) \]

and,

\[ \eta((\nabla_w \overline{\nabla})(X, Y)\xi) = 0. \quad (2.4.23) \]

Using (2.4.22) and (2.4.23) in (2.4.21), we have

\[ - R(X, Y)W - \frac{1}{n-2}[-(n-1)g(Y, W)X - S(Y, W)X + (n-1)g(X, W)Y
\]
\[ + S(X, W)Y] = A(W)\overline{\nabla}(X, Y)\xi + B(W)[\eta(Y)X - \eta(X)Y]. \quad (2.4.24) \]

Using (2.4.4) in (2.4.24) we get

\[ - R(X, Y)W - \frac{1}{n-2}[-(n-1)g(Y, W)X - S(Y, W)X + (n-1)g(X, W)Y
\]
\[ + S(X, W)Y] = A(W)[R(X, Y)\xi - \frac{1}{n-2}[S(Y, \xi)X - S(X, \xi)Y
\]
\[ + g(Y, \xi)QX - g(X, \xi)QY] + B(W)[\eta(Y)X - \eta(X)Y]. \quad (2.4.25) \]

Again using (2.1.6) and (2.1.9) in (2.4.25), we have

\[ - R(X, Y)W - \frac{1}{n-2}[-(n-1)g(Y, W)X - S(Y, W)X + (n-1)g(X, W)Y
\]
\[ + S(X, W)Y] = A(W)[\eta(X)Y - \eta(Y)X - \frac{1}{n-2}(-(n-1)\eta(Y)X
\]
\[ + (n-1)\eta(X)Y + \eta(X)QX - \eta(X)QY] + B(W)[\eta(Y)X - \eta(X)Y]. \quad (2.4.26) \]

Noting that we may assume that all vector fields \( X, Y, Z, W \) are orthogonal to \( \xi \) in the above equation, we get

\[ - R(X, Y)W - \frac{1}{n-2}[-(n-1)g(Y, W)X - S(Y, W)X
\]
\[ + (n-1)g(X, W)Y + S(X, W)Y] = 0. \quad (2.4.27) \]
From (2.4.27) it follows that,

\[ g(R(X, Y)W, U) = -\frac{1}{n-2}[-(n-1)g(Y, W)g(X, U) - S(Y, W)g(X, U) \\
+ (n-1)g(X, W)g(Y, U) + S(X, W)g(Y, U)]. \]  

Let \( \{e_i\}, i = 1, 2, \ldots, n \) be an orthonormal basis of the tangent space at any point of the manifold. Let us put \( Y = W = e_i \) in (2.4.28), where \( 1 \leq i \leq n \), we get

\[ S(X, U) = (n-1)g(X, U) + \frac{r}{n-2}g(X, U). \]  

(2.4.29)

Using (2.4.29) in (2.4.27) we obtain

\[ R(X, Y)W = \frac{2(n-1)^2 + r}{(n-1)(n-2)}[g(Y, W)X - g(X, W)Y]. \]  

(2.4.30)

Hence we can state the following theorem:

**Theorem 2.4.3.** A generalized conharmonic \( \phi \)-recurrent Kenmotsu manifold is a manifold of constant curvature.

### 2.5 3-dimensional Generalized Conharmonic \( \phi \)-recurrent Kenmotsu Manifold

In a 3-dimensional Kenmotsu manifold \((M^3, g)\) we have

\[ R(X, Y)Z = \left( \frac{r+4}{2} \right) [g(Y, Z)X - g(X, Z)Y] \left( \frac{r+6}{2} \right) [g(Y, Z)\eta(X)\xi \\
- g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \]  

(2.5.1)

\[ S(X, Y) = \frac{1}{2}[(r+2)g(X, Y) - \eta(X)\eta(Y)]. \]  

(2.5.2)
\[ QX = \frac{1}{2}[(r + 2)X - (r + 6)\eta(X)\xi]. \]  

(2.5.3)

Thus from (2.4.4), (2.5.1), (2.5.2) and (2.5.3), we get

\[ \overline{C}(X, Y)Z = \left[-\frac{r}{2}\right][g(Y, Z)X - g(X, Z)Y]. \]  

(2.5.4)

Taking the covariant differentiation to (2.5.4) with respect to W, we have

\[ \nabla_W \overline{C}(X, Y)Z = \left[-\frac{dr(W)}{2}\right][g(Y, Z)X - g(X, Z)Y]. \]  

(2.5.5)

Applying \( \phi^2 \) to the both sides of equation (2.5.5), we get

\[ \phi^2(\nabla_W \overline{C})(X, Y)Z = \left[-\frac{dr(W)}{2}\right][g(Y, Z)\phi^2 X - g(X, Z)\phi^2 Y]. \]  

(2.5.6)

And using (2.1.1) we get

\[ \phi^2(\nabla_W \overline{C})(X, Y)Z = \left[\frac{dr(W)}{2}\right][g(Y, Z)X - g(X, Z)Y]. \]  

(2.5.7)

By (2.4.5) the equation (2.5.7) reduces to

\[
A(W)\overline{C}(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y] = \left[\frac{dr(W)}{2}\right][g(Y, Z)X - g(X, Z)Y].
\]  

(2.5.8)

Putting \( W = \{e_i\} \), where \( \{e_i\}, i = 1, 2, 3 \) is an orthonormal basis of the tangent space at any point of the manifold and taking summation over \( i, 1 \leq i \leq 3 \), we obtain

\[ \overline{C}(X, Y)Z = \lambda[g(Y, Z)X - g(X, Z)Y], \]  

(2.5.9)

where \( \lambda = \frac{r dr(e_i)}{2A(e_i)} - \frac{B(e_i)}{A(e_i)} \) is a scalar, since \( A \) and \( B \) are non-zero 1-form. Then by Schur's theorem \( \lambda \) will be a constant on the manifold. Therefore, \((M^3, g)\) is of curvature \( \lambda \).

Thus we get the following theorem:

**Theorem 2.5.1.** A 3-dimensional generalized conharmonic \( \phi \)-recurrent Kenmotsu manifold is of constant curvature.
2.6 Relation between the Levi-civita Connection and the Quarter-symmetric Metric Connection in a Kenmotsu Manifold

Let $\tilde{\nabla}$ be a linear connection and $\nabla$ be a Riemannian connection of an almost Contact metric manifold $M$. Then we have

$$\tilde{\nabla}_XY = \nabla_XY + H(X,Y), \quad (2.6.1)$$

where $H$ is a tensor of type (1, 1). For $\tilde{\nabla}$ to be a quarter-symmetric metric connection in $M$, we have

$$H(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)], \quad (2.6.2)$$

and

$$g(T'(X,Y), Z) = g(T(Z,Y), X). \quad (2.6.3)$$

By virtue of (1.4.1) and (2.6.3), we get

$$T'(X,Y) = g(\phi X, Y)\xi - \eta(X)\phi Y. \quad (2.6.4)$$

Again using (1.4.1) and (2.6.4) in (2.6.2), we obtain

$$H(X,Y) = -\eta(X)\phi Y. \quad (2.6.5)$$

Therefore a quarter-symmetric metric connection $\tilde{\nabla}$ in a Kenmotsu manifold is given by

$$\tilde{\nabla}_XY = \nabla_XY - \eta(X)\phi Y. \quad (2.6.6)$$
Equation (2.6.6) is the relation between the Levi-Civita connection and the quarter-symmetric metric connection in a Kenmotsu manifold.

A relation between the curvature tensor of $M$ with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection $\nabla$ is given by [43]

$$
\tilde{R}(X,Y)Z = R(X,Y)Z - 2d\eta(X,Y)\phi Z + [\eta(X)g(\phi Y, Z)
- \eta(Y)\phi X - \eta(X)\phi Y]\eta(Z),
$$

(2.6.7)

where $\tilde{R}$ and $R$ are the Riemannian curvatures of the connections $\tilde{\nabla}$ and $\nabla$ respectively. From (2.6.7), it follows that

$$
S(Y, Z) = S(Y, Z) - 2d\eta(\phi Z, Y) + g(\phi Y, Z) + \psi \eta(Y)\eta(Z),
$$

(2.6.8)

where $\tilde{S}$ and $S$ are the Ricci tensors of the connection $\tilde{\nabla}$ and $\nabla$, respectively and $\psi = \sum_{i=1}^{n} g(\phi e_i, e_i) =$ Trace of $\phi$. From (2.6.8) it is clear that in a Kenmotsu manifold the Ricci tensor with respect to the quarter-symmetric metric connection is not symmetric.

Contracting (2.6.8), we get

$$
\tilde{r} = r + 2(n - 1),
$$

(2.6.9)

where $\tilde{r}$ and $r$ are the scalar curvatures of the connections $\tilde{\nabla}$ and $\nabla$ respectively.

**Definition 16.** A Kenmotsu manifold $M$ is said to be locally $\phi$-symmetric [37] if

$$
\phi^2((\nabla_W R)(X,Y))Z = 0,
$$

(2.6.10)

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$. This notion was introduced by T. Takahashi [114] for Sasakian manifolds.
Definition 17. A Kenmotsu manifold $M$ is said to be locally projective $\phi$-symmetric if

$$\phi^2((\nabla_w P)(X,Y)Z) = 0, \quad (2.6.11)$$

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$, where $P$ is the projective curvature tensor [119] given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}(S(Y,Z)X - S(X,Z)Y). \quad (2.6.12)$$

Here $R$ and $r$ are the Riemannian curvature tensor and the scalar curvature tensor respectively.

2.7 Generalized $\phi$-recurrent Kenmotsu Manifold with Respect to the Quarter-symmetric Metric Connection

A Kenmotsu manifold $M$ is said to be generalized $\phi$-recurrent[17] with respect to the quarter-symmetric metric connection if

$$\phi^2((\tilde{\nabla}_w \tilde{R})(X,Y)Z) = A(W)\tilde{R}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y], \quad (2.7.1)$$

for arbitrary vector fields $X, Y, Z$ and $W$.

Let us consider a generalized $\phi$-recurrent Kenmotsu manifold with respect to quarter-symmetric metric connection. Then by virtue of (2.1.1) and (2.7.1) we have

$$- ((\tilde{\nabla}_w \tilde{R})(X,Y)Z + \eta((\tilde{\nabla}_w \tilde{R})(X,Y)Z)\xi$$

$$= A(W)\tilde{R}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y]. \quad (2.7.2)$$
from which it follows that

$$- g((\tilde{\nabla}_w \tilde{R})(X, Y)Z, U) + \eta((\tilde{\nabla}_w \tilde{R})(X, Y)Z)g(\xi, U)$$

$$= A(W)g(\tilde{R}(X, Y)Z, U) + B(W)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)].$$  (2.7.3)

Let \{e_i\}, \(i = 1, 2, ..., n\), be an orthonormal basis of the tangent space at any point of the manifold. Then putting \(X = U = e^i\) in (2.7.3) and taking summation over \(i, 1 \leq i \leq n\), we get

$$- (\tilde{\nabla}_w \tilde{S})(Y, Z) + \sum_{i=1}^{n} \eta((\tilde{\nabla}_w \tilde{R})(e_i, Y)Z)\eta(e_i)$$

$$= A(W)\tilde{S}(Y, Z) + (n - 1)B(W)g(Y, Z).$$sorte  (2.7.4)

The second term of (2.7.4) by putting \(Z = e^t\) takes the form

$$g((\tilde{\nabla}_w \tilde{R})(e_i, Y)e^t, \xi) = g(\tilde{\nabla}_w \tilde{R}(e_i, Y)e^t, \xi) - g(\tilde{R}(\tilde{\nabla}_w e_i, Y)e^t, \xi)$$

$$- g(\tilde{R}(e_i, (\tilde{\nabla}_w Y)e^t, \xi) - g(\tilde{R}(e_i, Y)(\tilde{\nabla}_w \xi), \xi).$$  (2.7.5)

On simplification we obtain

$$g((\tilde{\nabla}_w \tilde{R})(e_i, Y)e^t, \xi) = 0.$$  (2.7.6)

Taking \(Z = e^t\) in (2.7.4), and using (2.1.5), we have

$$- (\tilde{\nabla}_w \tilde{S})(Y, \xi) = A(W)[-(n - 1)\eta(Y) + \psi(Y)] + (n - 1)B(W)\eta(Y).$$  (2.7.7)

We know that

$$(\tilde{\nabla}_w \tilde{S})(Y, \xi) = \tilde{\nabla}_w \tilde{S}(Y, \xi) - \tilde{S}(\tilde{\nabla}_w Y, \xi) - \tilde{S}(Y, \tilde{\nabla}_w \xi).$$  (2.7.8)
By making use of (2.1.2), (2.1.5), (2.6.6) and (2.6.8) we get

\[
(\nabla_W S)(Y, \xi) = -S(Y, W) + 2d\eta(\phi Y, W) - g(\phi Y, W) \\
+ \{\psi - (n - 1)\}g(Y, W) - \psi \eta(Y)\eta(W).
\]  

(2.7.9)

In view of (2.7.7) and (2.7.9), we have

\[
- S(Y, W) + 2d\eta(\phi Y, W) - g(\phi Y, W) \\
+ \{\psi - (n - 1)\}g(Y, W) - \psi \eta(Y)\eta(W) \\
= A(W)\{(n - 1)\eta(Y) - \psi \eta(Y)\} - (n - 1)B(W)\eta(Y).
\]  

(2.7.10)

Again putting \( Y = \xi \) in the above equation and using (2.1.1) and (2.1.5) we get

\[
[(n - 1) - \psi]A(W) = (n - 1)B(W).
\]  

(2.7.11)

This implies,

\[
B(W) = \left[1 - \frac{\psi}{n - 1}\right]A(W).
\]  

(2.7.12)

This leads to the following theorem:

**Theorem 2.7.1.** In a generalized \( \phi \)-recurrent Kenmotsu manifold with respect to quarter-symmetric metric connection, the 1-forms \( A \) and \( B \) are related as in (2.7.12).
2.8 Locally Projective $\phi$-symmetric Kenmotsu Manifold with respect to the Quarter-symmetric Metric Connection

A Kenmotsu manifold $M$ is said to be locally projective $\phi$-symmetric with respect to quarter-symmetric metric connection $[23]$ if

$$\phi^2((\bar{\nabla}_W P)(X,Y)Z) = 0,$$

(2.8.1)

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$, where $P$ is the projective curvature tensor with respect to quarter-symmetric metric connection given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y],$$

(2.8.2)

where $R$ is the Riemannian curvature tensor with respect to quarter-symmetric metric connection $\bar{\nabla}$.

Using (2.6.6), we can write

$$(\bar{\nabla}_W P)(X,Y)Z = (\nabla_W P)(X,Y)Z - \eta(W)\phi P(X,Y)Z$$

$$+ \eta(W)\tilde{P}(\phi X,Y)Z + \tilde{P}(X,\phi Y)Z + \tilde{P}(X,Y)\phi Z.$$  

(2.8.3)

Now differentiating (2.8.2) covariantly with respect to $W$, we obtain

$$(\nabla_W \tilde{R})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z$$

$$- \frac{1}{(n-1)}[(\nabla_W \tilde{S})(Y,Z)X - (\nabla_W \tilde{S})(X,Z)Y].$$

(2.8.4)

Using (2.8.3), we can write

$$(\bar{\nabla}_W \tilde{R})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z - \eta(W)\phi \tilde{R}(X,Y)Z.$$  

(2.8.5)
Now differentiating (2.6.7) covariantly and using (2.1.3) and (2.1.5) in (2.7.2) we get

\[
(\nabla_w \tilde{R})(X, Y)Z = (\nabla_w R)(X, Y)Z - 2d\eta(X, Y)\{g(\phi W, Z)\xi - \eta(Z)\phi W\}
\]

\[
+ \{g(X, W)g(\phi Y, Z) - g(Y, W)g(\phi X, Z)\}\xi
- 2\{\eta(X)\eta(W)g(\phi Y, Z) - \eta(Y)\eta(W)g(\phi X, Z)\}\xi
+ \{g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y)\}(W)
\]

\[
+ \{g(W, Z) - \eta(W)\eta(Z)\} \times \{\eta(Y)\phi X - \eta(X)\phi Y\}
+ \{g(Y, W)\phi X - g(X, W)\phi Y + g(\phi W, X)\eta(Y)\xi
\]

\[
- g(\phi W, Y)\eta(X)\xi - \eta(Y)\eta(W)\phi X - \eta(X)\eta(W)\phi Y
\]

\[
- 2\eta(X)\eta(Y)\phi W\} \eta(Z).
\] (2.8.6)

By making use of (2.8.6) in (2.8.4), we have

\[
(\nabla_w \tilde{R})(X, Y)Z = (\nabla_w R)(X, Y)Z - 2d\eta(X, Y)\{g(\phi W, Z)\xi - \eta(Z)\phi W\}
\]

\[
+ \{g(X, W)g(\phi Y, Z) - g(Y, W)g(\phi X, Z)\}\xi - 2\{\eta(X)\eta(W)g(\phi Y, Z)
\]

\[
- \eta(Y)\eta(W)g(\phi X, Z)\}\xi + \{g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y)\}(W)
\]

\[
+ \{g(W, Z) - \eta(W)\eta(Z)\} \times \{\eta(Y)\phi X - \eta(X)\phi Y\}
+ \{g(Y, W)\phi X - g(X, W)\phi Y + g(\phi W, X)\eta(Y)\xi
\]

\[
- g(\phi W, Y)\eta(X)\xi - \eta(Y)\eta(W)\phi X - \eta(X)\eta(W)\phi Y
\]

\[
- 2\eta(X)\eta(Y)\phi W\} \eta(Z) - \frac{1}{n-1}[(\nabla_w S)(Y, Z)X + \psi[g(W, Y)
\]

\[
- \eta(W)\eta(Y)]\eta(Z)X + \psi[g(W, Z) - \eta(W)\eta(Z)]\eta(Y)X
\]

\[
- (\nabla_w S)(X, Z)Y - \psi[g(W, X) - \eta(W)\eta(X)]\eta(Z)Y
\]

\[
- \psi[g(W, Z) - \eta(W)\eta(Z)]\eta(X)Y].
\] (2.8.7)
Taking account of (2.8.2), we can write (2.8.7) as

\[
(\nabla_w\tilde{F})(X,Y)Z = (\nabla_w P)(X,Y)Z - 2d\eta(X,Y)\{g(\phi W, Z)\xi - \eta(Z)\phi W\}
\]
\[
+ \{g(X,W)g(\phi Y, Z) - g(Y,W)g(\phi X, Z)\} \xi - 2\{\eta(X)\eta(W)g(\phi Y, Z)
\]
\[- \eta(Y)\eta(W)g(\phi X, Z)\}\xi + \{g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y)\}\{W\}
\]
\[
+ \{g(W, Z) - \eta(W)\eta(Z)\} \times \{\eta(Y)\phi X - \eta(X)\phi Y\} + \{g(Y, W)\phi X
\]
\[- g(X, W)\phi Y + g(\phi W, X)\eta(Y)\xi - g(\phi W, Y)\eta(X)\xi
\]
\[- \eta(Y)\eta(W)\phi X - \eta(X)\eta(W)\phi Y - 2\eta(X)\eta(Y)\phi W\eta(Z)
\]
\[- \frac{1}{n-1}[\psi[g(W, Y) - \eta(W)\eta(Y)]\eta(Z)X + \psi[g(W, Z) - \eta(W)\eta(Z)]\eta(Y)X
\]
\[- \psi[g(W, X) - \eta(W)\eta(X)]\eta(Z)Y
\]
\[- \psi[g(W, Z) - \eta(W)\eta(Z)]\eta(Y)\eta(Z)Y.
\]
\[(2.8.8)\]

Now applying (2.1.1) and (2.8.8) in (2.8.3), we get

\[
\phi^2(\nabla_w\tilde{F})(X,Y)Z = \phi^2(\nabla_w P)(X,Y)Z + 2d\eta(X,Y)\eta(Z)\phi^2(\phi W) + \{g(\phi Y, Z)\eta(X)
\]
\[- g(\phi X, Z)\eta(Y)\}\phi^2(W) + \{g(W, Z) - \eta(W)\eta(Z)\} \times \{\eta(Y)\phi^2(\phi X)
\]
\[- \eta(X)\phi^2(\phi Y)\} + \{g(Y, W)\phi^2(\phi X) - g(X, W)\phi^2(\phi Y)
\]
\[- \eta(Y)\eta(W)\phi^2(\phi X) - \eta(X)\eta(W)\phi^2(\phi Y) - 2\eta(X)\eta(Y)\phi^2(\phi W)\}\eta(Z)
\]
\[- \frac{1}{n-1}[\psi[g(W, Y) - \eta(W)\eta(Y)]\eta(Z)\phi^2 X + \psi[g(W, Z)
\]
\[- \eta(W)\eta(Z)\eta(Y)\phi^2 X - \psi[g(W, X) - \eta(W)\eta(X)]\eta(Z)\phi^2 Y
\]
\[- \psi[g(W, Z) - \eta(W)\eta(Z)]\eta(X)\phi^2 Y - \eta(W)\phi^2(\phi P(X, Y)Z)
\]
\[+ \eta(W)\{\phi^2(\tilde{P}(\phi X, Y)Z) + \phi^2(\tilde{P}(X, \phi Y)Z)
\]
\[+ \phi^2(\tilde{P}(X, Y)\phi Z)\}.
\]
\[(2.8.9)\]
If we consider \( X, Y, Z, W \) orthogonal to \( \xi \), (2.8.9) reduces to

\[
\phi^2(\nabla_W \tilde{P})(X, Y)Z = \phi^2(\nabla_W P)(X, Y)Z.
\]  

(2.8.10)

Hence we can state the following theorem:

**Theorem 2.8.1.** A Kenmotsu manifold is locally projective \( \phi \)-symmetric with respect to \( \tilde{\nabla} \) if and only if it is so with respect to Levi-Civita connection \( \nabla \).

Using (2.1.1) and (2.8.7) in (2.8.3), we get

\[
\phi^2(\nabla_W \tilde{P})(X, Y)Z = \phi^2(\nabla_W R)(X, Y)Z + 2d\eta(X, Y)\eta(Z)\phi^2(\phi W) + \{g(\phi Y, Z)\eta(X) \\
- \eta(X)\phi^2(\phi Y)\} X + \{g(Y, W)\phi^2(\phi X) - g(X, W)\phi^2(\phi Y) \\
- \eta(Y)\phi^2(\phi X) + \eta(X)\phi^2(\phi Y) - 2\eta(X)\eta(Y)\phi^2(\phi W)\} \eta(Z) \\
+ \frac{1}{n-1}\{\phi^2((\nabla_W S)(Y, Z))X + \psi[g(W, Y) - \eta(W)\eta(Y)]\eta(Z)\phi^2 X \\
+ \psi[g(W, Z) - \eta(W)\eta(Z)]\eta(Y)\phi^2 X \\
- \psi[g(\phi Y, Z) - \eta(\phi W)\eta(\phi Y)]\eta(Z)\phi^2 Y \\
- \psi[g(\phi W, Z) - \eta(\phi W)\eta(\phi Y)]\eta(X)\phi^2 Y - \phi^2((\nabla_W S)(X, Z))Y\} \\
+ \eta(W)\phi^2(\phi \tilde{P}(X, Y)Z) + \eta(W)\{\phi^2(\tilde{P}(\phi X, Y)Z) \\
+ \phi^2(\tilde{P}(X, \phi Y)Z) + \phi^2(\tilde{P}(X, \phi Y))\phi Z\}.\]  

(2.8.11)

Taking \( X, Y, Z \) and \( W \) orthogonal to \( \xi \) in (2.8.11) followed by a simplification we get

\[
\phi^2(\nabla_W \tilde{P})(X, Y)Z = \phi^2(\nabla_W R)(X, Y)Z.
\]  

(2.8.12)
Thus we can state the following theorem:

**Theorem 2.8.2.** A $\phi$-symmetric Kenmotsu manifold admitting the quarter-symmetric metric connection $\tilde{\nabla}$ is locally projective $\phi$-symmetric with respect to quarter-symmetric metric connection $\tilde{\nabla}$ if and only if it is locally $\phi$-symmetric with respect to the Levi-Civita connection $\nabla$.

### 2.9 3-dimensional Locally Generalized $\phi$-recurrent Kenmotsu Manifold with respect to the Quarter-symmetric Metric Connection

A Kenmotsu manifold $M$ is said to be a locally generalized $\phi$-recurrent with respect to quarter-symmetric metric connection if

$$
\phi^2((\tilde{\nabla}_WR)(X, Y)Z) = A(W)\tilde{R}(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y],
$$

(2.9.1)

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.

Using (2.6.6) we can write

$$
(\tilde{\nabla}_WR)(X, Y)Z = (\nabla_W\tilde{R})(X, Y)Z - \eta(W)\phi\tilde{R}(X, Y)Z.
$$

(2.9.2)

We know that for a 3-dimensional Kenmotsu manifold

$$
R(X, Y)Z = \left(\frac{r+4}{2}\right)[g(Y, Z)X - g(X, Z)Y]
- \left(\frac{r+6}{2}\right)[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi]
+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y.
$$

(2.9.3)
The curvature tensor \((2.9.3)\) with respect to quarter-symmetric metric connection is given by

\[
\tilde{R}(X, Y)Z = R(X, Y)Z + \eta(X)g(\phi Y, Z)\xi - \eta(Y)g(\phi X, Z)\xi - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X. \tag{2.9.4}
\]

In view of \((2.9.3)\), \((2.9.4)\) yields

\[
\tilde{R}(X, Y)Z = \left(\frac{r + 4}{2}\right) [g(Y, Z)X - g(X, Z)Y] - \left(\frac{r + 6}{2}\right) [g(Y, Z)\eta(X)\xi \\
- g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] + \eta(X)g(\phi Y, Z)\xi \\
- \eta(Y)\eta(\phi X, Z)\xi - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X. \tag{2.9.5}
\]

Now differentiating \((2.9.5)\) covariantly with respect to \(W\) and using \((2.1.3)\) we get from \((2.9.2)\) that

\[
(\tilde{\nabla}_W \tilde{R})(X, Y)Z = \frac{d\rho(W)}{2} [g(Y, Z)X - g(X, Z)Y] \\
- \frac{d\rho(W)}{2} [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\
- \left(\frac{r + 6}{2}\right) [g(Y, Z)(\tilde{\nabla}_W \eta)(X)\xi - g(X, Z)(\tilde{\nabla}_W \eta)(Y)\xi \\
+ g(Y, Z)\eta(X)\tilde{\nabla}_W \xi - g(X, Z)\eta(Y)\tilde{\nabla}_W \xi + (\tilde{\nabla}_W \eta)(Y)\eta(Z)X \\
+ \eta(Y)(\tilde{\nabla}_W \eta)(Z)X - (\tilde{\nabla}_W \eta)(X)\eta(Z)Y - \eta(X)(\tilde{\nabla}_W \eta)(Z)Y] \\
+ (\tilde{\nabla}_W \eta)(X)g(\phi Y, Z)\xi + \eta(X)g(\phi Y, Z)W - \eta(X)g(\phi Y, Z)\eta(W)\xi \\
- (\tilde{\nabla}_W \eta)(Y)g(\phi X, Z)\xi - \eta(Y)g(\phi X, Z)W + \eta(Y)g(\phi X, Z)\eta(W)\xi \\
- g(W, X)\eta(Z)\phi Y + 2\eta(W)\eta(X)\eta(Z)\phi Y - \eta(X)g(W, Z)\phi Y \\
- \eta(X)\eta(Z)g(\phi W, Y)\xi + g(W, Y)\eta(Z)\phi X - 2\eta(W)\eta(Y)\eta(Z)\phi X \\
+ \eta(Y)g(W, Z)\phi X + \eta(Y)\eta(Z)g(\phi W, X)\xi - \eta(W)\phi \tilde{R}(X, Y)Z. \tag{2.9.6}
\]
Now taking $X, Y, Z, W$ orthogonal to $\xi$ and using (2.9.1) in the above equation, we get

$$A(W)\tilde{R}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y] = -\left[\frac{dr(W)}{2}\right][g(Y,Z)X - g(X,Z)Y]. \quad (2.9.7)$$

Again using (2.9.4) in the above equation we have

$$A(W)[R(X,Y)Z + \eta(X)g(\phi Y, Z)\xi - \eta(Y)g(\phi X, Z)\xi - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X] = -\left[\frac{dr(W)}{2}\right][g(Y,Z)X - g(X,Z)Y] - B(W)[g(Y,Z)X - g(X,Z)Y]. \quad (2.9.8)$$

Taking $X, Y, Z$ and $W$ orthogonal to $\xi$ in (2.9.8), we get

$$R(X,Y)Z = \frac{dr(W)}{2A(W)} - \frac{B(W)}{A(W)}[g(X,Z)Y - g(Y,Z)X]. \quad (2.9.9)$$

Putting $W = \{e_i\}$, where $\{e_i\}, i = 1, 2, 3$, is an orthonormal basis of the tangent space at any point of the manifold and taking summation over $i, 1 \leq i \leq 3$, we obtain

$$R(X,Y)Z = \lambda[g(X,Z)Y - g(Y,Z)X], \quad (2.9.10)$$

where $\lambda = [\frac{dr(e_i)}{2A(e_i)} - \frac{B(e_i)}{A(e_i)}]$ is a scalar. Then by Schur's theorem $\lambda$ will be a constant on the manifold. Therefore, $(M^3, g)$ is of constant curvature $\lambda$.

Thus we state the following theorem:

**Theorem 2.9.1.** A 3-dimensional Kenmotsu manifold is locally generalized $\phi$-recurrent with respect to quarter-symmetric connection if and only if the scalar curvature $\lambda$ is constant.
2.10 Conclusion

- A generalized pseudo-projective $\phi$-recurrent Kenmotsu manifold $(M^{2n+1}, g)$ is an Einstein manifold.

- In a generalized pseudo-projective $\phi$-recurrent Kenmotsu manifold $(M^{2n+1}, g)$, the 1-forms $A$ and $B$ are related as in (2.2.11).

- In a generalized pseudo-projective $\phi$-recurrent Kenmotsu manifold, the characteristic field $\xi$ and the vector fields $\rho_1$ and $\rho_2$ associated to the 1-forms $A$ and $B$ respectively are co-directional and the 1-forms $A$ and $B$ are given by (2.2.17).

- A generalized pseudo-projective $\phi$-recurrent Kenmotsu manifold $(M^{2n+1}, g)$, $n \geq 1$, is a space of constant curvature, provided that $X$ and $Y$ are orthogonal to $\xi$.

- A 3-dimensional generalized pseudo-projective $\phi$-recurrent Kenmotsu manifold $M^3$ is of constant curvature.

- A generalized conharmonic $\phi$-recurrent Kenmotsu manifold is an Einstein manifold.

- In a generalized conharmonic $\phi$-recurrent Kenmotsu manifold $(M^{2n+1}, g)$, $n \geq 1$ the characteristic vector field $\xi$ and the vector fields $\rho_1$ and $\rho_2$ associated to the 1-forms $A$ and $B$ respectively are co-directional and the 1-forms $A$ and $B$ are given by (2.4.19).

- A generalized conharmonic $\phi$-recurrent Kenmotsu manifold is a manifold of constant curvature.
• A 3-dimensional generalized conharmonic $\phi$-recurrent Kenmotsu manifold is of constant curvature.

• Let $M$ be a generalized $\phi$-recurrent Kenmotsu manifolds with respect to quarter-symmetric metric connection, then the 1-forms $A$ and $B$ are related as in (2.7.12).

• A Kenmotsu manifold is locally projective $\phi$-symmetric with respect to $\tilde{\nabla}$ if and only if it is so with respect to Levi-Civita connection $\nabla$.

• If $M$ is $\phi$-symmetric with respect to quarter-symmetric metric connection then a Kenmotsu manifold is locally projective symmetric with respect to quarter-symmetric metric connection $\tilde{\nabla}$ if and only if it is $\phi$-symmetric with respect to Levi-Civita connection $\nabla$.

• A 3-dimensional Kenmotsu manifold is locally generalized $\phi$-recurrent with respect to quarter-symmetric connection if and only if scalar curvature $\lambda$ is constant.