CHAPTER 4

Publications based on this Chapter

• On Generalized pseudo-projective $\phi$-recurrent Lorentzian $\alpha$-Sasakian manifold, (Communicated).


• On $D$-conformal curvature tensor of Lorentzian $\alpha$-Sasakian manifold, (Communicated).
Chapter 4

On Lorentzian $\alpha$-Sasakian Manifolds

If a differentiable manifold has a Lorentzian metric $g$, i.e., a symmetric non-degenerated $(0,2)$ tensor field of index 1, then it is called a Lorentzian manifold. Generally, a differentiable manifold has a Lorentzian metric if and only if it has a 1-dimensional distribution. Hence odd dimensional manifold is able to have a Lorentzian metric. A differentiable manifold of dimension $(2n + 1)$ is called Lorentzian $\alpha$-Sasakian manifold if it admits a $(1,1)$ tensor field $\phi$, a vector field $\xi$ and 1-form $\eta$ and a Lorentzian metric $g$ which satisfies,

$$\nabla_X \xi = \alpha \phi X \quad (4.0.1)$$

$$\nabla_X \eta(Y) = \alpha g(X, \phi Y). \quad (4.0.2)$$

The notion of Lorentzian $\alpha$-Sasakian manifolds has been introduced by Yildiz, and Murathan [130]. The properties of Lorentzian $\alpha$-Sasakian manifolds has been studied by many geometers like Yildiz, Mine Turan and Murathan[131], Yildiz and Mine Turan and Bilal etal Acet[132], D.G.Prakasha, Bagewadi, N.S.Basavarajappa[93], D.G.Prakasha and Yildiz[95], and many others. On the other hand, G.P.Pokhariyal[29] have introduced new curvature tensor called $W_2$-curvature tensor in a Riemannian manifold and studied their properties. K.Matsumoto, S.Ianus and I.Mihai and Ahmet Yildiz and U.C.De[128] have studied $W_2$-curvature tensor in a para-Sasakian and Kenmotsu manifold respectively.
4.1 Preliminaries

Let $M^{2n+1} (\phi, \xi, \eta, g)$ be a Lorentzian $\alpha$-Sasakian manifold with the structure $(\phi, \xi, \eta, g)$.

Then the following relations hold [130]:

\[
\phi^2(X) = X + \eta(X)\xi, \quad \phi\xi = 0, \tag{4.1.1}
\]
\[
\eta(\xi) = -1, \quad g(X, \xi) = \eta(X), \quad \eta(\phi X) = 0, \tag{4.1.2}
\]
\[
g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{4.1.3}
\]
\[
(\nabla_X \phi)Y = \alpha [g(X, Y)\xi + \eta(Y)X], \tag{4.1.4}
\]
\[
(a) \nabla_X \xi = \alpha \phi X, \quad (b) (\nabla_X \eta)(Y) = \alpha g(X, \phi Y), \tag{4.1.5}
\]
\[
\eta(H(X, Y)Z) = \alpha^2 [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{4.1.6}
\]
\[
R(X, Y)\xi = \alpha^2 [\eta(Y)X - \eta(X)Y], \tag{4.1.7}
\]
\[
S(X, \xi) = 2\alpha^2 \eta(X), \tag{4.1.8}
\]
\[
Q\xi = 2\alpha^2 \xi, \tag{4.1.9}
\]
\[
S(\phi X, \phi Y) = S(X, Y) + 2\alpha^2 \eta(X)\eta(Y), \tag{4.1.10}
\]

for all vector fields $X, Y, Z$, where $\nabla$ denotes the operator of covariant differentiation with respect to $g$, $\phi$ is a $(1, 1)$ tensor field, $S$ is the Ricci tensor of type $(0, 2)$ and $R$ is the Riemannian curvature tensor of the manifold.
Definition 28. [121] The D-conformal curvature tensor $B$ on a Riemannian manifold $(M^n, g) \ (n > 4)$ is defined as

\[
B(X,Y)Z = R(X,Y)Z + \frac{1}{n-3} \{ S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX \\
- S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX \\\n- \frac{k-2}{n-3}\{g(X,Z)Y - g(Y,Z)X\} + \frac{k}{n-3}\{g(X,Z)\eta(Y)\xi \\\n- g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}. \tag{4.1.11}
\]

where $r$ is the scalar curvature, $Q$ is the Ricci operator and $k = \frac{(r+2(n-1))}{(n-2)}$.

Definition 29. [128] The curvature tensor $W_2$ is defined by

\[
W_2(X,Y,U,V) = R(X,Y,U,V) + \frac{1}{n-1}[g(X,U)S(Y,V) \\
- g(Y,U)S(X,V)]. \tag{4.1.12}
\]

where $S$ is a Ricci tensor of type $(0,2)$.

4.2 Lorentzian $\alpha$-Sasakian manifold satisfying

\[
B(X,Y)Z = 0
\]

Assume that in a Lorentzian $\alpha$-Sasakian manifold

\[
B(X,Y)Z = 0. \tag{4.2.1}
\]
Then it follows from (4.1.11) and (4.2.1) that

\[
R(X,Y)Z = -\frac{1}{n-3} \{ S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX \\
- S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QX + \eta(Y)\eta(Z)QX \}
+ \frac{k-2}{n-3} \{ g(X,Z)Y - g(Y,Z)X \} - \frac{k}{n-3} \{ g(X,Z)\eta(Y)\xi \}
- g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \},
\]

(4.2.2)

or,

\[
g(R(X,Y)Z,U) = -\frac{1}{n-3} \{ S(X,Z)g(Y,U) - S(Y,Z)g(X,U) \\
+ g(X,Z)S(Y,U) - g(Y,Z)S(X,U) - S(X,Z)\eta(Y)\eta(U) \\
+ S(Y,Z)\eta(X)\eta(U) - \eta(X)\eta(Z)S(Y,U) + \eta(Y)\eta(Z)S(X,U) \}
+ \frac{k-2}{n-3} \{ g(X,Z)g(Y,U) - g(Y,Z)g(X,U) \}
- \frac{k}{n-3} \{ g(X,Z)\eta(Y)\eta(U) - g(Y,Z)\eta(X)\eta(U) \}
+ \eta(X)\eta(Z)g(Y,U) - \eta(Y)\eta(Z)g(X,U) \}.
\]

(4.2.3)

Taking \( X = U = \xi \) in (4.2.3) and then using (4.1.2), (4.1.7) and (4.1.8) it is seen that,

\[
S(Y,Z) = (k - \alpha^2 - 1)g(Y,Z) - (\alpha^2 n - k + 1)\eta(Y)\eta(Z).
\]

(4.2.4)

This leads to the following theorem:

**Theorem 4.2.1.** An \( n \)-dimensional \( D \)-conformally flat Lorentzian \( \alpha \)-Sasakian manifold

is an \( \eta \)-Einstein manifold.
4.3 Lorentzian $\alpha$-Sasakian manifold satisfying

$$B(X, Y).S = 0$$

A Riemannian manifold $(M^n, g)$ is termed a Ricci $D$-conformal semi-Symmetric, if

$$B(X, Y).S = 0. \quad (4.3.1)$$

By virtue of $(4.1.11)$ and using $(4.1.2), (4.1.7), (4.1.8)$ we obtain

$$B(\xi, X)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X] + \frac{1}{n-3}\{S(\xi, Y)X - S(X, Y)\xi + \eta(Y)QX - S(\xi, Y)\xi - S(X, Y) + \eta(Y)QX$$

$$+ \eta(\xi)\eta(Y)Q\xi - \frac{k-2}{n-3}\{\eta(Y)X - \eta(Y)\xi - \eta(Y)X - \eta(X)\eta(Y)\xi\}. \quad (4.3.2)$$

As $M^n$ is a Lorentzian $\alpha$-Sasakian manifold satisfying the condition

$$B(U, X).S(Y, Z) = 0. \quad (4.3.3)$$

By virtue of (4.3.2) we get

$$S[B(U, X)Y, Z] + S(Y, B(U, X)Z) = 0. \quad (4.3.4)$$

Now putting $U = \xi$ in (4.3.3) and using (4.3.1), (4.1.2), (4.1.8) and (4.1.9) and after a brief simplification, we get

$$S(X, Y) = \left[\frac{-\alpha^4(n-1) + \alpha^2(k(n-1) + 1 - n)}{\alpha^2(n-2) + k - 1}\right]g(X, Y). \quad (4.3.5)$$

**Theorem 4.3.1.** A Lorentzian $\alpha$-Sasakian manifold $(M^n, g), (n > 4)$ satisfying the condition $B(X, Y).S = 0$ is an Einstein manifold.
4.4 Lorentzian $\alpha$-Sasakian Manifold Satisfying $B(X, Y).R = 0$

Assume that in a Lorentzian $\alpha$-Sasakian manifold satisfies the condition $B(X, Y).R = 0$.

This equation implies

$$B(X, Y)R(Z, U)V - R(B(X, Y)Z, U)V - R(Z, B(X, Y)U)V - R(Z, U)B(X, Y)V = 0. \quad (4.4.1)$$

Putting $X = \xi$ in (4.4.1), we have

$$B(\xi, Y)R(Z, U)V - R(B(\xi, Y)Z, U)V - R(Z, B(\xi, Y)U)V - R(Z, U)B(\xi, Y)V = 0. \quad (4.4.2)$$

Using (4.1.11) in (4.4.2), we get

$$L[g(Y, R(Z, U)V)V - g(R(Z, U)V, Y - g(Y, Z)R(\xi, U)V + \eta(Z)R(\xi, U)V
- g(\xi, U)R(Z, V)V + \eta(U)R(Z, \xi)V + \eta(V)R(\xi, U)V
+ S(Y, Z)R(\xi, U)V + \eta(Z)R(QY, U)V
+ S(Y, U)R(Z, \xi)V + \eta(U)R(Z, QY)V + S(Y, V)R(Z, U)\xi
+ \eta(V)R(Z, U)QY] = 0. \quad (4.4.3)$$

where $L = \frac{2(\alpha^2-k+1)}{n-3}$ and $M = \frac{2}{n-3}$. 
Taking the inner product with $\xi$ and using (4.1.7) in (4.4.3) and after a brief simplification using $Z = V = \xi$ we get

$$S(QU, Y) = \left[ -\frac{L}{M} + k - 2 \right] S(U, Y) + \left[ \frac{L}{M} (k - 2) \right] g(U, Y)$$

$$+ \left[ \left( \frac{L}{M} + \alpha^2 (n - 1) \right) (k - 2 - \alpha^2 (n - 1)) \right] \eta(U) \eta(Y). \quad (4.4.4)$$

Hence we have the following theorem:

**Theorem 4.4.1.** In an $n$-dimensional Lorentzian $\alpha$-Sasakian manifold $(M^n, g)$, ($n > 4$) satisfying the condition $B(X, Y).R = 0$, then the equation (4.4.4) is satisfied.

### 4.5 Lorentzian $\alpha$-Sasakian Manifold Satisfying

$$B(X, Y).W_2 = 0$$

Assume that Lorentzian $\alpha$-Sasakian manifold satisfies the condition

$$B(X, Y).W_2 = 0.$$ 

This equation implies

$$B(X, Y)W_2(Z, U)V - W_2(B(X, Y)Z, U)V$$

$$- W_2(Z, B(X, Y)U)V - W_2(Z, U)B(X, Y)V = 0. \quad (4.5.1)$$

Putting $X = \xi$ in (4.5.1), we have

$$B(\xi, Y)W_2(Z, U)V - W_2(B(\xi, Y)Z, U)V$$

$$- W_2(Z, B(\xi, Y)U)V - W_2(Z, U)B(\xi, Y)V = 0. \quad (4.5.2)$$
Using (4.1.11), (4.1.12) in (4.5.2), we get

\[
L[g(Y, W_2(Z, U)V), \xi] - g(W_2(Z, U)V, \xi)Y - g(Y, Z)W_2(\xi, U)V
+ \eta(Z)W_2(Y, U)V - g(Y, U)W_2(Z, \xi)V + \eta(U)W_2(Z, Y)V
- g(Y, V)W_2(Z, U)\xi + \eta(V)W_2(Z, U)Y + M[-S(Y, W_2(Z, U)V)\xi
- S(Y, Z)W_2(\xi, U)V + \eta(Z)W_2(QY, U)V
+ S(Y, U)W_2(Z, \xi)V + \eta(U)W_2(Z, QY)V + S(Y, V)W_2(Z, U)\xi
+ \eta(V)W_2(Z, U)QY] = 0,
\]

(4.5.3)

where \( L = \left[ \alpha^2 + \frac{2(k-1)-a^2(n-1)}{n-3} \right] \) and \( M = \frac{2}{n-3} \).

On the other hand assume

\[
W_2(X, Y, U, \xi) = 0.
\]

(4.5.4)

Taking the inner product with \( \xi \) and using (4.5.4) in (4.5.3), we get

\[
L[-g(Y, W_2(Z, U)V)] + M[S(Y, W_2(Z, U)V) = 0.
\]

(4.5.5)

Putting \( Z = V = \xi \) in (4.5.5) and using (4.1.7) and (4.1.8), we have

\[
S(QY, U) = \left[ \frac{L}{M} + \alpha^2(n-1) \right] S(U, Y) - \left[ \frac{La^2(n-1)}{M} \right] g(U, Y).
\]

(4.5.6)

Hence we have the following theorem:

**Theorem 4.5.1.** In an \( n \)-dimensional Lorentzian \( \alpha \)-Sasakian manifold \((M^n, g), (n > 4)\) satisfying the condition \( B(X, Y).W_2 = 0 \) holds on \( M \), then the equation (4.5.6) is satisfied on \( M \).
4.6 Extended Generalized Concircularly $\phi$-recurrent Lorentzian $\alpha$-Sasakian Manifold

**Definition 30.** A Lorentzian $\alpha$-Sasakian manifold is said to be locally $\phi$-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0,$$

(4.6.1)

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.

**Definition 31.** A Lorentzian $\alpha$-Sasakian manifold is said to be generalized $\phi$-recurrent if its curvature tensor $R$ satisfies the condition

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y],$$

(4.6.2)

where, $A$ and $B$ are 1-forms, $B$ is non-zero and these are defined by

$$A(W) = g(W, \rho_1) \text{ and } B(W) = g(W, \rho_2),$$

where $\rho_1$ and $\rho_2$ are vector fields associated with 1-forms $A$ and $B$ respectively.

**Definition 32.** A Lorentzian $\alpha$-Sasakian manifold is said to be concircularly $\phi$-recurrent if there exists a nowhere vanishing unique 1-form $A$ such that

$$\phi^2((\nabla_W C)(X,Y)Z) = A(W)\overline{C}(X,Y)Z,$$

(4.6.3)

for all vector fields $X, Y, Z, W \in \chi(M)$.

A Lorentzian $\alpha$-Sasakian manifold $M^\alpha(\phi, \xi, \eta, g)$, is said to be an extended generalized concircularly $\phi$-recurrent Lorentzian $\alpha$-Sasakian manifold if its concircular curvature tensor $\overline{C}$ satisfies the relation

$$\phi^2((\nabla_W \overline{C})(X,Y)Z) = A(W)\phi^2(\overline{C}(X,Y)Z) + B(W)\phi^2(G(X,Y)Z),$$

(4.6.4)
where $A$ and $B$ are nonvanishing 1-forms, $\nabla$ denotes the operator of covariant differentiation with respect to the metric $g$ i.e., $\nabla$ is the Riemannian connection and $C$ is the concircular curvature tensor given by

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n - 1)} G(X, Y)Z,$$ (4.6.5)

where $r$ is the scalar curvature of the manifold.

Let us consider an extended generalized concircularly $\phi$-recurrent Lorentzian $\alpha$-Sasakian manifold $M^n(\phi, \xi, \eta, g)$. Then by virtue of (4.1.1), if follows from (4.6.4) that

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)\phi^2(R(X, Y)Z) - B(W)\phi^2(G(X, Y)Z)$$

$$= \frac{dr(W) - rA(W)}{n(n - 1)} [g(Y, Z)X - \eta(X)g(Y, Z)\xi - g(X, Z)Y + \eta(Y)g(X, Z)\xi].$$ (4.6.6)

This leads to the following theorem:

**Theorem 4.6.1.** An extended generalized concircularly $\phi$-recurrent Lorentzian $\alpha$-Sasakian manifold $(M^n, g)$, $(n \geq 3)$ is generalized $\phi$-recurrent if and only if

$$\frac{dr(W) - rA(W)}{n(n - 1)} [g(Y, Z)X + \eta(X)g(Y, Z)\xi - g(X, Z)Y - \eta(Y)g(X, Z)\xi] = 0.$$ (4.6.7)

Now taking inner product of (4.6.7) with $U$ we have

$$\frac{dr(W) - rA(W)}{n(n - 1)} [g(Y, Z)g(X, U) + \eta(X)g(Y, Z)\eta(U) - g(X, Z)g(Y, U)$$

$$- \eta(Y)g(X, Z)\eta(U)] = 0.$$ (4.6.8)

Contracting over $X$ and $U$ we get

$$\{dr(W) - rA(W)\}[ng(Y, Z) - \eta(Y)\eta(Z)] = 0.$$ (4.6.9)
Again contracting (4.6.8) over $Y$ and $Z$, we get

$$\{dr(W) - rA(W)\}[n^2 - 1] = 0. \quad (4.6.10)$$

which implies that

$$A(W) = \frac{1}{r}dr(W) \text{ for all } W \text{ and } r \neq 0$$

i.e., $\rho = \frac{1}{r}\text{grad } r$, where $A(W) = g(W, \rho)$. \quad (4.6.11)

This leads to the following theorem:

**Theorem 4.6.2.** If an extended generalized concircularly $\phi$-recurrent Lorentzian $\alpha$-Sasakian manifold $(M^n, g)$, $(n \geq 3)$ is an extended generalized $\phi$-recurrent Lorentzian $\alpha$-Sasakian manifold, then the associated vector field corresponding to the 1-form $A$ is given by $\rho = \frac{1}{r}\text{grad } r$, $r$ being the non-zero and non-constant scalar curvature of the manifold.

Now by virtue of (4.1.1), it follows from (4.6.6) that

$$\{V_i R\}(X, Y)Z = -\frac{1}{r}[(V_i 'R)(X, Y)Z - A(W)R(X, Y)Z + (R(X, Y)Z)\xi]$$

$$+ B(W)[g(X, Y)Z + \eta(X)g(Y, Z)\xi]$$

$$+ \frac{dr(W) - rA(W)}{n(n - 1)}[g(Y, Z)X + \eta(X)g(Y, Z)\xi]$$

$$- g(X, Z)Y - \eta(Y)g(X, Z)\xi]. \quad (4.6.12)$$

Taking inner product of (4.6.12) with $U$ and then contracting over $X$ and $U$, and then using (4.1.4) and (4.1.6) we get

$$\{\nabla_w S\}(Y, Z) = A(W)S(Y, Z) + \eta(B(W) + \alpha^2A(W))g(Y, Z)$$

$$+ \frac{dr(W)}{n(n - 1)}[ng(Y, Z) - \xi(Y)\eta(Z)]$$

$$+ A(W) \left[ (-\alpha^2 + \frac{r}{n(n - 1)})\eta(Y)\eta(Z) - \frac{rn}{n(n - 1)}g(Y, Z) \right]$$

$$- B(W)\eta(Y)\eta(Z). \quad (4.6.13)$$
Again taking contraction over $Y$ and $Z$ in (4.6.13), we get

$$dr(W) = (r - \alpha^2 n(n - 1))A(W) - n(n^2 - 1)B(W). \quad (4.6.14)$$

From (4.6.14), we can state the following theorem:

**Theorem 4.6.3.** In an extended generalized concircularly $\phi$-recurrent Lorentzian $\alpha$-Sasakian manifold $(M^n, g)$, $(n \geq 3)$ the associated 1-forms $A$ and $B$ are related by the relation (4.6.14).

**Corollary 4.6.4.** In an extended generalized concircularly $\phi$-recurrent Lorentzian $\alpha$-Sasakian manifold $(M^n, g)$, $(n \geq 3)$ with constant scalar curvature, the associated 1-forms $A$ and $B$ are related by

$$\{r - \alpha^2 n(n - 1)\}A - n(n^2 - 1)B = 0. \quad (4.6.15)$$

Now using (4.6.14) in (4.6.13) we get

$$\nabla WS(Y, Z) = A(W)S(Y, Z) + \left\{ (-n^2)B(W) + \alpha^2(1 - n)A(W) \right\}g(Y, Z)$$

$$+ \ nB(W)\eta(Y)\eta(Z). \quad (4.6.16)$$

From (4.6.16), it follows that the Ricci tensor $S$ satisfies the condition

$$\nabla S = \alpha \otimes S + \beta \otimes g + \gamma \otimes \pi, \quad (4.6.17)$$

where $\alpha(W) = A(W), \beta(W) = -n^2 B(W) + \alpha^2(1 - n)A(W), \gamma(W) = nB(W)$

and $\pi = \eta \otimes \eta$. 
From (4.6.17), we can state the following theorem:

**Theorem 4.6.5.** An extended generalized concircular $\phi$-recurrent Lorentzian $\alpha$-Sasakian manifold $(M^n, g)$, $(n \geq 3)$ is super generalized Ricci-recurrent manifold.

Now contracting (4.6.16) over $W$ and $Z$, we get

$$
\frac{1}{2} dr(Y) = S(Y, p\_1) - n^2 B(Y) + \alpha^2 (1 - n) A(Y) + n \eta(Y) B(\xi). \tag{4.6.18}
$$

By virtue of (4.6.14), the above relation takes the form

$$
S(Y, p\_1) = \left[ \frac{r - \alpha^2 n^2 + 3 \alpha^2 n - 2 \alpha^2}{2} \right] A(Y) - \left[ \frac{n^3 - n + 2n^2}{2} \right] B(Y) - n \eta(Y) B(\xi). \tag{4.6.19}
$$

From (4.6.19), we can state the following:

**Theorem 4.6.6.** In an extended generalized concircularly $\phi$-recurrent Lorentzian $\alpha$-Sasakian manifold $(M^n, g)$, $(n \geq 3)$ the Ricci tensor in the direction of $p\_1$ is given by (4.6.19).

Now setting $Z = \xi$ in (4.6.16) and then using (4.1.8) we get

$$
\alpha S(\phi W, Y) = \alpha^3 (n - 1) g(W, \phi Y) + n(n + 1) B(W) \eta(Y). \tag{4.6.20}
$$

Replacing $Y$ by $\phi Y$ in (4.6.20) and using (4.1.9) and (4.1.10) we have

$$
\alpha S(W, Y) = \alpha^3 (n - 1) g(W, Y). \tag{4.6.21}
$$

Replacing $W$ by $\phi W$ in (4.6.20) and then using (4.1.2) we get

$$
\alpha S(W, Y) = \alpha^3 (n - 1) g(W, Y) + n(n + 1) B(\phi W) \eta(Y). \tag{4.6.22}
$$
From (4.6.21) and (4.6.22) we have

\[ B(\phi W) = 0, \quad (4.6.23) \]

which implies that

\[ B(W) = \eta(W)B(\xi). \quad (4.6.24) \]

This leads to the following theorem:

**Theorem 4.6.7.** In an extended generalized concircularly \( \phi \)-recurrent Lorentzian \( \alpha \)-Sasakian manifold \( (M^n, g), (n \geq 3) \), the vector field \( p_2 \) associated with the 1-form \( B \) and the characteristic vector field \( \xi \) are co-directional.

### 4.7 Generalized Pseudo-projective \( \phi \)-recurrent Lorentzian \( \alpha \)-Sasakian Manifold

**Definition 33.** [130] A Lorentzian \( \alpha \)-Sasakian manifold is said to be locally \( \phi \)-symmetric if

\[ \phi^2((\nabla_W R)(X, Y)Z) = 0, \quad (4.7.1) \]

for all vector fields \( X, Y, Z, W \) orthogonal to \( \xi \).

**Definition 34.** A Lorentzian \( \alpha \)-Sasakian manifold is said to be locally pseudo-projective \( \phi \)-symmetric if

\[ \phi^2((\nabla_W \overline{P})(X, Y)Z) = 0, \quad (4.7.2) \]

for all vector fields \( X, Y, Z, W \) orthogonal to \( \xi \).
Definition 35. A Lorentzian $\alpha$-Sasakian manifold is said to be pseudo-projective $\phi$-recurrent manifold if there exists a non-zero 1-form $A$ such that

$$\phi^2((\nabla_w \bar{P})(X,Y)Z) = A(W)\bar{P}(X,Y)Z,$$

for arbitrary vector fields $X, Y, Z, W$, where $\bar{P}$ is a pseudo-projective curvature tensor.

Definition 36. A Lorentzian $\alpha$-Sasakian manifold is said to be generalized pseudo-projective $\phi$-recurrent if its curvature tensor $\bar{P}$ satisfies the condition

$$\phi^2((\nabla_w \bar{P})(X,Y)Z) = A(W)\bar{P}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y],$$

where $A$ and $B$ are two 1-forms, $B$ is non-zero and these are defined by

$$A(W) = g(W, \rho_1) \text{ and } B(W) = g(W, \rho_2)$$

where $\rho_1, \rho_2$ are vector fields associated with 1-forms $A$ and $B$, respectively.

Let us consider a generalized pseudo-projective $\phi$-recurrent Lorentzian $\alpha$-Sasakian manifold. Then by virtue of (4.1.1) and (4.7.4) we have

$$((\nabla_w \bar{P})(X,Y)Z) + \eta((\nabla_w \bar{P})(X,Y)Z)\xi = A(W)\bar{P}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y],$$

from which it follows that

$$g((\nabla_w \bar{P})(X,Y)Z, U) + \eta((\nabla_w \bar{P})(X,Y)Z)\eta(U) = A(W)g(\bar{P}(X,Y)Z, U) + B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].$$

Let $\{e_i\}, i = 1, 2, \ldots, 2n + 1$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (4.7.6) and taking summation over $i$,
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1 ≤ i ≤ 2n + 1, we get

\[
[a + (2n + 1)b](\nabla_W S)(Y, Z) - b(\nabla_W S)(\xi, Z)\eta(Y) = \\
(a + 2nb)A(W) \left\{ S(Y, Z) - \frac{r}{(2n + 1)}g(Y, Z) \right\} + 2nB(W)g(Y, Z). \tag{4.7.7}
\]

Replacing \(Z\) by \(\xi\) in (4.7.7) and using (4.1.2) and (4.1.8), we get

\[
[a + (2n + 1)b][\nabla_W S)(Y, \xi)] \\
= (a + 2nb)A(W)\eta(Y) \left\{ 2n\alpha^2 - \frac{r}{2n + 1} \right\} + 2nB(W)\eta(Y). \tag{4.7.8}
\]

Now we have

\[
(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi). \tag{4.7.9}
\]

Using (4.1.5) and (4.1.8) in the above relation, it follows that

\[
(\nabla_W S)(Y, \xi) = -2n\alpha^2 g(\phi W, Y) + \alpha S(Y, \phi W). \tag{4.7.10}
\]

In view of (4.7.8) and (4.7.10), we have

\[
[a + (2n + 1)b][-2n\alpha^2 g(\phi W, Y) + \alpha S(Y, \phi W)] \\
= (a + 2nb)A(W)\eta(Y) \left\{ 2n\alpha^2 - \frac{r}{(2n + 1)} \right\} + 2nB(W)\eta(Y). \tag{4.7.11}
\]

Replacing \(Y\) by \(\phi Y\) in (4.7.11) and using (4.1.3) and (4.1.10), we obtain

\[
S(Y, W) = 2n\alpha^2 g(Y, W). \tag{4.7.12}
\]

This leads to the following theorem:

**Theorem 4.7.1.** A generalized pseudo-projective \(\phi\)-recurrent Lorentzian \(\alpha\)-Sasakian manifold \((M^{2n+1}, g)\) is an Einstein manifold.
Now putting $Y = Z = e_i$ in (4.7.6) and taking summation over $i$, $i = 1, 2, ..., 2n + 1$, we get

$$
(a - b)(\nabla_w S)(X, U) + b\nabla_w r g(X, U) - \frac{\nabla_w r}{2n + 1} (a + 2nb)g(X, U)
$$

$$
+ (a - b)(\nabla_w S)(X, \xi)\eta(U) + b\nabla_w r\eta(X)\eta(U) - \frac{\nabla_w r}{2n + 1} (a + 2nb)\eta(X)\eta(U)
$$

$$
= A(W) [aS(X, U) + br g(X, U) - bS(X, U)] + 2nB(W)g(X, U) = 0. \quad (4.7.13)
$$

Replacing $U = \xi$ in (4.7.13), we have

$$
A(W) \left[ 2na \alpha^2 \eta(X) + br \eta(X) - 2nb \alpha^2 \eta(X) - \frac{r}{2n + 1} (a + 2nb)\eta(X) \right]
$$

$$
+ 2nB(W)\eta(X) = 0. \quad (4.7.14)
$$

Replacing $X = \xi$ in (4.7.14) we have,

$$
B(W) = \left[ -a \alpha^2 - \frac{br}{2n} + b \alpha^2 + \frac{r}{2n(2n + 1)} (a + 2nb) \right] A(W). \quad (4.7.15)
$$

This leads to the following result.

**Theorem 4.7.2.** In a generalized pseudo-projective $\phi$-recurrent Lorentzian $\alpha$-Sasakian manifold $(M^{2n+1}, g)$, the 1-forms $A$ and $B$ are related as in (4.7.15).

Now from (4.7.5) we have

$$
(\nabla_w \overline{P})(X, Y)Z = -\eta((\nabla_w \overline{P})(X, Y)Z)\xi - A(W)\overline{P}(X, Y)Z
$$

$$
- B(W)[g(Y, Z)X - g(X, Z)Y]. \quad (4.7.16)
$$
This implies,

\[
an(\nabla_w R)(X, Y)Z = -a\eta((\nabla_w R)(X, Y)Z)\xi - aA(W)R(X, Y)Z
\]

\[
- b[\nabla_w S)(Y, Z)\eta(X) - (\nabla_w S)(X, Z)\eta(Y)]\xi
\]

\[
- b[\nabla_w S)(Y, Z)X - \nabla_w S)(X, Z)Y]
\]

\[
- bA(W)[S(Y, Z)X - S(X, Z)Y]
\]

\[
+ \frac{r}{(2n + 1)} \left[ \frac{a}{2n} + b \right] A(W)[g(Y, Z)X - g(X, Z)Y]
\]

\[
- B(W)[g(Y, Z)X - g(X, Z)Y]. \tag{4.7.17}
\]

From (4.7.17) and the Bianchi identity we get

\[
aA(W)\eta(R(X, Y)Z) + aA(X)\eta(R(Y, W)Z) + aA(Y)\eta(R(W, X)Z)
\]

\[
= bA(W)[S(X, Z)\eta(Y) - S(Y, Z)\eta(X)]
\]

\[
- \frac{r}{(2n + 1)} \left[ \frac{a}{2n} + b \right] A(W)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]
\]

\[
+ bA(X)[S(Y, Z)\eta(W) - S(W, Z)\eta(Y)]
\]

\[
- \frac{r}{(2n + 1)} \left[ \frac{a}{2n} + b \right] A(X)[g(Y, Z)\eta(W) - g(W, Z)\eta(Y)]
\]

\[
+ bA(Y)[S(W, Z)\eta(X) - S(X, Z)\eta(W)]
\]

\[
- \frac{r}{(2n + 1)} \left[ \frac{a}{2n} + b \right] A(Y)[g(W, Z)\eta(X) - g(X, Z)\eta(W)]
\]

\[
+ B(W)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]
\]

\[
+ B(X)[g(W, Z)\eta(Y) - g(Y, Z)\eta(W)]
\]

\[
+ B(Y)[g(X, Z)\eta(W) - g(W, Z)\eta(X)]. \tag{4.7.18}
\]
By virtue of (4.1.6), we obtain from (4.7.18) that

\[aA(W)\alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + aA(X)\alpha^2[g(W, Z)\eta(Y) - g(Y, Z)\eta(W)] + aA(Y)\alpha^2[g(Y, Z)\eta(W) - g(W, Z)\eta(X)] = bA(W)[S(X, Z)\eta(Y) - S(Y, Z)\eta(X)] - \frac{r}{2n+1} \left[\frac{a}{2n} + b\right] A(W)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]
- bA(X)[S(Y, Z)\eta(W) - S(W, Z)\eta(Y)] - \frac{r}{2n+1} \left[\frac{a}{2n} + b\right] A(Y)[g(W, Z)\eta(X) - g(X, Z)\eta(W)]
- bA(Y)[S(W, Z)\eta(X) - S(X, Z)\eta(W)] + bA(Y)[g(W, Z)\eta(Y) - g(X, Z)\eta(Y)] + B(W)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + B(X)[g(W, Z)\eta(Y) - g(Y, Z)\eta(W)].\]  

(4.7.19)

Putting \(Y = Z = r,\) in (4.7.19) and taking summation over \(i, 1 \leq i \leq 2n + 1,\) we get

\[(a)A(W)\eta(X) = A(X)\eta(W)\quad \text{and}\]
\[(b)B(W)\eta(X) = B(X)\eta(W),\]  

(4.7.20)

for all vector fields \(X, W.\) Replacing \(X\) by \(\xi\) in (4.7.20), we get

\[(a)A(W) = -\eta(W)\eta(\rho_1),\]
\[(b)B(W) = -\eta(W)\eta(\rho_2).\]  

(4.7.21)

From (4.7.20) and (4.7.21) we can state the following theorem:

**Theorem 4.7.3.** In a generalized pseudo-projective \(\phi\)-recurrent Lorentzian \(\alpha\)-Sasakian manifold the characteristic field \(\xi\) and the vector fields \(\rho_1\) and \(\rho_2\) associated to the 1-forms \(A\) and \(B\) respectively are anti-directional and the 1-forms \(A\) and \(B\) are given by (4.7.21).
4.8 3-dimensional Locally Generalized Pseudo-projective \(\phi\)-recurrent Lorentzian \(\alpha\)-Sasakian Manifold

It is known that in a three dimensional Lorentzian \(\alpha\)-Sasakian manifold the curvature tensor has the following form [30]

\[
R(X, Y)Z = \left(\frac{r}{2} - 2\alpha^2\right) [g(Y, Z)X - g(X, Z)Y] + \left(\frac{r}{2} - 3\alpha^2\right) [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y],
\]

and,

\[
S(X, Y) = \left(\frac{r}{2} - \alpha^2\right) g(X, Y) + \left(\frac{r}{2} - 3\alpha^2\right) \eta(X)\eta(Y).
\]

Then from (4.8.1), (4.8.2) and (2.1.14) we have

\[
\begin{align*}
\Pi(X, Y)Z &= \left[a\left(\frac{r}{2} - 2\alpha^2\right) + b\left(\frac{r}{2} - \alpha^2\right) - \frac{r}{2n+1}\left(\frac{a}{2n} + b\right)\right] [g(Y, Z)X - g(X, Z)Y] - g(X, Z)Y + a\left(\frac{r}{2} - 3\alpha^2\right) [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] + \left[a\left(\frac{r}{2} - 3\alpha^2\right) - b\left(\frac{r}{2} - 3\alpha^2\right)\right] [\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].
\end{align*}
\]

Taking covariant differentiation of (4.8.3) with respect to \(W\), we obtain

\[
(\nabla_W \Pi)(X, Y)Z = dr(W) \left[\frac{10a}{21} + \frac{5b}{14}\right] [g(Y, Z)X - g(X, Z)Y] + \frac{adr(W)}{2} [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] + \frac{dr(W)}{2} [a - b][\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].
\]

Noting that we may assume that all vector fields \(X, Y, Z, W\) are orthogonal to \(\xi\) in the above relation, we have

\[
(\nabla_W \Pi)(X, Y)Z = dr(W) \left[\frac{10a}{21} + \frac{5b}{14}\right] [g(Y, Z)X - g(X, Z)Y].
\]
Applying $\phi^2$ to the both sides of (4.8.5) and using (4.7.5) and (4.7.6) we get

$$
\phi^2(\nabla_W \overline{P})(X,Y)Z = dr(W) \left[ \frac{10a}{21} + \frac{5b}{14} \right] [g(Y,Z)X - g(X,Z)Y].
$$

(4.8.6)

Using (4.7.5) the equation (4.8.6) reduces to

$$
A(W)\overline{P}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y]
= dr(W) \left[ \frac{10a}{21} + \frac{5b}{14} \right] [g(Y,Z)X - g(X,Z)Y].
$$

(4.8.7)

Putting $W = \{e_i\}$, where $\{e_i\}$, $i = 1, 2, 3$, is an orthonormal basis of the tangent space at any point of the manifold and taking summation over $i$, $1 \leq i \leq 3$, we obtain

$$
\overline{P}(X,Y)Z = \lambda [g(Y,Z)X - g(X,Z)Y],
$$

(4.8.8)

where

$$
\lambda = \left[ \frac{dr(e_i)}{A(e_i)} \left( \frac{10a}{21} + \frac{5b}{14} \right) - \frac{B(e_i)}{A(e_i)} \right]
$$

(4.8.9)

is a scalar, since $A$ is a non-zero 1-form. Then by Schur's theorem $\lambda$ will be a constant on the manifold. Therefore, $(M^3, g)$ is of constant curvature $\lambda$. Thus we get the following theorem:

**Theorem 4.8.1.** A 3-dimensional generalized pseudo-projective $\phi$-recurrent Lorentzian $\alpha$-Sasakian manifold is of constant curvature.
4.9 Conclusion

- An n-dimensional D-conformally flat Lorentzian α-Sasakian is an η-Einstein manifold.

- A Lorentzian α-Sasakian manifold \((M^n, g), (n > 4)\) satisfying the condition \(B(X, Y).S = 0\) is an Einstein manifold.

- A Lorentzian α-Sasakian manifold \((M^n, g), (n > 4)\) satisfying the condition \(B(X, Y).S = 0\) is an Einstein manifold.

- In an n-dimensional Lorentzian α-Sasakian manifold \((M^n, g), (n > 4)\) satisfying the condition \(B(X, Y).R = 0\) holds on \(M\), then the equation (4.4.4) is satisfied on \(M\).

- In an n-dimensional Lorentzian α-Sasakian manifold \((M^n, g), (n > 4)\) satisfying the condition \(B(X, Y).W_2 = 0\) holds on \(M\), then the equation (4.5.6) is satisfied on \(M\).

- An extended generalized concircularly \(\phi\)-recurrent Lorentzian α-Sasakian manifold \((M^n, g), (n > 3)\) is generalized \(\phi\)-recurrent if and only if

\[
\frac{d\rho(W)}{n(n-1)} [g(Y, Z)X + \eta(X)g(Y, Z)\xi - g(X, Z)Y - \eta(Y)g(X, Z)\xi] = 0.
\]

- If an extended generalized concircularly \(\phi\)-recurrent Lorentzian α-Sasakian manifold \((M^n, g), (n > 3)\) is an extended generalized \(\phi\)-recurrent Lorentzian α-Sasakian manifold, then the associated vector field corresponding to the 1-form \(A\) is given by \(\rho_1 = \frac{1}{r}grad r\), \(r\) being the non-zero and non-constant scalar curvature of the manifold.
• In an extended generalized concircularly \( \phi \)-recurrent Lorentzian \( \alpha \)-Sasakian manifold \((M^n, g), (n \geq 3)\) the associated 1-forms \( A \) and \( B \) are related by the relation (4.6.14).

• In an extended generalized concircularly \( \phi \)-recurrent Lorentzian \( \alpha \)-Sasakian manifold \((M^n, g), (n \geq 3)\) with constant scalar curvature, the associated 1-forms \( A \) and \( B \) are related by \( \{r - \alpha^2 n(n - 1)\} A - n(n^2 - 1)B = 0 \).

• An extended generalized concircular \( \phi \)-recurrent Lorentzian \( \alpha \)-Sasakian manifold \((M^n, g), (n \geq 3)\) is super generalized Ricci-recurrent manifold.

• In an extended generalized concircularly \( \phi \)-recurrent Lorentzian \( \alpha \)-Sasakian manifold \((M^n, g), (n \geq 3)\) the Ricci tensor in the direction of \( \rho_1 \) is given by (4.6.19).

• In an extended generalized concircularly \( \phi \)-recurrent Lorentzian \( \alpha \)-Sasakian manifold \((M^n, g), (n \geq 3)\) the vector field \( \rho_2 \) associated with the 1-form \( B \) and the characteristic vector field \( \xi \) are co-directional.

• A generalized pseudo-projective \( \phi \)-recurrent Lorentzian \( \alpha \)-Sasakian manifold \((M^{2n+1}, g)\) is an Einstein manifold.

• In a generalized pseudo-projective \( \phi \)-recurrent Lorentzian \( \alpha \)-Sasakian manifold \((M^{2n+1}, g)\) the 1-forms \( A \) and \( B \) are related as in (4.7.15).

• In a generalized pseudo-projective \( \phi \)-recurrent Lorentzian \( \alpha \)-Sasakian manifold the characteristic field \( \xi \) and the vector fields \( \rho_1 \) and \( \rho_2 \) associated to the 1-forms \( A \) and \( B \) respectively are co-directional and the 1-forms \( A \) and \( B \) are given by (4.7.21).

• A 3-dimensional generalized pseudo-projective \( \phi \)-recurrent Lorentzian \( \alpha \)-Sasakian manifold is of constant curvature.