CHAPTER 2

Publications based on this Chapter;

- A Geometry on Ricci solitons in \((LCS)_a\) manifolds, Differential Geometry
  - Dynamical Systems, 16 (2014), 50-62.
Chapter 2

A Geometry On Ricci Solitons in \((LCS)_n\) Manifolds

2.1 Introduction

In 2003 A.A. Shaikh [160], introduced the notion of Lorentzian concircular structure manifolds (briefly \((LCS)_n\)-manifolds) with an example, which generalize the notion of LP-Sasakian manifolds introduced by Matsumoto. In 2012 H.G. Nagaraja [119], worked on the Ricci solitons in Kenmotsu manifolds using the semi symmetric conditions. Motivation of these works have helped us to obtain the following results on \((LCS)_n\) manifolds.

2.2 Ricci soliton in a \((LCS)_n\) manifolds satisfying

\[ R(\xi, X) \cdot \tilde{P} = 0 \]

The Pseudo-projective curvature tensor \( \tilde{P} \) is defined by

\[
\tilde{P}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] \\
- \frac{r}{n} \left( \frac{a}{n-1} + b \right) [g(Y, Z)X - g(X, Z)Y],
\]

(2.2.1)
where \( a, b \neq 0 \) are constants. Taking \( Z = \xi \) in (2.2.1) and using (1.5.14), (1.5.16), (1.5.17), we get

\[
P(X, Y)\xi = \left[a(\alpha^2 - \rho) - b\lambda - \frac{r}{n} \left( \frac{a}{n - 1} + b \right) \right] \eta(\xi)X - \eta(Y)Y. \tag{2.2.2}
\]

Similarly using (1.5.11), (1.5.16), (1.5.17), (1.5.18) in (2.2.1), we get

\[
\eta(\tilde{P}(X, Y)Z) = \left[a(\alpha^2 - \rho) - b\lambda - \frac{r}{n} \left( \frac{a}{n - 1} + b \right) \right] \eta(Y)\eta(X)
- g(X, Z)\eta(Y). \tag{2.2.3}
\]

We assume that the condition \( R(\xi, X) \cdot \tilde{P} = 0 \), then we have

\[
R(\xi, X)\tilde{P}(U, V)W - \tilde{P}(R(\xi, X)U, V)W
- \tilde{P}(U, R(\xi, X)V)W - \tilde{P}(U, V)R(\xi, X)W = 0. \tag{2.2.4}
\]

Using (1.5.13) in (2.2.4), we find

\[
(a^2 - \rho)[\eta(\tilde{P}(U, V)W)X - g(X, \tilde{P}(U, V)W)\xi - \eta(U)\tilde{P}(X, V)W)
+ g(X, U)\tilde{P}(\xi, V)W - \eta(V)\tilde{P}(U, X)W + g(X, V)\tilde{P}(U, \xi)W
- \eta(W)\tilde{P}(U, V)X + g(X, W)\tilde{P}(U, V)\xi] = 0. \tag{2.2.5}
\]

By taking an inner product with \( \xi \), we get

\[
\eta(\tilde{P}(U, V)W)\eta(X) + g(X, \tilde{P}(U, V)W) - \eta(U)\eta(\tilde{P}(X, V)W)
+ g(X, U)\eta(\tilde{P}(\xi, V)W) - \eta(V)\eta(\tilde{P}(U, X)W) + g(X, V)
\eta(\tilde{P}(U, \xi)W) - \eta(W)\eta(\tilde{P}(U, V)X) + g(X, W)\eta(\tilde{P}(U, V)\xi) = 0. \tag{2.2.6}
\]

By using (2.2.2), (2.2.3) in (2.2.6), we have

\[
g(X, \tilde{P}(U, V)W) + \left[a(\alpha^2 - \rho) - b\lambda - \frac{r}{n} \left( \frac{a}{n - 1} + b \right) \right]
[g(X, V)g(U, W) - g(X, U)g(V, W)] = 0. \tag{2.2.7}
\]
In view of (2.2.1) in (2.2.7), we have

\[
ag(X, R(U, V)W) + b[(\alpha + \lambda)\{g(V, X)g(U, W) - g(V, W)g(U, X)\} \\
+ \alpha [g(V, X)\eta(U)\eta(W) - \eta(V)\eta(W)g(U, X)] + [a(\alpha^2 - \rho) \\
- b(\alpha + \lambda)]g(X, V)g(U, W) - g(X, U)g(V, W)] = 0. \tag{2.2.8}
\]

Taking \(X = U = e_i\) in (2.2.8) and summing over \(i = 1, 2, \ldots, n\), and on simplification, we get

\[
aS(V, W) = -a(\alpha^2 - \rho)(1 - n)g(V, W) - b\alpha(1 - n)\eta(V)\eta(W). \tag{2.2.9}
\]

Putting \(V = W = \xi\) in (2.2.9) and by virtue of (1.5.18), (1.5.19), we get the following equation

\[
\lambda = \frac{(a(\alpha^2 - \rho) - b\alpha)(1 - n)}{\alpha}. \tag{2.2.10}
\]

Since that, Ricci soliton in \((LCS)_n\) manifold satisfying \(R(\xi, X) \cdot \vec{P} = 0\).

Hence we state the following theorem:

**Theorem 2.2.1.** A Ricci soliton in a \((LCS)_n\) manifold satisfying \(R(\xi, X) \cdot \vec{P} = 0\) and \(\alpha\) a positive function is

1. shrinking if \(\alpha > \frac{b}{a}\), expanding if \(\alpha < \frac{b}{a}\) and steady if \(\alpha = \frac{b}{a}\), provided that \(\xi\) is orthogonal to \(\nabla\alpha\).

2. shrinking if \(\alpha^2 + k|\nabla\alpha| > \frac{b}{a}\alpha\), expanding if \(\alpha^2 + k|\nabla\alpha| < \frac{b}{a}\alpha\) and steady if \(\alpha^2 + k|\nabla\alpha| = \frac{b}{a}\alpha\), provided that angle between \(\xi\) and \(\nabla\alpha\) is acute.

3. shrinking if \(\alpha^2 > k|\nabla\alpha| + \frac{b}{a}\alpha\), expanding if \(\alpha^2 < k|\nabla\alpha| + \frac{b}{a}\alpha\) and steady if \(\alpha^2 = k|\nabla\alpha| + \frac{b}{a}\alpha\), provided that angle between \(\xi\) and \(\nabla\alpha\) is obtuse.
Proof. 1. From the Remark 1.5.1 (1) and 2.2.10

\[ \lambda = \frac{-(n-1)(a\alpha^2 - b\alpha)}{a}. \]

Hence \( \lambda < 0 \) if \( \alpha > \frac{b}{a} \), \( \lambda > 0 \) if \( \alpha < \frac{b}{a} \) and \( \lambda = 0 \) if \( \alpha = \frac{b}{a} \).

2. From the Remark 1.5.1 (2) and 2.2.10

\[ \lambda = \frac{-(n-1)[a(\alpha^2 + k|\nabla\alpha|) - b\alpha]}{a}. \]

Hence \( \lambda < 0 \) if \( \alpha^2 + k|\nabla\alpha| > \frac{b}{a} \alpha \), \( \lambda > 0 \) if \( \alpha^2 + k|\nabla\alpha| < \frac{b}{a} \alpha \) and \( \lambda = 0 \) if \( \alpha^2 + k|\nabla\alpha| = \frac{b}{a} \alpha \).

3. From the Remark 1.5.1 (3) and 2.2.10

\[ \lambda = \frac{-(n-1)[a(\alpha^2 - k|\nabla\alpha|) - b\alpha]}{a}. \]

Hence \( \lambda < 0 \) if \( \alpha^2 > k|\nabla\alpha| + \frac{b}{a} \alpha \), \( \lambda > 0 \) if \( \alpha^2 < k|\nabla\alpha| + \frac{b}{a} \alpha \) and \( \lambda = 0 \) if \( \alpha^2 = k|\nabla\alpha| + \frac{b}{a} \alpha \).

\[ \square \]

Corollary 2.2.2. An LP–Sasakian manifold is shrinking if \( a > b \), expanding if \( a < b \) and steady if \( a = b \).

Follows by putting \( \alpha = 1 \) in (1) of Theorem 2.2.1.
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2.3 Ricci soliton in a $(LCS)_n$ manifolds satisfying $R(\xi, X)B = 0$

The C-Bochner curvature tensor $B$ is defined by

\[
B(X,Y)Z = R(X,Y)Z + \frac{1}{n+3}[g(X,Z)QY - S(Y,Z)X - g(Y,Z)QX + S(X,Z)Y
+ g(\phi X, Z)Q\phi Y - S(\phi Y, Z)\phi X - g(\phi Y, Z)Q\phi X + S(\phi X, Z)\phi Y
+ 2S(\phi X, Y)\phi Z + 2g(\phi X, Y)Q\phi Z + \eta(Y)\eta(Z)QX - \eta(Y)S(X, Z)\xi
+ \eta(X)S(Y, Z)\xi - \eta(X)\eta(Z)QY] - \frac{D + n - 1}{n+3}[g(\phi X, Z)\phi Y
- g(\phi Y, Z)\phi X + 2g(\phi X, Y)\phi Z] + \frac{D}{n+3}[\eta(Y)g(X, Z)\xi - \eta(Y)\eta(Z)X
+ \eta(X)\eta(Z)Y - \eta(X)g(Y, Z)\xi] - \frac{D - 4}{n+3}[g(X, Z)Y - g(Y, Z)X],
\]

where $D = \frac{r+n-1}{n+1}$.

Taking $Z = \xi$ in (2.3.1) and using (1.5.14), (1.5.16), (1.5.17), we get

\[
B(X,Y)\xi = \left[(\alpha^2 - \rho) + \frac{2\alpha + 3\lambda}{n+3} + \frac{2r + 2n - 2 - 4n - 4}{(n+1)(n+3)}\right] [\eta(Y)X - \eta(X)Y].
\]

Similarly using (1.5.11), (1.5.16), (1.5.17), (1.5.18) in (2.3.2), we get

\[
\eta(B(X,Y)Z) = \left[(\alpha^2 - \rho) + \frac{\lambda}{n+3} - \frac{4 - 2D}{n+3} + \frac{2(\alpha + \lambda)}{n+3}\right] [g(Y,Z)\eta(X)
- g(X,Z)\eta(Y)].
\]

We assume that the condition $R(\xi, X)B = 0$, then we have

\[
R(\xi, X)B(Y,Z)W - B(R(\xi, X)Y,Z)W
- B(Y,R(\xi, X)Z)W - B(Y,Z)R(\xi, X)W = 0.
\]
By using (1.5.13) and taking an inner product with $\xi$ then, by using (2.3.2) and (2.3.3) in (2.3.4), then we have

$$4S(Z, W) = \left[ (\alpha^2 - \rho)(n - 1)(n + 3) + \lambda(n - 1) - (4 - 2D)(n - 1) \right.$$ 

$$+ 2(\alpha + \lambda)(n - 1) + \mathbf{r} + 4(\alpha + \lambda) - D + 3(D + n - 1) 

- (D - 4)(n - 1)g(Z, W) + [-r - 2\lambda + 4(\alpha + \lambda) 

+ D(n - 2) + 3(D + n - 1)]\mathbf{g}(Z)\eta(W). \quad (2.3.5)$$

Taking $Z = W = \xi$, and by virtue of (1.5.18), (1.5.19), we get the following equation

$$\lambda = \left[ -\frac{(\alpha^2 - \rho)(n - 2) - 4\alpha}{5} \right]. \quad (2.3.6)$$

Hence we state the following theorem:

**Theorem 2.3.1.** A Ricci soliton in a $(LCS)_n$ manifold satisfying $R(\xi, X) \cdot B = 0$ is shrinking if $\alpha$ is a positive function whose gradient vector field is orthogonal to $\xi$.

**Proof.** By the Remark 1.5.1 (1) and 2.3.6

$$\lambda = -\frac{\alpha^2(n - 2) + 4\alpha}{5} \quad (2.3.7)$$

$$\implies \lambda < 0. \quad (2.3.8)$$

However, results for cases of acute and obtuse angles between $\xi$ and $\nabla \alpha$ are little complex.
2.4 Ricci soliton in a \((LCS)_n\) manifolds satisfying

\[ R(\xi, X) \cdot \tilde{M} = 0 \]

The \(M\)-projective curvature tensor \(\tilde{M}\) is defined by

\[ \tilde{M}(X, Y)Z = R(X, Y)Z - \frac{1}{2(n-1)}[S(Y, Z)X - S(X, Z)Y \]
\[ + g(Y, Z)QX - g(X, Z)QY], \quad (2.4.1) \]

Taking \(Z = \xi\) in (2.4.1) and using (1.5.14), (1.5.16), (1.5.17), we get

\[ \tilde{M}(X, Y)\xi = \left[(\alpha^2 - \rho) + \frac{(\alpha + \lambda)}{2(n-1)} - \frac{\alpha}{2(n-1)}\right][\eta(Y)X - \eta(X)Y]. \quad (2.4.2) \]

Similarly using (1.5.11), (1.5.16), (1.5.17), (1.5.18) in (2.4.2), we get

\[ \eta(\tilde{M}(X, Y)Z) = \left[(\alpha^2 - \rho) + \frac{(\alpha + \lambda)}{2(n-1)} - \frac{\alpha}{2(n-1)}\right][g(Y, Z)\eta(X) \]
\[ - g(X, Z)\eta(Y)]. \quad (2.4.3) \]

We assume that the condition \(R(\xi, X) \cdot \tilde{M} = 0\), then we have

\[ R(\xi, X)\tilde{M}(Y, Z)W - \tilde{M}(R(\xi, X)Y, Z)W \]
\[ - \tilde{M}(Y, R(\xi, X)Z)W - \tilde{M}(Y, Z)R(\xi, X)W = 0. \quad (2.4.4) \]

By using (1.5.13) in (2.4.4), we have

\[ [g(X, \tilde{M}(Y, Z)W)\xi] = g(X, Y)\tilde{M}(\xi, Z)W + \]
\[ \eta(Y)\tilde{M}(X, Z)W - \tilde{M}(Y, \xi)Wg(X, Z) + \eta(Z)\tilde{M}(Y, X)W \]
\[ - g(X, W)\tilde{M}(Y, Z)\xi + \eta(W)\tilde{M}(Y, Z)X = 0, \quad (2.4.5) \]
By taking an inner product with $\xi$, we get

$$
[g(X, \tilde{M}(Y, Z)W) + \eta(\tilde{M}(Y, Z)W)\eta(X) + g(X, Y)\eta(\tilde{M}(\xi, Z)W) - \eta(Y)\eta(\tilde{M}(X, Z)W) + \eta(\tilde{M}(Y, \xi)W)g(X, Z) - \eta(Z)\eta(\tilde{M}(Y, X)W) + g(X, W)\eta(\tilde{M}(Y, Z)\xi) - \eta(W)\eta(\tilde{M}(Y, Z)X)] = 0. \quad (2.4.6)
$$

By using (2.4.2) and (2.4.3) in (2.4.6), we have

$$
\left[\left(\alpha^2 - \rho\right) + \frac{(\alpha + \lambda)}{2(n - 1)} - \frac{\alpha}{2(n - 1)}\right] g(Y, W)g(X, Z) - g(X, Y)g(Z, W) + g(X, M(Y, Z)W) = 0. \quad (2.4.7)
$$

In view of (2.4.1) in (2.4.7) and taking $X = Y = e_i$, we have

$$
S(Z, W) = \left[\frac{r + 2(n - 1)(\alpha + \lambda) + 2(n - 1)^2(\alpha^2 - \rho) - \alpha(n - 1)}{n}\right] g(Z, W). \quad (2.4.8)
$$

Taking $Z = W = \xi$, then we get

$$
\lambda = -\left[(n - 1)(\alpha^2 - \rho)\right]. \quad (2.4.9)
$$

Hence we state the following theorem:

**Theorem 2.4.1.** A Ricci soliton in a $(LCS)_n$ manifold satisfying $R(\xi, X) \cdot \tilde{M} = 0$ is

1. shrinking if characteristic vector field $\xi$ is orthogonal to $\nabla \alpha$.

2. shrinking of the angle between characteristic vector field $\xi$ and the gradient vector field $\nabla \alpha$ is acute.

3. If the angle between characteristic vector field $\xi$ and the gradient vector field $\nabla \alpha$ is obtuse then it is shrinking if $\alpha^2 > k|\nabla \alpha|$, expanding if $\alpha^2 < k|\nabla \alpha|$ and steady $\alpha^2 = k|\nabla \alpha|$. 

Proof.  1. According to Remak 1.5.1 (1) and 2.4.9, we have

$$ \lambda = -(n - 1) \alpha^2, \ \lambda < 0. \quad (2.4.10) $$

2. According to Remak 1.5.1 (3) and 2.4.9, we have

$$ \lambda = -(n - 1)(\alpha^2 + k|\nabla \alpha|), \ \lambda < 0. \quad (2.4.11) $$

3. According to Remak 1.5.1 (4) and 2.4.9, we have

$$ \lambda = -(n - 1)(\alpha^2 - k|\nabla \alpha|), \ \lambda < 0. \quad (2.4.12) $$

2.5 Ricci soliton in a \((LCS)_n\) manifolds satisfying

$$ R(\xi, X) \cdot H = 0 $$

The Conharmonic curvature tensor \(H\) is defined by

$$ H(X, Y)Z = R(X, Y)Z - \frac{1}{(n - 2)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]. \quad (2.5.1) $$

Taking \(Z = \xi\) in (2.5.1) and using (1.5.14), (1.5.16), (1.5.17), we get

$$ H(X, Y)\xi = \left[ (\alpha^2 - \rho) + \frac{2(\alpha + \lambda)}{(n - 2)} - \frac{\alpha}{(n - 2)} \right] [\eta(Y)X - \eta(X)Y]. \quad (2.5.2) $$

Similarly using (1.5.11), (1.5.16), (1.5.17), (1.5.18) in (2.5.2), we get

$$ \eta(H(X, Y)Z) = \left[ (\alpha^2 - \rho) + \frac{2(\alpha + \lambda)}{(n - 2)} - \frac{\alpha}{(n - 2)} \right] [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \quad (2.5.3) $$
We assume that the condition $R(\xi, X)\cdot H = 0$, then we have

$$R(\xi, X)H(Y, Z)W - H(R(\xi, X)Y, Z)W - H(Y, R(\xi, X)Z)W - H(Y, Z)R(\xi, X)W = 0. \quad (2.5.4)$$

By using (1.5.13) in (2.5.4), we have

$$[g(X, H(Y, Z)W)\xi - \eta(H(Y, Z)W)X) - g(X, Y)H(C Z)W - H(Y, Z)W + \eta(Z)H(Y, X)W]
+ \eta(Y)H(X, Z)W - H(Y, e)VKg(X Z) + v(Z)H(Y, X)W
= 0. \quad (2.5.5)$$

By taking an inner product with $\xi$, then we get

$$[g(X, H(Y, Z)W)\xi - \eta(H(Y, Z)W)X) + g(X, Y)H(C Z)W - H(Y, e)VKg(X Z) + v(Z)H(Y, X)W]
+ \eta(Y)H(X, Z)W - H(Y, e)VKg(X Z) + v(Z)H(Y, X)W
= 0. \quad (2.5.6)$$

By using (2.5.2) and (2.5.3) in (2.5.6), we have

$$[(\alpha^2 - \rho)(1 - n)(n - 2) + 2(\alpha + \lambda)(1 - n) - \alpha(1 - n) - r]
g(Z, W) = 0. \quad (2.5.7)$$

In view of (2.5.1) in (2.4.7) and taking $X = Y = e_i$, we have

$$\left[(\alpha^2 - \rho)(1 - n)(n - 2) + 2(\alpha + \lambda)(1 - n) - \alpha(1 - n) - r\right]
g(Z, W) = 0, \quad (2.5.8)$$

where $g(Z, W) \neq 0$, therefore we get

$$\lambda = [(\alpha^2 - \rho)(1 - n)]. \quad (2.5.9)$$

Hence we state the following theorem:

**Theorem 2.5.1.** A Ricci soliton in $\text{(LCS)}_n$ manifold satisfying $R(\xi, X)\cdot H = 0$ is (2.5.9).
2.6 Ricci soliton in a \((LCS)_n\) manifolds satisfying

\[
R(\xi, X) \cdot \tilde{C} = 0
\]

The Quasi-conformal curvature tensor \(\tilde{C}\) is defined by

\[
\tilde{C}(X,Y)Z = aR(X,Y)Z + b(S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX
\]

\[
- g(X, Z)QY - \frac{r}{n} \left( \frac{a}{n-1} + 2b \right) [g(Y, Z)X - g(X, Z)Y].
\]

(2.6.1)

where \(a, b \neq 0\) are constants. Taking \(Z = \xi\) in (2.6.1) and using (1.5.14), (1.5.16), (1.5.17), we get

\[
\eta(\tilde{C}(X,Y)\xi) = \left[ a(\alpha^2 - \rho) - b(\beta + \alpha) - \frac{r}{n} \left( \frac{a}{n-1} + 2b \right) \right] [\eta(Y)X - \eta(X)Y].
\]

(2.6.2)

Similarly using (1.5.11), (1.5.16), (1.5.17), (1.5.18) in (2.6.1), we get

\[
\eta(\tilde{C}(X,Y)Z) = \left[ a(\alpha^2 - \rho) - b(\beta + \alpha) - \frac{r}{n} \left( \frac{a}{n-1} + 2b \right) \right] [g(Y, Z)\eta(X) - g(X, Z)\eta(\xi)].
\]

(2.6.3)

We assume that the condition \(R(\xi, X) \cdot \tilde{C} = 0\), then we have

\[
R(\xi, X)\tilde{C}(U, V)W - \tilde{C}(R(\xi, X)U, V)W
\]

\[
-\tilde{C}(U, R(\xi, X)V)W - \tilde{C}(U, V)R(\xi, X)W = 0.
\]

(2.6.4)

Using (1.5.13) in (2.6.4), we find

\[
(\alpha^2 - \rho)[\eta(\tilde{C}(U, V)W)X - g(X, \tilde{C}(U, V)W)\xi - \eta(U)\tilde{C}(X, V)W
\]

\[
+ g(X, U)\tilde{C}(\xi, V)W - \eta(V)\tilde{C}(U, X)W + g(X, V)\tilde{C}(U, \xi)W
\]

\[
- \eta(W)\tilde{C}(U, V)X + g(X, W)\tilde{C}(U, V)\xi] = 0.
\]

(2.6.5)
By taking an inner product with $\xi$, then we get

\begin{equation}
\eta(\tilde{C}(U, V)W)\eta(X) + g(X, \tilde{C}(U, V)W) - \eta(U)\eta(\tilde{C}(X, V)W) \\
+ g(X, U)\eta(\tilde{C}(\xi, V)W) - \eta(V)\eta(\tilde{C}(U, X)W) + g(X, V)\eta(\tilde{C}(U, \xi)W) \\
- \eta(W)\eta(\tilde{C}(U, V)X) + g(X, W)\eta(\tilde{C}(U, V)\xi) = 0. 
\end{equation}

(2.6.6)

By using (2.6.2), (2.6.3) in (2.6.6), we have

\begin{equation}
g(X, \tilde{C}(U, V)W) + a(a^2 - \rho) - b(2\lambda + \alpha) - \frac{r}{n}(\frac{a}{n - 1} + 2b) \\
[g(X, V)g(U, W) - g(X, U)g(V, W)] = 0. 
\end{equation}

(2.6.7)

In view of (2.6.1) in (2.6.7) and Taking $X = U = e_i$, summing over $i = 1, 2, \ldots, n$, and on simplification, we get

\begin{equation}
a + b(n - 2)S(V, W) = -[a(\alpha^2 - \rho)(1 - n)g(V, W) - \alpha b(n - 2)]g(V, W). 
\end{equation}

(2.6.8)

Putting $V = W = \xi$ in (2.6.8) and by virtue of (1.5.18), (1.5.19), we get the following equation

\begin{equation}
\lambda = (\alpha^2 - \rho)(1 - n). 
\end{equation}

(2.6.9)

Since, the Ricci soliton in $(LCS)_n$ manifold satisfying $R(\xi, X) \cdot \tilde{C} = 0$.

Hence we state the following theorem:

**Theorem 2.6.1.** A Ricci soliton in a $(LCS)_n$ manifold satisfying $R(\xi, X) \cdot \tilde{C} = 0$ is (2.6.9).
2.7 Ricci soliton in a \((LCS)_n\) manifolds satisfying

\[ R(\xi, X) \cdot C = 0 \]

The Concircular curvature tensor \(C\) is defined by

\[
C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]. \tag{2.7.1}
\]

Taking \(Z = \xi\) in (2.7.1) and using (1.5.14), (1.5.16), (1.5.17), we get

\[
C(X, Y)\xi = \left(\alpha^2 - \rho\right) - \frac{r}{n(n-1)}[g(Y, Z)\eta(Y)X - g(X, Z)\eta(X)Y]. \tag{2.7.2}
\]

Similarly using (1.5.11), (1.5.16), (1.5.17), (1.5.18) in (2.7.2), we get

\[
\eta(C(X, Y)Z) = \left(\alpha^2 - \rho\right) - \frac{r}{n(n-1)}[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \tag{2.7.3}
\]

We assume that the condition \(R(\xi, X) \cdot C = 0\), then we have

\[
R(\xi, X)C(Y, Z)W - C(R(\xi, X)Y, Z)W
- C(Y, R(\xi, X)Z)W - C(Y, Z)R(\xi, X)W = 0. \tag{2.7.4}
\]

By using (1.5.13) in (2.7.4), we have

\[
[g(X, C(Y, Z)W)\xi - \eta(C(Y, Z)W)X - g(X, Y)C(\xi, Z)W
+ \eta(Y)C(X, Z)W - C(Y, \xi)Wg(X, Z) + \eta(Z)C(Y, X)W
-g(X, W)C(Y, Z)\xi + \eta(W)C(Y, Z)X] = 0. \tag{2.7.5}
\]

By taking an inner product with \(\xi\), then we get

\[
[g(X, C(Y, Z)W) + \eta(C(Y, Z)W)\eta(X) + g(X, Y)\eta(C(\xi, Z)W
- \eta(Y)\eta(C(X, Z)W) + \eta(C(Y, \xi)W)g(X, Z) - \eta(Z)\eta(C(Y, X)W)
+ g(X, W)\eta(C(Y, Z)\xi) - \eta(W)\eta(C(Y, Z)X)] = 0. \tag{2.7.6}
\]
By using (2.7.2) and (2.7.3) in (2.7.6), we have

\[
\left[ (\alpha^2 - \rho) - \frac{r}{n(n-1)} \right] [g(U, W)g(X, V) - g(X, U)g(V, W)]
+ g(X, C(U, V)W) = 0.
\]

(2.7.7)

In view of (2.4.1) in (2.7.7) and taking \( X = Y = e_i \), we have

\[
S(V, W) = -(\alpha^2 - \rho)(1 - n)g(V, W).
\]

(2.7.8)

Taking \( V = W = \xi \), then we get

\[
\lambda = (\alpha^2 - \rho)(1 - n).
\]

(2.7.9)

Hence we state the following theorem:

**Theorem 2.7.1.** A Ricci soliton in a \((LCS)_n\) manifold satisfying \( R(\xi, X) \cdot C = 0 \) is (2.7.9).

Based on the above all results we conclude that the value of \( \lambda \) in (2.5.9), (2.6.9) and (2.7.9) is same as (2.4.9). Hence the results for Theorems 2.5.1, 2.6.1 and 2.7.1 may be explained as in Theorem 2.4.1.