PREFACE
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Geometry is extremely important and used in surveying, astronomy, navigation and construction of building. It was studied well over 2000 years ago by Ancient Greece Mathematicians like Euclid and others Pythagoras, Thales, Plato and Aristotle. The most fascinating and accurate geometry text was written by Euclid and was called Elements. Euclid’s text has been used for over 2000 years. It is the study of angles, triangles, perimeter, area and volume. It differs from algebra in that one develops a logical structure where mathematical relationships are proved and applied.

Riemannian geometry studies Riemannian manifolds i.e., smooth manifolds with Riemannian metric, a notion of a distance expressed by means of a smooth positive definite symmetric bilinear form defined on the tangent space at each point. It generalizes Euclidean geometry. Various concepts based on length, such as the arc length of curve, area of plane regions and volume of solid all admit natural analogous in Riemannian geometry. The notion of directional derivative of a function from multivariate calculus is extended in Riemannian geometry to the notion of covariant derivative of a tensor. Many concepts and techniques of analysis and differential equations have been generalized to the setting of Riemannian manifolds.
In Riemannian geometry, the use of integral formulas was initiated about fifty years ago by Bochner, Chern, Hodge, Thomas and others. Integral formulas were found to be very powerful tools for obtaining global results in Riemannian geometry. Application of integral formulas are also found in Hermitian and Kaehlerian geometries, as well as in the geometry of almost-contact manifolds.

A Manifold is an important object in Mathematics and Physics because it allows more complicated structure to be expressed and understood in terms of the relatively well understood properties of simpler spaces. For example, a circle can be constructed by bending two line segments into arcs which overlap at their ends and gluing them together where they overlap. There are lots of applications of manifolds in physics i.e., differentiable manifolds serve as the phase space in classical mechanics and four-dimensional pseudo-Riemannian manifolds are used to model spacetime in general relativity.

A manifold is an abstract mathematical space in which every point has a neighborhood which resembles Euclidean space. In discussing manifolds, the idea of dimension is important. In an one-dimensional manifold, every point has a neighborhood that looks like a segment of a line. Examples of one-manifolds include lines, circles and two separate circles. In a two-manifold, every point has a neighborhood that looks like a disc. Examples include plane, the surface of a sphere and the surface of a torus.

A manifold can be described by an atlas i.e., collection of charts. Each chart specifies a coordinate system on a piece of the manifold, which is a function from that piece of the manifold into a Euclidean space, the notion of a differentiable manifold refines the notion of a manifold by requiring the transition from one chart to another to be differentiable.
A Differentiable Manifold is a topological manifold with a globally defined differentiable structure i.e., topological manifold is given a differentiable structure locally by using the homeomorphisms in its atlas, combined with the standard differentiable structure on the Euclidean space. In other words, the homeomorphism can be used to give a local coordinate system.

A Riemannian manifold is a smooth manifold $M$ together with a covariant tensor field $g$ of degree 2 (called Riemannian metric) i.e., of type $(0, 2)$ which satisfies the following conditions:

- $g$ is symmetric i.e., $g(X,Y) = g(Y,X)$ for all $X,Y \in \chi(M)$,
- $g$ is positive definite, i.e., $g(X,X) \geq 0$ for all $X \in \chi(M)$ and $g(X,X) = 0$ if and only if $X = 0$.

A contact manifold is an $n(n = 2m + 1)$-dimensional $C^\infty$ manifold $M^n$ equipped with a global 1-form $\eta$ called a contact form of $M^n$ such that $\eta \wedge (d\eta)^m \neq 0$ everywhere on $M^n$. In physics thermodynamics space is an example of contact manifold.

An $n$-dimensional differentiable manifold is said to have an almost contact structure $(\phi, \xi, \eta)$ if it carries a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and a 1-form $\eta$ such that

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, d\eta(X,\xi) = 0 \text{ and } \eta \circ \phi = 0.$$ 

The manifold $M$ equipped with almost contact structure is said to be almost contact manifold.
Ricci Solitons in contact geometry were first studied by J.T. Cho and Ramesh Sharma [50], over the last few years. Ricci flow and gradient Ricci solitons arise in geometry and physics. In differential geometry they introduce a systematic framework to find canonical metrics on Riemannian manifolds and make advances towards their classification by proving the isomerization conjecture. Geometric flows have become important tools in Riemannian geometry and general relativity. List [99] has studied a geometric flow whose fixed points correspond to static Ricci flat space (space time) which is nothing but Ricci flow pullback by a certain diffeomorphism. The association of each Ricci flat space time gives notion of local Ricci soliton in higher dimension. The importance of geometric flow in Riemannian geometry is due to Hamilton who has given flow equation and List [99] generalized Hamilton’s equation and extended it to space (spacetime) for static metrics. He has given the system, namely Einstein free-scalar field system. This observation is useful for the corresponding of solution of system i.e., Ricci soliton and symmetric property of spacetime, that how Riemannian space (or spacetime) with Ricci solitons deal different kind of symmetry properties.

The entire work represented in the thesis has been partitioned into Seven chapters.

The chapter 1 deals with basic concepts and preliminaries of \((LCS)_n\) manifolds, Kenmotsu Manifolds, \(C(\lambda)\) manifolds, \(\alpha\)-Sasakian Manifolds, Lorentzian Para-Sasakian Manifolds, \(K\)-contact manifolds, Ricci flow, Ricci solitons, \(\eta\)-Ricci solitons and also gradient Ricci solitons are given in detail.
The chapter 2 is devoted to "A Geometry On Ricci Solitons in $(LCS)_n$ Manifolds" and we have obtained following results:

- A Ricci soliton in a $(LCS)_n$ manifold satisfying $R(\xi, X) \cdot \tilde{P} = 0$, $\tilde{P}$ is Pseudo-projective curvature tensor and $\alpha$ is a positive function is

  1. shrinking if $\alpha > \frac{b}{a}$, expanding if $\alpha < \frac{b}{a}$ and steady if $\alpha = \frac{b}{a}$, provided that $\xi$ is orthogonal to $\nabla \alpha$.

  2. shrinking if $\alpha^2 + k|\nabla \alpha| > \frac{k}{a} \alpha$, expanding if $\alpha^2 + k|\nabla \alpha| < \frac{k}{a} \alpha$ and steady if $\alpha^2 + k|\nabla \alpha| = \frac{k}{a} \alpha$, provided that angle between $\xi$ and $\nabla \alpha$ is acute.

  3. shrinking if $\alpha^2 > k|\nabla \alpha| + \frac{k}{a} \alpha$, expanding if $\alpha^2 < k|\nabla \alpha| + \frac{k}{a} \alpha$ and steady if $\alpha^2 = k|\nabla \alpha| + \frac{k}{a} \alpha$, provided that angle between $\xi$ and $\nabla \alpha$ is obtuse.

- A Ricci soliton in a $(LCS)_n$ manifold satisfying $R(\xi, X) \cdot B = 0$, $B$ is $C$–Bochner curvature tensor, is shrinking if $\alpha$ is a positive function whose gradient vector field is orthogonal to $\xi$, i.e.,

  \[ \lambda = -\frac{\alpha^2(n - 2) + 4\alpha}{5} \]  

  \[ \Rightarrow \lambda < 0. \]  

- A Ricci soliton in $(LCS)_n$ manifold satisfying $R(\xi, X) \cdot \tilde{M} = 0$, $\tilde{M}$ is $M$–Projective curvature tensor is

  1. shrinking if characteristic vector field $\xi$ is orthogonal to $\nabla \alpha$.

  2. shrinking if the angle between characteristic vector field $\xi$ and the gradient vector field $\nabla \alpha$ is acute.
3. shrinking if $\alpha^2 > k|\nabla \alpha|$, expanding if $\alpha^2 < k|\nabla \alpha|$ and steady, if $\alpha^2 = k|\nabla \alpha|$ when the angle between characteristic vector field $\xi$ and the gradient vector field $\nabla \alpha$ is obtuse.

- Similar results for Ricci solitons we obtained when

1. $R(\xi, X) \cdot H = 0$, $H$ is Conharmonic curvature tensor,

2. $R(\xi, X) \cdot \bar{C} = 0$, $\bar{C}$ is Quasi-conformal curvature tensor,

3. $R(\xi, X) \cdot C = 0$, $C$ is Concircular curvature tensor.


In chapter 3 we study “On Ricci Solitons in Kenmotsu Manifolds” and we have the following results:

- A Ricci soliton in Kenmotsu manifold satisfying condition $S(\xi, X) \cdot W_2 = 0$, is expanding.

- A Ricci soliton in Kenmotsu manifold which satisfies the condition $H(\xi, X) \cdot W_2 = 0$, $H$ is Conhormonic curvature tensor, is:

  1. $\eta$–Einstein manifold.

  2. expanding solution.

- A Ricci solitoin in Kenmotsu manifold satisfying the condition $\bar{W}(\xi, X) \cdot W_2 = 0$, $\bar{W}$ is pseudo quasi-conformal curvature tensor, is expanding for all $n > 3$. 
• A Ricci soliton in a $\beta$–Kenmotsu manifold satisfying semi-symmetric conditions on $P$–projective curvature tensor and $C$–Conformal curvature tensor is expanding.

• A Ricci soliton in a $\beta$–Kenmotsu manifold satisfying $R(\xi, X) \cdot \tilde{W} = 0$, $\tilde{W}$ is Pseudo quasi-conformal curvature tensor is

1. expanding if $(n - 1)[\beta^2 + (\xi \beta)] > \frac{\delta(n(p+d)-2p+q(n-2)^2)}{n(p+d)}$,
2. shrinking if $(n - 1)[\beta^2 + (\xi \beta)] < \frac{\delta(n(p+d)-2p+q(n-2)^2)}{n(p+d)}$,
3. steady if $(n - 1)[\beta^2 + (\xi \beta)] = \frac{\delta(n(p+d)-2p+q(n-2)^2)}{n(p+d)}$.

• Finally at the end of the chapter we illustrate the examples on Ricci solitons in Kenmotsu and $\beta$–Kenmotsu manifolds.

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The **Chapter-4** of the thesis consists "On $C(\lambda)$ manifolds" and we have the following:

• A Quasi-conformally and $\xi$–Quasi-conformally flat almost $C(\lambda)$ manifold, is either cosymplectic or is special type of $\eta$–Einstein.
• Pseudo-projectively and $\xi$-Pseudo-projectively curvature flat almost $C(\lambda)$ manifold, is either cosymplectic or is special type of $\eta$-Einstein.

• A Ricci soliton on $\tau$ flat $C(\lambda)$ manifold is shrinking/expanding. Also if the Ricci soliton $(g, V, \lambda)$, $n \geq 3$ is expanding at $\infty$ then it has cone structure at $\infty$, provided asymptotic curvature $A(G)$ is finite or otherwise it is asymptotically flat.

• A Ricci soliton in $C(\lambda)$ manifold satisfying $B = 0$ is shrinking.

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The Chapter-5 of the thesis deals with “On Ricci solitons in $\alpha$-Sasakian Manifolds” and we have the following:

• Ricci solitons in $\alpha$-Sasakian manifolds are either shrinking or expanding.

• A Ricci soliton in $3$-dimensional $\alpha$-Sasakian manifolds is either shrinking or expanding but cannot be steady, if $V$ is conformal Killing vector field.

• On a $\alpha$-Sasakian manifold if the symmetric $(0.2)$-tensor field $h = (L_\xi g)(\xi, \xi) + 2S(\xi, \xi)$ is parallel with respect to the Levi-Civita connection associated to $g$, then the $\eta$-Ricci soliton relation defines a Ricci soliton on $M$.

• Finally we illustrate the above results by means of example on $\alpha$-Sasakian manifolds.
**Publication/Communicated:**


In **Chapter-6** we study “On Lorentzian Para-Sasakian Manifolds” and we have the following results:

- A Pseudo-Projective $\phi$–Recurrent and $\phi$–Symmetric Lorentzian para-Sasakian Manifold is an Einstein Manifold.

- A special weakly Ricci-symmetric $LP$–manifold, the Ricci tensor is parallel.

- Let $(M^n, g)$ be a special Weakly Ricci-Symmetric $LP$–manifold with a cyclic parallel Ricci tensor. Then the 1–form $\gamma$ must vanish.

- If a special Weakly Ricci-symmetric $LP$–manifold is not an Einstein manifold, then 1–form $\gamma \neq 0$.

- A weakly conformally $\phi$–symmetric $LP$–Sasakian manifold is an $\eta$–Einstein manifold.

- If an $n$–dimensional $LP$–Sasakian manifold $M$ is:

  - Pseudo-projectively flat, then the manifold is an $\eta$–Einstein manifold and it is an space of constant curvature.
$M$—projectively flat and locally symmetric then it is an Einstein manifold.

- Finally, we study $\phi$—symmetric, locally $\phi$—symmetric and locally concircular $\phi$—symmetric admitting the semi-symmetric metric connection $\tilde{\nabla}$ with respective to the Levi-civita connection in LP-Sasakian manifold.

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The *Chapter-7* is devoted to “A Study of Ricci solitons Using Integral Formulas” and we have proved the following results:

- A Ricci soliton in $K$–contact manifold, $(M, g)$ is shrinking.

- A gradient Ricci soliton in $K$–contact manifold, $(M, g)$ shows that it is Einstein.

- Let $(M, g)$ be a Ricci soliton with respect to vector field $V$ on $M$. Then
  \[
  \int Rd\mu_g = -n\lambdaVol(M).
  \]
  \[
  div V = 0 \text{ if and only if either } R \leq -n\lambda \text{ or } R \geq -n\lambda \text{ on } M, \text{ then } R = -n\lambda.
  \]

- Let $(M, g)$ be a compact gradient Ricci soliton in $K$–contact manifold. If $(M, g)$ is also a gradient Ricci soliton in $K$–contact manifold with potential $f$, then up to a constant it agrees with Hodge-de Rham potential $h$.

- Let $(M, g)$ be a closed gradient Ricci soliton in $K$–contact manifold further, assume that $(M, g)$ is conformally flat $K$–contact manifold. Then $(M, g)$ is of constant sectional curvature.

- Let $(M, g)$ be a gradient Ricci soliton in Riemannian manifold which is shrinking in nature. Assume that if $(M, g)$ is conformally conservative Riemannian manifold. Then Riemannian manifold $(M, g)$ must be Einstein. Moreover conformal curvature tensor reduces to concircular curvature.

- Let $(M, g)$ be a gradient Ricci soliton in a Riemannian manifold which is shrinking in nature. Assume that if $(M, g)$ is conformally conservative Riemannian manifold. Then $(M, g)$ is concircularly flat if it is a space of constant sectional curvature.
Let \((M, g)\) be a gradient Ricci soliton in Riemannian manifold which is shrinking in nature. Assume that if \((M, g)\) is concircularly conservative Riemannian manifold. Then Riemannian manifold \((M, g)\) must be Einstein and the scalar curvature \(R\) is constant.

Let \((M, g)\) be a gradient Ricci soliton in Riemannian manifold which is shrinking in nature. Assume that if \((M, g)\) is pseudo-projective conservative Riemannian manifold. Then Riemannian manifold \((M, g)\) must be Einstein.

Let \((M, g)\) be a gradient Ricci soliton in Riemannian manifold which is shrinking in nature. Assume that if \((M, g)\) is conharmonic conservative Riemannian manifold. Then Riemannian manifold \((M, g)\) must be Einstein. Moreover conharmonic curvature tensor reduces to concircular curvature.

Let \((M, g)\) be a gradient Ricci soliton in a Riemannian manifold which is shrinking in nature. Assume that if \((M, g)\) is conharmonic conservative Riemannian manifold. Then \((M, g)\) is conharmonic flat if it is a space of constant sectional curvature.

**Publication/Communicated:**


Finally, the thesis ends with a list of bibliography and publications.