CHAPTER 5

Publications based on this Chapter;


- $\eta$-Ricci solitons in $\alpha$-Sasakian Manifolds, (Communicated).
Chapter 5

On Ricci solitons in $\alpha$–Sasakian Manifolds

5.1 Introduction

In this chapter we study both Ricci solitons and $\eta$–Ricci solitons using integral techniques. A Ricci soliton on a compact manifold is a gradient Ricci soliton. A Ricci soliton on a compact manifold has constant curvature in dimension 2 [82] and also in dimension 3 [88]. In [133], Perelman proved that a Ricci soliton on a compact $n$-manifold is a gradient Ricci soliton. In [174], R. Sharma studied Ricci solitons in K-contact manifolds, where the structure field $\xi$ is Killing and he proved that a complete K-contact gradient soliton is compact Einstein and Sasakian. In [191], M.M. Tripathi studied Ricci solitons in $N(k)$-contact metric and $(k,\mu)$ manifolds. In [8], Amadendu Ghosh and Ramesh Sharma studied $K$–contact metrics as Ricci solitons. In [120], H.G. Nagaraja and C.R. Premalatha studied Ricci Solitons in $f$-Kenmotsu Manifolds and 3-dimensional trans-Sasakian manifolds. Recently, C.S. Bagewadi and Gurupadavva Ingalahalli [16] studied Ricci solitons in Lorentzian $\alpha$-Sasakian Manifolds. Motivated by the above studies on Ricci solitons, we study Ricci solitons in an $\alpha$-Sasakian manifolds, and we have the following:
5.2 Ricci solitons in $\alpha$-Sasakian manifold

In this section we prove some theorems on Ricci solitons in $\alpha$-Sasakian manifold:

**Proposition 5.2.1.** A complete Einstein $\alpha$-Sasakian manifold is Compact.

**Proof.** Let $M$ be a complete Einstein $\alpha$-Sasakian manifold, then the general form is given by

$$S(X, Y) = \frac{r}{n}g(X, Y) \Rightarrow Q = \frac{r}{n}I. \quad (5.2.1)$$

Operating $\xi$ in (5.2.1) and by using the equation (1.10.8) shows $r = n(n - 1)\alpha^2$. Hence we get $Q = \alpha^2(n - 1)I$. So the Ricci curvatures are equal to $\alpha^2(n - 1)$ which is a positive constant. By Myer's theorem [117] we conclude that $M$ is compact. $\square$

**Theorem 5.2.2.** If the metric $g$ of an $\alpha$-Sasakian manifold $(M, g)$ is a gradient Ricci soliton, then the Ricci soliton is a shrinking soliton and $(M, g)$ is compact Einstein.

**Proof.** Equation (1.3.1) can be written as

$$\nabla_Y Df = QY + \lambda Y, \quad (5.2.2)$$

where $D$ denotes the gradient operator of $g$ and $Y$ an arbitrary vector field on $M$. Using this we derive

$$R(X, Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X. \quad (5.2.3)$$

Taking its inner product with $\xi$, substituting $X = \xi$ and using the equation (1.10.4) and (1.10.8), we have

$$Df = (\xi f)\xi. \quad (5.2.4)$$
Substituting (5.2.4) in (5.2.2), we get

$$Y(\xi f)\eta(X) - \alpha(\xi f)g(\phi Y, X) = S(X, Y) + \lambda g(X, Y).$$ (5.2.5)

Interchanging $X$ and $Y$ in (5.2.5), we have

$$X(\xi f)\eta(Y) - \alpha(\xi f)g(\phi X, Y) = S(Y, X) + \lambda g(Y, X).$$ (5.2.6)

Adding (5.2.5) and (5.2.6), we have

$$X(\xi f)\eta(Y) + Y(\xi f)\eta(X) = 2S(Y, X) + 2\lambda g(Y, X).$$ (5.2.7)

Putting $Y = \xi$ in (5.2.7), we have

$$X(\xi f) = [\alpha^2(n - 1) + \lambda]\eta(X).$$ (5.2.8)

The use of the above two equations provides

$$S(X, Y) = [\alpha^2(n - 1) + \lambda]\eta(X)\eta(Y) - \lambda g(Y, X).$$ (5.2.9)

Consequently, (5.2.2) assumes the form

$$\nabla_Y Df = [\alpha^2(n - 1) + \lambda]\eta(Y)\xi.$$ (5.2.10)

Using this we compute $R(X, Y)Df$ and taking inner product with $\xi$ (bearing in mind that $Df = (\xi f)\xi$) we obtain $[\alpha^2(n - 1) + \lambda] = 0$. Therefore, from equation (5.2.7) we have $X(\xi f) = 0$. That is, $\xi f$ is constant or $\xi f = c$. Hence equation (5.2.4) can be written as $df = c\eta$. Its exterior derivative implies $c\eta = 0$. Hence $c = 0$. Thus $f$ is constant.

Consequently, equation (5.2.2) reduces to $S = \alpha^2(n - 1)g$, that is an $\alpha$-Sasakian manifolds is an Einstein. Also, as $\lambda = -\alpha^2(n - 1)$ is negative for $\alpha > 0$ or $\alpha < 0$, that is Ricci soliton in $\alpha$-Sasakian manifolds is shrinking.
From above mentioned theorem we state the following corollary:

**Corollary 5.2.3.** If a metric $g$ of a compact $\alpha$-Sasakian manifold $(M, g)$ is a Ricci soliton, then $g$ is a shrinking soliton and the manifold is Einstein.

**Theorem 5.2.4.** If a metric $g$ in an $\alpha$-Sasakian manifold is a Ricci soliton with $V = \xi$, then it is Einstein.

*Proof.* Putting $V = \xi$ in (1.2.1), then we have

\[
(\mathcal{L}_\xi g + 2S + 2\lambda g)(X, Y) = 0. \tag{5.2.11}
\]

where

\[
(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi). \tag{5.2.12}
\]

Substituting (5.2.12) in (5.2.11), then we get the result. \(\square\)

**Proposition 5.2.5.** If an $\alpha$-Sasakian manifold is a Ricci soliton with $V$ point-wise collinear with $\xi$, then $V$ is a constant multiple of $\xi$ and the manifold is Einstein.

*Proof.* From (1.2.1), we have

\[
(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{5.2.13}
\]

where

\[
(\mathcal{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V). \tag{5.2.14}
\]

Substituting (5.2.14) in (5.2.13), then we obtain

\[
g(\nabla_X V, Y) + g(X, \nabla_Y V) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{5.2.15}
\]

Putting $V = \alpha \xi$ in (5.2.15), we get

\[
(X\alpha)\eta(Y) + (Y\alpha)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{5.2.16}
\]
Putting $X = Y = \xi$ in (5.2.16), we have

$$(\xi a) + \alpha^2(n - 1) + \lambda = 0.$$  \hfill (5.2.17)

Again putting $X = \xi$ in (5.2.16), we obtain

$$(Ya) = [-\alpha^2(n - 1) - \lambda] \eta(Y).$$  \hfill (5.2.18)

Equation (5.2.19) implies that

$$da = [-\alpha^2(n - 1) - \lambda] \eta.$$  \hfill (5.2.19)

Applying $d$ on both sides

$$d^2a = [-\alpha^2(n - 1) - \lambda] d\eta.$$  \hfill (5.2.20)

Equation (5.2.20) implies that $d^2a = 0$ but $d\eta$ is nowhere vanishing. Therefore,

$$-\lambda - \alpha^2(n - 1) = 0$$ which implies $da = 0$ that is, $a$ is constant. As $\xi$ is Killing, we conclude that the manifold is Einstein which completes the proof. \hfill \Box

**Definition 5.2.1.** A vector field $V$ is said to be Conformal Killing vector field if it satisfies

$$\mathcal{L}_V g = 2\rho g$$ \hfill (5.2.21)

for some scalar function $\rho$.

**Theorem 5.2.6.** Let $(g, V, \lambda)$ be a Ricci soliton in an $\alpha$-Sasakian manifolds $(M, g)$. Then $(M, g)$ is Ricci-semi symmetric if and only if $V$ is conformal Killing.

**Proof.** Suppose $V$ is a conformal killing vector field and from (1.2.1), we have

$$2\rho g(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0.$$ \hfill (5.2.22)
The above equation implies that

\[ S(X, Y) = [-\rho - \lambda]g(X, Y). \]  

(5.2.23)

This shows that the Ricci soliton is Einstein.

\[ QX = [-\rho - \lambda]X. \]  

(5.2.24)

Let \( M \) be an \( \alpha \)-Sasakian manifolds, then we have [80]

1. Einstein,

2. locally Ricci symmetric,

3. Ricci semi-symmetric that is \( R \cdot S = 0 \).

The implication (1) \( \rightarrow \) (2) \( \rightarrow \) (3) is trivial. Now, we prove the implication (3) \( \rightarrow \) (1).

Now,

\[ (R(X, Y) \cdot S)(U, V) = -S(R(X, Y)U, V) - S(U, R(X, Y)V). \]  

(5.2.25)

Considering \( R \cdot S = 0 \) and putting \( X = \xi \) in (5.2.25), we have

\[ S(R(\xi, Y)U, V) + S(U, R(\xi, Y)V) = 0. \]  

(5.2.26)

By using (1.10.4) in (5.2.26), we obtain

\[ \alpha^2[g(Y, U)S(\xi, V) - \eta(U)S(Y, V)] + \alpha^2[g(Y, V)S(U, \xi) - \eta(V)S(U, Y)] = 0. \]  

(5.2.27)

Putting \( U = \xi \) in (5.2.27) and by using (1.9.1), (1.10.6) and (1.10.7) on simplification, we obtain

\[ S(Y, V) = (n - 1)\alpha^2g(Y, V). \]  

(5.2.28)
Substituting (5.2.28) in (1.2.1), we have

$$\mathcal{L}_V g(X,Y) = \rho g(X,Y) \quad (5.2.29)$$

where $\rho = 2[-\alpha^2(n-1) - \lambda]$, that is $V$ is conformal Killing.

Now we study Ricci solitons in 3-dimensional $\alpha$-Sasakian manifolds:

**Theorem 5.2.7.** In a 3-dimensional $\alpha$-Sasakian manifolds, a Ricci soliton $(g,V,\lambda)$, where $V$ is conformal Killing vector field, is shrinking if $\rho = 2\alpha^2$ and expanding if $\rho < 2\alpha^2$ and $\rho > 2\alpha^2$.

**Proof.** Suppose $(M,g)$ is a 3-dimensional $\alpha$-Sasakian manifolds and $(g,V,\lambda)$ is a Ricci soliton in $(M,g)$. If $V$ is a conformal Killing vector field, then

$$\mathcal{L}_V g = 2\rho g \quad (5.2.30)$$

for some scalar function $\rho$.

In a 3-dimensional $\alpha$-Sasakian manifolds and from (1.2.1), we have

$$S(X,Y) = [-\rho - \lambda]g(X,Y), \quad (5.2.31)$$

$$QX = [-\rho - \lambda]X, \quad (5.2.32)$$

$$r = 3[-\rho - \lambda]. \quad (5.2.33)$$

In a 3-dimensional $\alpha$-Sasakian manifolds, the curvature tensor $R$ is given by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y$$

$$- \frac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \quad (5.2.34)$$

Using (5.2.31), (5.2.32) and (5.2.33) in (5.2.34), we get

$$R(X,Y)Z = \{2[-\rho - \lambda] - \frac{r}{2}\}[g(Y,Z)X - g(X,Z)Y] \quad (5.2.35)$$
Put $X = Z = \xi$ in (5.2.35), we get

$$R(\xi, Y)\xi = \{2[-\rho - \lambda - \frac{r}{2}]\eta(Y)\xi - Y\}. \quad (5.2.36)$$

In an $\alpha$-Sasakian manifolds $R(\xi, Y)\xi$ is given by

$$R(\xi, Y)\xi = \alpha^2[\eta(Y)\xi - Y]. \quad (5.2.37)$$

From (5.2.36) and (5.2.37), we have

$$\{2[-\rho - \lambda - \frac{r}{2} - \alpha^2]\eta(Y)\eta(W) - g(Y, W)\} = 0. \quad (5.2.38)$$

The above equation implies that

$$\{2[-\rho - \lambda - \frac{r}{2} - \alpha^2]\} = 0. \quad (5.2.39)$$

From (5.2.33) and (5.2.39), we have

$$\lambda = -[\rho + 2\alpha^2]. \quad (5.2.40)$$

- Let $\rho = 2\alpha^2$ implies $\lambda = -4\alpha^2$, that is, $\lambda < 0$. Hence Ricci soliton is shrinking.

- Let $\rho < 2\alpha^2$, suppose $\rho = -2\alpha^2 - 1$ which is $< 2\alpha^2$ implies $\rho + 2\alpha^2 = -1$, that is, $\lambda = 1 > 0$. Hence Ricci soliton is expanding.

- Let $\rho > 2\alpha^2$, that is, $-2\alpha^2 < -\rho$ if $-2\alpha^2 - 2\alpha^2 < -\rho - 2\alpha^2$ then $-4\alpha^2 < -\lambda$ implies $\lambda > 4\alpha^2$. Hence Ricci soliton is expanding.

If $\rho = 0$ in (5.2.30) then $\mathcal{L}_V g = 0$, i.e., conformal vector field doesn’t exists then the Ricci soliton is generalization of Einstein metric i.e., $V$ is a Killing vector field. On the bases of this condition we state the following:
Remark 5.2.1. A Ricci soliton in $\alpha$-Sasakian manifold is shrinking, if $V$ is a Killing vector field.

5.3 Example

Let $M = \{(x, y, z) \in R^3\}$. Let $(E_1, E_2, E_3)$ be linearly independent vector fields given by

$$E_1 = e^z \frac{\partial}{\partial y}, \quad E_2 = e^z \left[ \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial z} \right], \quad E_3 = \frac{\partial}{\partial z}. \quad (5.3.1)$$

Let $g$ be the Riemannian metric defined by $g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0, g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1$, where $g$ is given by

$$g = \frac{1}{e^{2z}}[(1 - 4e^{2z}y^2)dx \otimes dx + dy \otimes dy + e^{2z}dz \otimes dz].$$

Let $\eta$ be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \mathfrak{X}(M)$. Let $\phi$ be the $(1,1)$ tensor field defined by $\phi E_1 = E_2, \phi E_2 = -E_1, \phi E_3 = 0$. Then using the linearity of $\phi$ and $g$ yields that $\eta(E_3) = 1, \phi^2 U = -U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$ for any vector fields $U, W \in \mathfrak{X}(M)$. Thus for $E_3 = \xi, (\phi, \xi, \eta, g)$ defines a Sasakian structure on $M$. By definition of Lie bracket, we have

$$[E_1, E_2] = -e^z E_1 + 2e^{2z} E_3, \quad [E_1, E_3] = [E_2, E_3] = 0.$$

Let $\nabla$ be the Levi-Civita connection with respect to above metric $g$ Koszula formula is given by

$$2g(\nabla_x Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y))$$

$$-g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \quad (5.3.2)$$
Then

\[ \nabla_{E_1} E_1 = e^x E_2, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_3} E_3 = 0, \]
\[ \nabla_{E_1} E_2 = -e^x E_1 + e^{2x} E_3, \quad \nabla_{E_2} E_1 = -e^{2x} E_3, \quad \nabla_{E_2} E_3 = e^{2x} E_1, \]
\[ \nabla_{E_1} E_3 = -e^{2x} E_2, \quad \nabla_{E_2} E_2 = -e^{2x} E_2, \quad \nabla_{E_2} E_3 = e^{2x} E_1. \] (5.3.3)

Clearly \((\phi, \xi, \eta, g)\) structure is an \(\alpha\)-Sasakian structure and satisfy,

\[ \nabla_{\nabla X \phi} Y = \alpha (g(X, Y) \xi - \eta(Y) X), \quad \nabla_X \xi = -\alpha \phi X, \] (5.3.4)

where \(\alpha = e^{2x} \neq 0\). Hence \((\phi, \xi, \eta, g)\) structure defines \(\alpha\)-Sasakian structure. Thus \(M\) equipped with \(\alpha\)-Sasakian structure is a \(\alpha\)-Sasakian manifold. The tangent vectors \(X\) and \(Y\) to \(M\) are expressed as linear combination of \(E_1, E_2, E_3\), that is \(X = \sum_{i=1}^{3} a_i E_i\) and \(Y = \sum_{i=1}^{3} b_i E_i\), where \(a_i\) and \(b_i(i = 1, 2, 3)\) are scalars.

On \(\alpha\)-Sasakian manifold \((M, g)\), we have

\[ (\mathcal{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V) \] (5.3.5)

where \(\nabla\) denotes the Levi-Civita connection of \(M\). Hence if \((M, g)\) is a \(\eta\)-Ricci soliton with potential vector field \(V\), then (1.4.1) and (5.3.5), we have

\[ 2S(X, Y) = -g(\nabla_X V, Y) - g(X, \nabla_Y V) - 2\lambda g(X, Y) - 2\mu \eta(X) \eta(Y). \] (5.3.6)

By taking \(X = Y = e_i\) where \(e_i\) is an orthonormal basis and \(1 \leq i \leq n\), then we have

\[ \int_M \left[ \text{div} V + r + n\lambda + \mu \right] = 0. \] (5.3.7)

On integrating the above equation we have by Green's theorem \(\int \text{div} V = 0\) and for scalar curvature \(r\), then we have

\[ (r + n\lambda + \mu) \text{Vol}(M) = 0. \] (5.3.8)
The above equation implies that
\[ r = -(n\lambda + \mu). \tag{5.3.9} \]

For Ricci solitons \( \mu = 0 \), then
\[ \lambda = -\frac{r}{n}. \tag{5.3.10} \]

In \( \alpha \)-Sasakian manifolds scalar curvature \( r = \alpha^2(n - 1) \), we have
\[ \lambda = -\frac{\alpha^2(n - 1)}{n} < 0. \tag{5.3.11} \]

Hence, we state the following:

**Theorem 5.3.1.** A \( \eta \)-Ricci soliton in an \( \alpha \)-Sasakian is shrinking.

**Corollary 5.3.2.** If a metric \( g \) in an \( \alpha \)-Sasakian manifold is a \( \eta \)-Ricci soliton with \( V = \xi \) then it is \( \eta \)-Einstein.

**Proof.** Putting \( V = \xi \) in (1.4.1), then we have
\[
(L_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu \eta(X)\eta(Y). \tag{5.3.12}
\]
where
\[
(L_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi). \tag{5.3.13}
\]
Substituting (5.3.13) in (5.3.12), then we get the result. \( \square \)

**Proposition 5.3.3.** If an \( \alpha \)-Sasakian manifold is a \( \eta \)-Ricci soliton with \( V \) point-wise collinear with \( \xi \), then \( V \) is a constant multiple of \( \xi \) and the manifold is Einstein.

**Proof.**
\[
(L_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu \eta(X)\eta(Y) = 0 \tag{5.3.14}
\]
where
\[(L_v g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V).\tag{5.3.15}\]

Substituting (5.3.15) in (5.3.14), then we have
\[g(\nabla_X V, Y) + g(X, \nabla_Y V) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu \eta(X) \eta(Y) = 0.\tag{5.3.16}\]

Putting \(V = a\xi\) in (5.3.16), we have
\[(Xa)\eta(Y) + (Ya)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu \eta(X) \eta(Y) = 0.\tag{5.3.17}\]

Putting \(X = Y = \xi\) in (5.3.17), we have
\[(\xi a) + a^2(n - 1) + \lambda + \mu = 0.\tag{5.3.18}\]

Again putting \(X = \xi\) in (5.3.17), we have
\[(Ya) = [-\alpha (n - 1) - \lambda - \mu] \eta(Y).\tag{5.3.19}\]

Equation (5.3.20) implies that
\[da = [-\alpha^2(n - 1) - \lambda - \mu] \eta.\tag{5.3.20}\]

Applying \(d\) on both sides
\[d^2a = [-\alpha^2(n - 1) - \lambda - \mu] d\eta.\tag{5.3.21}\]

Since \(d^2a = 0\) but \(d\eta\) is nowhere vanishing. Therefore, \(-\lambda - \alpha^2(n - 1) - \mu = 0\) which implies \(da = 0\), i.e., \(a\) is constant. Hence on the bases of above hence we state that

\[\textbf{Theorem 5.3.4.} \text{ On an } \alpha\text{–Sasakian manifold, the contact form } \eta \text{ is closed if and only if }\]
\[\xi \text{ is integrable and the Nijenhuis tensor field of the structural endomorphism } \phi \text{ vanishes identically.}\]
Proof. From (1.10.2), we have

\[(d\eta)(X, Y) = \frac{1}{2}[X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])] \]

\[= \frac{1}{2}[g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi)] = -\alpha g(\phi X, Y). \] (5.3.22)

If \(\xi\) is integrable then \(d\eta = 0\).

Nijenhuis tensor field of the endomorphism is given by

\[N(\phi, X, Y) = \phi^3[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \]

\[= \phi^2\{\nabla_X Y - \nabla_Y X\} + \{\nabla_{\phi X} \phi Y - \nabla_{\phi Y} \phi X\} \]

\[- \phi\{\nabla_{\phi X} Y - \nabla_{\phi Y} X\} - \phi\{\nabla_X \phi Y - \nabla_{\phi Y} X\} \]

\[= 0. \] (5.3.23)

\[\square\]

5.4 Parallel symmetric second order tensors and Ricci Solitons in \(\alpha\)-Sasakian manifolds

Fix \(h\) a symmetric tensor field of \((0, 2)\)-type which we suppose to be parallel with respect to \(\nabla\) that is \(\nabla h = 0\). Applying the Ricci identity [136]

\[\nabla^2 h(X, Y; Z, W) - \nabla^2 h(X, Y; W, Z) = 0, \] (5.4.1)

we obtain the relation

\[h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0. \] (5.4.2)

Replacing \(Z = W = \xi\) in (5.4.2) and by using (1.10.3) and by the symmetry of \(h\), we have

\[2\alpha^2[h(Y)\eta(X) - \eta(X)h(Y, \xi)] = 0. \] (5.4.3)
Put \( X = \xi \) in (5.4.3) and by virtue of (1.9.1), we have

\[
2\alpha^2[\eta(Y)h(\xi, \xi) - h(Y, \xi)] = 0. \tag{5.4.4}
\]

Since \( \alpha^2 \neq 0 \), it results

\[
h(Y, \xi) = \eta(Y)h(\xi, \xi). \tag{5.4.5}
\]

Let us call a regular \( \alpha \)-Sasakian manifolds with \( \alpha^2 \neq 0 \) and remark that \( \alpha \)-Sasakian manifold is regular, where regularity means the nonvanishing of the Ricci curvature with respect to the generator of \( \alpha \)-Sasakian manifolds.

**Definition 5.4.1.** \( \xi \) is called semi-torse forming vector field for \( \alpha \)-Sasakian manifold if, for all vector fields \( X \):

\[
R(X, \xi)\xi = 0. \tag{5.4.6}
\]

From (1.10.3), we have \( R(X, \xi)\xi = \alpha^2[X - \eta(X)\xi] \) and therefore, if \( X \in \text{ker}\xi = \xi^\perp \), then \( R(X, \xi)\xi = \alpha^2 X \) and we obtain:

**Proposition 5.4.1.** *For an \( \alpha \)-Sasakian manifold the following are equivalent:*

1. is regular,
2. \( \xi \) is not semi-torse forming,
3. \( S(\xi, \xi) \neq 0 \), that is, \( \xi \) is non-degenerate with respect to \( S \).

Now, differentiating the equation (5.4.5) covariantly with respect to \( X \), we have

\[
(\nabla_X h)(Y, \xi) + h(\nabla_X Y, \xi) + h(Y, \nabla_X \xi) = [(\nabla_X \eta)(Y) + \eta(\nabla_X Y)]h(\xi, \xi)
\]

\[
+ \eta(Y)[(\nabla_X h)(Y, \xi) + 2h(\nabla_X \xi, \xi)]. \tag{5.4.7}
\]
By using the parallel condition $\nabla h = 0$, $\eta(\nabla_X \xi) = 0$ and (5.4.5) in (5.4.7), we have

$$h(Y, \nabla_X \xi) = (\nabla_X \eta)(Y) h(\xi, \xi). \quad (5.4.8)$$

By using (1.10.2) in (5.4.8), we get

$$-\alpha h(Y, \varphi X) = \alpha g(X, \varphi Y) h(\xi, \xi). \quad (5.4.9)$$

Replacing $X = \varphi X$ in (5.4.9), we get

$$\alpha [h(Y, X) - g(Y, X) h(\xi, \xi)] = 0. \quad (5.4.10)$$

Clearly $\alpha$ is a nonzero smooth function in $\alpha$--Sasakian manifold this implies that

$$h(X, Y) = g(X, Y) h(\xi, \xi), \quad (5.4.11)$$

the above equation implies that $h(\xi, \xi)$ is a constant, via (5.4.5). Now by considering the above condition we state the following theorem:

**Theorem 5.4.2.** A symmetric parallel second order covariant tensor in an $\alpha$--Sasakian manifold is a constant multiple of the metric tensor.

**Theorem 5.4.3.** Let $M$ be a $\alpha$--Sasakian manifold, the symmetric $(0, 2)$-tensor field $h := \{C^g\} X, Y) + 2S(X, Y) + 2\mu\nu(X)\eta(Y)$ is parallel with respect to the Levi-Civita connection associated to $g$. Then $(g, \xi, \lambda, \mu)$ yields an $\eta$--Ricci soliton.

**Proof.** Assume $h(\xi, \xi) = (\mathcal{L}_\xi g)(\xi, \xi) + 2S(\xi, \xi) + 2\mu\nu(\xi)\eta(\xi)$. Now (5.3.12), can be written in form

$$h(X, Y) = -2\lambda g(X, Y). \quad (5.4.12)$$
i.e.,

\[ \lambda = \frac{-1}{2} h(\xi, \xi). \]

(5.4.13)

Therefore, \((\mathcal{L}_\xi g)(\xi, \xi) + 2S(\xi, \xi) + 2\mu \eta(\xi)\eta(\xi) = -2\lambda g(\xi, \xi). \]

\[ \square \]

If \(\mu = 0\), then \((\mathcal{L}_\xi g)(\xi, \xi) + 2S(\xi, \xi) = -2\lambda g(\xi, \xi). \) Hence we conclude that

**Corollary 5.4.4.** On a \(\alpha\)-Sasakian manifold the symmetric \((0, 2)\)-tensor field \( h := (\mathcal{L}_\xi g)(\xi, \xi) + 2S(\xi, \xi) \) is parallel with respect to the Levi-Civita connection associated to \( g \), then the \(\eta\)-Ricci soliton relation defines a Ricci soliton on \( M \).