Chapter 1

Preliminaries

1.1 Introduction

In this section trigonometric and hyperbolic functions using differential equations, Gamma function, Laplace transform and Banach space are discussed. We consider a sequence of ordinary differential equations of the type $x^{(r)} \pm x = 0, \ r = 1, 2, \ldots$ with suitable initial conditions. Few properties of solutions of such equations are studied. For each value of $r$, we get solutions which are trigonometric or hyperbolic like functions. The trigonometric functions are widely used in all the areas of mathematics. We also introduce matrix trigonometric functions which takes trigonometry to new fronteirs.

Secondly, another important aspect is Euler Gamma function which play an important role in the future development of the results of our thesis. In 1729, Euler introduced the concept of Gamma function. At about same time J. Sterling published a different formula but was unable to show that it always converges. However, Hermite proved that the formula given by Sterling does work and defined the same as Eulers. We discuss Gamma function in section 1.3.
There are various methods of solving linear differential equations, but the integral transform technique is the simplest and the basic technique widely used for solving linear ordinary as well as partial differential equations with initial conditions and boundary conditions. In course of time different integral transforms are developed to tackle different physical phenomenas governed by IVP/BVP. Among the different transforms, Laplace transform is very useful in the study of differential equations. Some prerequisite is discussed in the last section of this chapter.

We organise the chapter as follows: The section 1.2 and 1.3 consists study of trigonometric and hyperbolic functions using differential equations. In the section 1.4 we study extended trigonometric functions. In section 1.5 we extend the idea of trigonometric and hyperbolic functions to matrix functions. The Euler Gamma function is introduced in section 1.6. The Laplace transform of some elementary functions is discussed in section 1.7. Linear spaces and Banach spaces are discussed in the last section.

1.2 Trigonometric Functions

The course of Elementary Trigonometry is covered at a high school level. The trigonometrical relations are widely used in all areas of mathematics. In this section we discuss an alternative approach in developing trigonometrical functions and their properties through methods of differential equations [7]. This approach is precise, elegant and rewarding in respect of matrix trigonometric functions.
It is known that the linear scaler homogeneous differential equation of second order
\[ x'' + x = 0, \quad x(0) = 1, \quad x'(0) = 0, \] (1.1)
\[ (\frac{d}{dt}, \quad t \in R, \text{ a set of real numbers}) \]
has basic solutions \( \cos t \) and \( \sin t \). Yet without resorting to solving this initial value problem (IVP), it is possible to establish the properties of solutions by employing analytical tools.

The IVP (1.1) is equivalent to the system of equations
\[ x_1' = -x_2, \quad x_2' = x_1 \]
\[ x_1(0) = 1, \quad x_2(0) = 0, \quad t \in R \]
(1.2)
or
\[ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \]

We derive below only few trigonometrical properties of (1.1) or (1.2).

(i) Clearly \( x_1 x_1' + x_2 x_2' = \frac{1}{2} d(x_1^2 + x_2^2) = 0. \)

After integration between 0 and \( t \) and employing the initial conditions, we have the Wronskian \( W \) is given by
\[ W(x_1, x_2) = \begin{bmatrix} x_1(t) & x_2(t) \\ -x_2(t) & x_1(t) \end{bmatrix} = x_1^2(t) + x_2^2(t) = 1. \] (1.3)

This is a basic trigonometrical relation.

(ii) For any constant \( \alpha \in R, \) \( x_1(t + \alpha), \) \( x_2(t + \alpha) \) are also solutions of the system in (1.2). Let
\[ x_1(t + \alpha) = c_1 x_1(t) + c_2 x_2(t). \]
For $t = 0$, $c_1 = x_1(\alpha)$. Differentiating this relation, we find for $t = 0$, $c_2 = -x_2(\alpha)$. Hence
\[ x_1(t + \alpha) = x_1(t)x_1(\alpha) - x_2(\alpha)x_2(t). \]

Similarly, $x_2(t + \alpha) = x_2(t)x_1(\alpha) + x_2(\alpha)x_1(t)$.

Many other trigonometrical relations can be derived by using these addition formulae. see [7]

(iii) We find that the characteristic equation for (1.1) is $r^2 + 1 = 0$ having roots $\pm i$. By using infinite series solution method, we have
\[ x_1(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots + (-1)^n \frac{t^{2n}}{(2n)!} + \cdots \]
\[ = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} \]
\[ = \cos t = M_{20}(t), \text{(say)}. \]
\[ x_2(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots + (-1)^n \frac{t^{2n+1}}{(2n + 1)!} + \cdots \]
\[ = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n + 1)!} \]
\[ = \sin t = M_{21}(t), \text{(say)}. \]

Note the nature of new appropriate notation introduced here. Clearly
\[ e^{it} = M_{20}(t) + iM_{21}(t), \]
\[ e^{-it} = M_{20}(t) - iM_{21}(t), \]

which when solved for $M_{20}$ and $M_{21}$ yield,
\[ M_{20}(t) = \frac{e^{it} + e^{-it}}{2} \text{ and } M_{21}(t) = \frac{e^{it} - e^{-it}}{2i} \]
(iv) Further for any integer \( n \)

\[
(e^{it})^n = (M_{20}(t) + iM_{21}(t))^n
= e^{i(tn)} = M_{20}(tn) + iM_{21}(tn).
\]

This is well-known De Moivre’s Theorem.

We aim at extending these results to matrix trigonometrical functions. In order to define such functions and their properties, we define below norm of a matrix.

**Definition 1.2.1** The norm of a matrix \( T = [a_{ij}] \) of order \( n \) is denoted by \( ||T|| \) and is defined as

\[
||T|| = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.
\]

We need the following proposition.

**Proposition 1.2.1** Suppose

\[
f_1(t) = \sum_{i=0}^{\infty} a_i t^i \quad \text{and} \quad f_2(t) = \sum_{i=0}^{\infty} b_i t^i
\]

be scalar functions convergent for \( |t| < \rho, \ \rho > 0 \).

\((a_i, b_i, i = 0, 1, 2, \cdots \) are known constants.\)

Suppose that \( p(t) = f_1(t) + f_2(t) \) and \( q(t) = f_1(t) \cdot f_2(t) \). Then the matrix functions defined below, namely

\[
p(T) = f_1(T) + f_2(T),
\]

and \( q(T) = f_1(T) \cdot f_2(T) \);

are convergent for \( ||T|| < r \).

The proof is obvious. see [24]
Theorem 1.2.1 Suppose $T$ is any matrix of order $n$. Then,

(i) $M_{20}^2(T) + M_{21}^2(T) = I,$

(ii) $M_{21}(2T) = 2M_{21}(T) M_{20}(T)$

Proof: (i) Consider the scalar trigonometric functions

$$M_{20}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!}$$

and

$$M_{21}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}, \quad t \in \mathbb{R}$$

The infinite series for $M_{20}(t)$ and $M_{21}(t)$ are convergent for $t \in \mathbb{R}$. Assume that

$$f_1(t) = M_{20}^2(t) = \sum_{i=0}^{\infty} a_i t^i,$$

$$and \ f_2(t) = M_{21}^2(t) = \sum_{i=0}^{\infty} b_i t^i.$$ 

Observe that the coefficients $a_i$ and $b_i$ are uniquely determined by squaring the series for $M_{20}(t)$ and $M_{21}(t)$. Now from (1.3)

$$p(t) = f_1(t) + f_2(t)$$

$$= M_{20}^2(t) + M_{21}^2(t) = \sum_{i=0}^{\infty} (a_i + b_i) t^i = 1$$

Hence, $a_0 + b_0 = 1$ and $a_i + b_i = 0, \ i = 1, 2, \cdots.$ (1.4)

Consider the infinite series involving $T$ given by

$$M_{20}(T) = \sum_{n=0}^{\infty} (-1)^n \frac{T^{2n}}{(2n)!} \quad and \quad (1.5)$$

$$M_{21}(T) = \sum_{n=1}^{\infty} (-1)^n \frac{T^{2n+1}}{(2n+1)!}. \quad (1.6)$$
These series are well defined. Set
\[ P(T) = f_1(T) + f_2(T) \]
\[ = M_{20}^2(T) + M_{21}^2(T) \]
\[ = \sum_{i=0}^{\infty} (a_i + b_i)T^i = T^0 = I, \quad (I : n \times n \text{ Identity matrix}) \]
in view of (1.4).

Thus we have established property (i) of matrix trigonometric functions, namely
\[ M_{20}^2(T) + M_{21}^2(T) = I, \quad (1.7) \]
for any \( n \times n \) matrix \( T \).

(ii) From the addition formula for trigonometric functions we have
\[ M_{21}(2t) = 2M_{21}(t)M_{20}(t), \]
\[
\text{Let } M_{21}(t)M_{20}(t) = \sum_{i=0}^{\infty} \eta_i t^i, (\eta_i \text{ exists uniquely}), \quad t \in \mathbb{R}.
\]
\[ 2M_{21}(T)M_{20}(T) = \sum_{i=0}^{\infty} 2\eta_i T^i = M_{21}(2T) \]

The following properties of matrix trigonometric functions (addition formulae) are immediate.

(i) \[ M_{20}(T + \alpha) = M_{20}(T)M_{20}(\alpha) - M_{21}(T)M_{20}(\alpha), \]
\[ M_{21}(T + \alpha) = M_{21}(T)M_{20}(\alpha) + M_{21}(\alpha)M_{20}(T), \]
(\( \alpha \in \mathbb{R}^{n \times n} \) is a constant \( n \times n \) matrix and \( T\alpha = \alpha T \));

(ii) \[ M_{20}(T) = \frac{e^{iT} + e^{-iT}}{2}, \quad M_{21}(T) = \frac{e^{iT} - e^{-iT}}{2}; \]

(iii) \[ (M_{20}(T) + iM_{21}(T))^n = M_{20}(Tn) + iM_{21}(Tn) \quad (1.8) \]
This is De Moivre’s Theorem for matrix trigonometric functions and is true for any rational number \( n \).

**Example 1.2.1** Let \( T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \), find \( M_{20}(T) \) and \( M_{21}(T) \) and show that \( M_{20}^2(T) + M_{21}^2(T) = I \). Verify the relation (1.8)

Observe that \( T^n = 0, \ n \geq 3 \). From the relations (1.5) and (1.6), we have
\[
M_{20}(T) = I - \frac{T^2}{2!} \quad \text{and} \quad M_{21}(T) = T.
\]

Further,
\[
M_{20}^2(T) + M_{21}^2(T) = \left( I - \frac{T^2}{2!} \right)^2 + T^2 = I - T^2 + \frac{T^4}{4} + T^2 = I.
\]

For \( n = 2 \),
\[
(M_{20}(T) + iM_{21}(T))^2 = \left( I + iT - \frac{T^2}{2!} \right)^2 = (I - 2T)^2 + i(2T) = M_{20}(2T) + iM_{21}(2T).
\]

To prove the above result for any integer \( n \), use the method of mathematical induction.

**Example 1.2.2** For \( A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \), verify that
\[
M_{20}(A + B) = M_{20}(A)M_{20}(B) - M_{21}(A)M_{20}(B),
\]
\[
M_{21}(A + B) = M_{21}(A)M_{20}(B) + M_{21}(B)M_{20}(A)
\]
Here, \( M_{20}(A) = I - \frac{A^2}{2!} \), \( M_{21}(A) = A, (A^n = B^n = 0 \text{ for } n \geq 3) \)

\[
M_{20}(B) = I - \frac{B^2}{2!}, \quad M_{21}(B) = B
\]

Now, L. H. S. = \( M_{20}(A + B) = I - \frac{(A + B)^2}{2} + \frac{(A + B)^4}{24} \)

\[
= I - \frac{A^2}{2} - \frac{AB}{2} + \frac{BA}{2} + \frac{A^2B^2}{8} + \frac{B^2A^2}{8}
\]

R. H. S. = \( \left( I - \frac{A^2}{2} \right) \left( I - \frac{B^2}{2} \right) - \frac{AB}{2} - \frac{BA}{2} \)

\[
= I - \frac{A^2}{2} - \frac{B^2}{2} + \frac{A^2B^2}{8} + \frac{B^2A^2}{8} - \frac{AB}{2} - \frac{BA}{2}
\]

Here, L. H. S. = R. H. S. Second statement follows. Observe that in this example \( AB \neq BA \), still the conclusion holds. Clearly commutivity condition on involved matrices is strong.

### 1.3 Hyperbolic Functions

Consider the IVP

\[
x'' - x = 0, \quad x(0) = 1, \quad x'(0) = 0, \quad (1.9)
\]

Related system of equations is given by

\[
x' = y, \quad y' = x, \quad t \in \mathbb{R}
\]

\[
x(0) = 1, \quad y(0) = 0, \quad (1.10)
\]

Assume that \( x_h(t), y_h(t), \quad t \in \mathbb{R} \) represents solution of (1.10).

Therefore, \( x'_h(t) = y_h(t), y'_h(t) = x_h(t) \)

\[
x_h(0) = 1, \quad y_h(0) = 0.
\]
We get \[ x_h(t) \ x'_h(t) - y_h(t) \ y'_h(t) = 0 \]

Integrating, \[ x^2_h(t) - y^2_h(t) = k \]

Using initial conditions we get \( k = 1 \) and hence

\[ x^2_h(t) - y^2_h(t) = 1 \] (1.11)

\( x_h(t) \) and \( y_h(t) \) are called hyperbolic functions.

(i) For any constant \( \alpha \in R \), \( x_h(t+\alpha) \), \( x_h(t+\alpha) \) are also solutions of the system in (1.10). Let

\[ y_h(t + \alpha) = c_1 y_h(t) + c_2 x_h(t). \]
\[ x_h(t + \alpha) = d_1 y_h(t) + d_2 x_h(t). \]

\( c_1 \), \( c_2 \), \( d_1 \), \( d_2 \) are constants.

For \( t = 0 \), \( c_2 = y_h(\alpha) \); \( d_2 = x_h(\alpha) \). Differentiating this relation, we find for \( t = 0 \), \( c_1 = x_h(\alpha) \); \( d_1 = y_h(\alpha) \). Hence

\[ y_h(t + \alpha) = y_h(t) x_h(\alpha) - y_h(\alpha) x_h(t). \]

Similarly, \( x_h(t + \alpha) = y_h(t) y_h(\alpha) + x_h(\alpha) x_h(t). \)

Many other relations can be derived by using these addition formulae. see [7]

(ii) We find that the characteristic equation for (1.9) is \( r^2 - 1 = 0 \) having roots \( \pm 1 \). By using infinite series solution method, we have
\[ x_h(t) = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots + \frac{t^{2n}}{(2n)!} + \cdots \]
\[ = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \]
\[ = \cosh t = N_{20}(t), \text{ (say).} \]
\[ y_h(t) = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots + \frac{t^{2n+1}}{(2n+1)!} + \cdots \]
\[ = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \]
\[ = \sinh t = N_{21}(t), \text{ (say).} \]

Note the nature of new appropriate notation introduced here for hyperbolic functions. Clearly

\[ e^t = N_{20}(t) + N_{21}(t), \]
\[ e^{-t} = N_{20}(t) - N_{21}(t), \]

which when solved for \( N_{20} \) and \( N_{21} \) yield,

\[ N_{20}(t) = \frac{e^t + e^{-t}}{2} \quad \text{and} \quad N_{21}(t) = \frac{e^t - e^{-t}}{2} \]

(iii) Further, for any integer \( n \)

\[ (e^t)^n = (N_{20}(t) + N_{21}(t))^n \]
\[ e^{(tn)} = N_{20}(tn) + N_{21}(tn). \]

Next, employing similar logic stated for matrix trigonometrical functions we get
for any $n \times n$ matrices $A$ and $B$,

\begin{align*}
(i) & \quad \cosh^2 A - \sinh^2 A = I, \\
(ii) & \quad \cosh (A + B) = \cosh A \cosh B + \sinh A \sinh B \\
& \quad \sinh (A + B) = \sinh A \cosh B + \cosh A \sinh B \\
(iii) & \quad \cosh A = \frac{e^A + e^{-A}}{2}, \quad \sinh A = \frac{e^A - e^{-A}}{2}, \\
(iv) & \quad (\cosh A + \sinh A)^n = \cosh nA + i \sinh nA.
\end{align*}

For a generalized notation for matrix hyperbolic functions $\cosh T$ and $\sinh T$ we have

**Definition 1.3.1** For any $n \times n$ matrix $T$,

\begin{align*}
\cosh T &= N_{20}(T) = \sum_{n=0}^{\infty} \frac{T^{2n}}{(2n)!} \quad \text{and} \quad (1.12) \\
\sinh T &= N_{21}(T) = \sum_{n=0}^{\infty} \frac{T^{2n+1}}{(2n+1)!} \quad (1.13)
\end{align*}

### 1.4 Extended Trigonometric Functions

In place of equation (1.1), let us consider a similar IVP for third order differential equation,

\begin{align*}
x''' + x &= 0, \quad x(0) = 1, \quad x'(0) = 0, \quad x''(0) = 0; \quad (1.14) \\
(\prime = \frac{d}{dt}, \quad t \in \mathbb{R}).
\end{align*}

It is easy to verify that the following linearly independent func-
tions satisfy (1.14). These are

\[ M_{30}(t) = 1 - \frac{t^3}{3!} + \frac{t^6}{6!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{3n}}{(3n)!}, \]

\[ M_{31}(t) = t - \frac{t^4}{4!} + \frac{t^7}{7!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+1}}{(3n+1)!}, \]

\[ M_{32}(t) = \frac{t^2}{2!} - \frac{t^5}{5!} + \frac{t^{11}}{11!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+2}}{(3n+2)!}. \]  

(1.15)

The system representation of (1.14) is

\[ x_1' = -x_3, \quad x_2' = x_1, \quad x_3' = x_2, \quad x_1(0) = 1, \quad x_2(0) = x_3(0) = 0; \]

i.e.

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
= \begin{bmatrix}
  0 & 0 & -1 \\
  1 & 0 & 0 \\
  0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix},
\begin{bmatrix}
  x_1(0) \\
  x_2(0) \\
  x_3(0)
\end{bmatrix}
= \begin{bmatrix}
  1 \\
  0 \\
  0
\end{bmatrix}.
\]

Observe that \( x_1(t) = M_{30}(t), x_2(t) = M_{31}(t), x_3(t) = M_{32}(t) \).

We list below some properties of these functions.

(i) \( W(M_{30}, M_{31}, M_{32})(t) = \)

\[
\begin{bmatrix}
  M_{30}(t) & M_{31}(t) & M_{32}(t) \\
  M'_{30}(t) & M'_{31}(t) & M'_{32}(t) \\
  M''_{30}(t) & M''_{31}(t) & M''_{32}(t)
\end{bmatrix}
= \begin{bmatrix}
  M_{30}(t) & M_{31}(t) & M_{32}(t) \\
  -M_{32}(t) & M_{30}(t) & M_{31}(t) \\
  -M_{31}(t) & -M_{32}(t) & M_{30}(t)
\end{bmatrix}
= M_{30}^3 - M_{31}^3 + M_{32}^3 + 3M_{30}M_{31}M_{32} = 1 \text{ for each } t. \]  

(1.16)

This is a basic relation [similar to 1.3] for the third order. This fact suggests that we can get similar basic relations for differential equations of higher orders \( n = 4, 5, \ldots \).
(ii) By following the method in the section 1.2, we can prove the addition formulae for any $t \in R$,

$$M_{30}(t + \alpha) = M_{30}(t)M_{30}(\alpha) - M_{31}(t)M_{32}(\alpha) - M_{32}(t)M_{31}(\alpha)$$  \hspace{1cm} (1.17)

$$M_{31}(t + \alpha) = M_{30}(t)M_{31}(\alpha) + M_{31}(t)M_{30}(\alpha) - M_{32}(t)M_{32}(\alpha)$$  \hspace{1cm} (1.18)

$$M_{32}(t + \alpha) = M_{30}(t)M_{32}(\alpha) + M_{31}(t)M_{31}(\alpha) + M_{32}(t)M_{30}(\alpha)$$  \hspace{1cm} (1.19)

Putting $\alpha = 1$, we can obtain $M_{30}(2t), M_{31}(2t)$ and $M_{32}(2t)$. \hspace{1cm} (1.20)

(iii) The characteristic equation of (1.14) is $Z^3 + 1 = 0$, having roots $-1, \frac{1}{2}(1 - i\sqrt{3}), \frac{1}{2}(1 + i\sqrt{3})$. Denote these roots by $-1, \omega$ and $-\omega^2$ respectively. Then

$$e^{-t} = M_{30}(t) - M_{31}(t) + M_{32}(t)$$

$$e^{\omega t} = M_{30}(t) + \omega M_{31}(t) + \omega^2 M_{32}(t)$$

$$e^{-\omega^2 t} = M_{30}(t) - \omega^2 M_{31}(t) - \omega M_{32}(\alpha)$$  \hspace{1cm} (1.21)

Solving these equations we get $M_{30}, M_{31}$ and $M_{32}$ in terms of $e^{-t}, e^{\omega t}$ and $e^{-\omega^2 t}$, [8].

(iv) To prove De Moivre’s theorem, we observe that for any integer $n$

$$[e^{\omega t}]^n = [M_{30}(t) + \omega M_{31}(t) + \omega^2 M_{32}(t)]^n$$

$$= e^{\omega(nt)} = e^{(n\omega)t}$$

$$= M_{30}(n\omega) + \omega M_{31}(n\omega) + \omega^2 M_{32}(n\omega).$$

We now define extended matrix trigonometric functions.
**Definition 1.4.1** The extended trigonometric functions are denoted by $M_{30}(T)$, $M_{31}(T)$, $M_{32}(T)$ and defined as

\[ M_{30}(T) = \sum_{n=0}^{\infty} (-1)^n \frac{T^{3n}}{(3n)!}, \]
\[ M_{31}(T) = \sum_{n=0}^{\infty} (-1)^n \frac{T^{3n+1}}{(3n + 1)!}, \text{ and} \]
\[ M_{32}(T) = \sum_{n=0}^{\infty} (-1)^n \frac{T^{3n+2}}{(3n + 2)!} \]  

(1.22)

for any matrix of order $n$

These infinite series are convergent for all $T$, $||T|| < \infty$. We can now rewrite (1.17), (1.18), (1.19) and (1.20) by replacing $t$ by $T$ and $\alpha$ by any matrix of order $n$. Then formulae (1.17), (1.18), (1.19) and (1.20) hold.

**Example 3.** For the matrix $T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

find $M_{30}(T)$, $M_{31}(T)$, $M_{32}(T)$ and verify (8.10).

Here we observe that $T^n = 0$ for $n \geq 4$. Hence from the definition, we have

\[ M_{30}(T) = E_4 - \frac{T^3}{3!}, M_{31}(T) = T, M_{32}(T) = \frac{T^2}{2!}. \]
To verify the basic relation (8.10) for $T$, we find that

$$M_{30}^3 - M_{31}^3 + M_{32}^3 + 3M_{30}M_{31}M_{32}$$

$$= \left(E_4 - \frac{T^3}{3!}\right)^3 - T^3 + \frac{T^6}{(2!)^3} + 3 \left(E_4 - \frac{T^3}{3!}\right)(T) \left(-\frac{T^2}{2!}\right)$$

$$= E_4.$$

**Example 4.** For the matrix $T$ as given in Example 3 above, find

$$M_{30}(2T), M_{31}(2T), M_{32}(2T).$$

Replacing $t$ by $T$ and $\alpha$ by $T$ in (8.11), we have

$$M_{30}(2T) = M_{30}^2(T) - 2M_{31}(T)M_{32}(T)$$

$$= \left(E_4 - \frac{T^3}{3!}\right)^2 - 2T \left(T^2 \frac{T^2}{2!}\right)$$

$$= E_4 - 2T^3 \frac{T^3}{3!} + \frac{T^6}{(3!)^2} - T^3 = E_4 - \frac{4T^3}{3};$$

$$M_{31}(2T) = 2M_{30}(T)M_{31}(T) - [M_{32}(T)]^2$$

$$= 2 \left(E_4 - \frac{T^3}{3!}\right)T - \frac{T^4}{4} = T;$$

$$M_{32}(2T) = 2M_{30}(T)M_{32}(T) + [M_{31}(T)]^2$$

$$= 2 \left(E_4 - \frac{T^3}{3!}\right) \frac{T^2}{2} + T^2 = 2T^2.$$

### 1.5 Matrix Functions

The matrix trigonometric function in the above situation is called the matrix extension of the scalar function. One important matrix
extension is the matrix exponential

\[ \exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n. \]

This function is defined for all matrices because

\[ \exp(t) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \] converges for all \( t \in \mathbb{R}. \)

The method for evaluating the functions of a matrix is based on the application of Cayley-Hamilton Theorem and this method is applicable to any matrix. We describe the procedure below and provide logical steps in the proof by considering simple matrices.

**Proposition 1.5.1** [8] If the degree of minimal polynomial of a matrix \( A \) is \( n \), any function \( f(A) \) can be expressed as a linear combination of the \( m \) linear independent matrices \( I, A, \ldots, A^{n-1} \)

\[
f(A) = \alpha_{n-1} A^{n-1} t^{n-1} + \alpha_{n-2} A^{n-2} t^{n-2} + \ldots \\
+ \cdots + \alpha_2 A^2 t^2 + \alpha_1 A t + \alpha_0 I_n.
\] (1.23)

where \( \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \) are functions of \( t \in I \). Define

\[
r(\lambda) = \alpha_{n-1} \lambda^{n-1} + \alpha_{n-2} \lambda^{n-2} + \cdots + \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0.
\] (1.24)

We first find the eigenvalues of the matrix \( A t \). Let \( \lambda_j \) be one of the distinct eigenvalues repeated \( k \)-times, the algebraic functions \( f(\lambda) \) and \( r(\lambda) \) satisfy the following \( k \) equations.
\[ f(\lambda_j) = r(\lambda_j) ; \]
\[ \frac{d}{d\lambda} f(\lambda_j) = \frac{d}{d\lambda} |r(\lambda)| \text{ at } \lambda = \lambda_j \] (1.25)

\[ \vdots \]
\[ \frac{d^{k-1}}{d\lambda^{k-1}} f(\lambda_j) = \frac{d^{k-1}}{d\lambda^{k-1}} |r(\lambda)| \text{ at } \lambda = \lambda_j . \]

\( \frac{d^k}{d\lambda^k} \) denotes the \( k^{th} \) derivative of \( r(\lambda) \) evaluated at \( \lambda = \lambda_k \). From the polynomial relation (1.24), it is easy to evaluate relations in (1.25). Relations (1.25) are to be written for each distinct eigenvalue \( \lambda_j \). Here (1.25) consists of \( k \) simultaneous equations which depend on \( \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \) as unknown terms. Hence \( \alpha_j(t), j = 0, 1, 2, \ldots, n \) are uniquely determined. Substitute these \( \alpha_j \)'s in (1.23) to get finally \( f(A) \). We now work out these steps for \( 2 \times 2 \) matrix \( A \) having eigenvalues \( \lambda_1, \lambda_2 \), real and distinct. We have the following examples illustrating various situations.

**Example 1.5.1** Given the matrix \( A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \) we find basic matrix trigonometric functions which we need in solving various types of differential and integral equations.

Using (1.23) and (1.24) above and using given matrix \( A \),

(i) We write, \( e^{At} = \alpha_0 E_2 + \alpha_1 A t, \quad r(\lambda) = \alpha_0 + \alpha_1 \lambda, \quad r(\lambda_1) = \alpha_0 + \alpha_1 \lambda_1 \) and \( r(\lambda_2) = \alpha_0 + \alpha_1 \lambda_2, \quad (\lambda_1 \neq \lambda_2) \)

For the given matrix \( A \) the eigenvalues are \( \lambda_1 = -t, \lambda_2 = 5t. \)
Hence \( e^{-t} = \alpha_o - \alpha_1 t \), \( e^{5t} = \alpha_0 + 5\alpha_1 t \) which yield
\[
\alpha_0 = \frac{1}{6}(e^{5t} + 5e^{-t}) \quad \text{and} \quad \alpha_1 = \frac{1}{6t}(e^{5t} - e^{-t})
\]
Substituting these values of \( \alpha_0 \) and \( \alpha_1 \) in the equation \( e^{At} = \alpha_0 E_2 + \alpha_1 At \) we get,
\[
e^{At} = \frac{1}{6} \left[ 2e^{5t} + 4e^{-t} \quad 2e^{5t} - 2e^{-t} \right]
\]
Similarly, we get
\[
a^{At} = \frac{1}{6} \left[ 2a^{5t} + 4a^{-t} \quad 2a^{5t} - 2a^{-t} \right]
\]
( ii ) We write, \( \sin At = \alpha_0 E_2 + \alpha_1 At \), \( r(\lambda) = \alpha_0 + \alpha_1 \lambda \), \( r(\lambda_1) = \alpha_0 + \alpha_1 \lambda_1 \) and \( r(\lambda_2) = \alpha_0 + \alpha_1 \lambda_2 \), \( (\lambda_1 \neq \lambda_2) \)
For the given matrix \( \mathbf{A} t \) the eigenvalues are \( \lambda_1 = -t \), \( \lambda_2 = 5t \).
Hence \( -\sin At = \alpha_o - \alpha_1 t \), \( \sin 5t = \alpha_0 + 5\alpha_1 t \) which yield
\[
\alpha_0 = \frac{1}{6}(\sin 5t - 5\sin t) \quad \text{and} \quad \alpha_1 = \frac{1}{6t}(\sin 5t + \sin t)
\]
Substituting these values of \( \alpha_0 \) and \( \alpha_1 \) in the equation \( \sin At = \alpha_0 E_2 + \alpha_1 At \) we get,
\[
\sin At = \frac{1}{6} \left[ 2\sin 5t - 4\sin t \quad 2\sin 5t + 2\sin t \right]
\]
Similarly, we get
\[
\cos At = \frac{1}{6} \left[ 2\cos 5t + 4\cos t \quad 2\cos 5t - 2\cos t \right]
\]
\[
tan At = \frac{1}{6} \left[ 2\tan 5t - 4\tan t \quad 2\tan 5t + 2\tan t \right]
\]
The above procedure can be used in finding other trigonometric, hyperbolic and extended matrix functions. See [24]

1.6 Euler Gamma Function.

The gamma function can be thought of as the natural way to generalize the concept of the factorial to non-integer arguments.

Euler’s original formula for the gamma function is

\[
\Gamma(t + 1) = \lim_{n \to \infty} \frac{n^{z+1} \ n!}{\prod_{k=0}^{n} \ (t + 1 + k)}
\]

However, it is now more commonly defined by

\[
\Gamma(\alpha + 1) = \int_{0}^{\infty} e^{-t} \ t^\alpha \ dt \quad (1.26)
\]

which converges for all \( \alpha > -1 \). Using integration by parts, we get the important property of the Gamma function namely,

\[
\Gamma(\alpha + 1) = \alpha \ \Gamma(\alpha) \quad (1.27)
\]

Let \( \alpha = 0 \) in (1.27) which gives \( \Gamma(1) = 1 \). For \( \alpha \), a positive integer, \( \Gamma(\alpha + 1) = \alpha! \). It is interesting to note that \( \Gamma(\alpha) \) is defined by (1.26) for all real values except \( \alpha = 0, -1, -2, \ldots \). Here (1.26) does not give \( \Gamma(\alpha + 1) \) for \( \alpha \leq -1 \) because the behaviour of \( t^\alpha \) at \( t = 0 \) makes the integral divergent. For details see [18] and [27]. The graph of \( \Gamma(\alpha) \) (Figure 1) has the appearance as given below and is given to indicate the nature of Gamma function which we use in subsequent development.

The Gamma function above is defined by a definite integral. The
incomplete gamma function is defined as an integral function of the same integrand. There are two varities of the incomplete gamma function defined as follows.

The upper incomplete gamma function is defined as

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} \, dt$$

The lower incomplete gamma function is defined as

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t} \, dt$$

1.7 Laplace Transform.

Integral transform is a method for solving linear differential equations arising in Science and Technology. It reduces the problem of solving a differential equation to an algebraic equations, [18], [27].
The Laplace transform converts a function of one variable to a function of the complex transform variable $s$. This transformation is achieved by the formula

$$F(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt$$  \hspace{1cm} (1.28)$$

where $s$ is such that the integral exists throughout its range. Laplace transform allows us to deal with differential equations. Using Laplace transform we convert differential equation in $t$ to algebraic equation in $s$ and then apply inverse transform to get the result.

The function $F(s)$ defined by the integral (1.28) is called the Laplace transform of the function $f(t)$. Note that this transformation is a linear transformation

$$L\{a f_1(t) + b f_2(t)\} = a L\{ f_1(t) \} + b L\{ f_2(t) \}$$

where $a$ and $b$ are constants.

**Definition 1.7.1** A function $f(t)$ is said to be piecewise continuous in a closed interval $[a, b]$ if that interval can be subdivided into finite number of intervals in each of which function $f(t)$ is continuous and has finite right and left hand limits.

**Definition 1.7.2** A function $f(t)$ is said to be exponential order $a$ as $t$ tends to infinity if there exists a positive constant $M$, a number $a$ and a finite number $t_o$ such that

$$|f(t)| < M e^{at}, \text{ for } t > t_o$$  \hspace{1cm} (1.29)$$
Theorem 1.7.1 [28] If \( f(t) \) be a continuous function for all \( t \geq 0 \) and satisfy condition (1.29) for some \( a \) and \( M \). Further let \( f'(t) \) exists and be continuous for all \( t > 0 \). Then the Laplace transform of the derivative \( f'(t) \) exists when \( s > a \) and

\[
L\{f'(t)\} = s L\{f(t)\} - \{f(0)\}. \quad (1.30)
\]

Applying equation (1.30) successively to \( n^{th} \), \((n-1)^{th}\) derivative of \( f(t) \), we get

\[
L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - \ldots - f^{n-1}(0) \quad (1.31)
\]

From (1.28), \( L \{ f(t) \} = F(s) \) then \( f(t) \) is called the inverse Laplace transform of the function \( F(s) \), written as \( f(t) = L^{-1} \{ F(s) \} \) For Laplace transform and inverse Laplace transform of exponential and trigonometric functions see [18], [27].

We next address the Laplace transform of extended trigonometric functions \( M_{30}(t) \), \( M_{31}(t) \) and \( M_{32}(t) \). Using (1.28), we have

\[
L \{ M_{30}(t) \} = \int_0^\infty e^{-st} M_{30}(t) \, dt
\]

\[
= \int_0^\infty e^{-st} \left\{ \sum_{n=0}^\infty (-1)^n \frac{t^{3n}}{(3n)!} \right\} \, dt
\]

\[
= \sum_{n=0}^\infty \frac{(-1)^n}{(3n)!} \int_0^\infty e^{-st} \, t^{3n} \, dt
\]

\[
= \sum_{n=0}^\infty \frac{(-1)^n}{s^{3n+1}}
\]

\[
= \frac{s^2}{s^3 + 1}
\]

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Also, \( L^{-1}\left(\frac{s^2}{s^3+1}\right) = M_{30}(t) \)

Similarly, \( L\{ M_{31}(t) \} = \frac{s}{s^3+1}, \quad L^{-1}\left(\frac{s}{s^3+1}\right) = M_{31}(t) \)

\( L\{ M_{32}(t) \} = \frac{1}{s^3+1}, \quad L^{-1}\left(\frac{1}{s^3+1}\right) = M_{32}(t) \)

1.8 Metric Space

Let \( X \) be any non-empty set.
A metric on \( X \) is a real function \( d \) of ordered pairs of elements of \( X \) satisfying following conditions

(i) \( d(x, y) \geq 0 \) and \( d(x, y) = 0 \) iff \( x = y \)

(ii) \( d(x, y) = d(y, x) \) (symmetry)

(iii) \( d(x, y) \leq d(x, z) + d(z, y) \) (triangle inequality)

d assigns to each pair \((x, y)\) of elements of \( X \) a non-negative real number \( d(x, y) \) which by symmetry does not depend on order of the elements. The function \( d(x, y) \) is called distance between \( x \) and \( y \). \((X, d)\) is called metric space.

Example 1.8.1 Let \( X \) be an arbitrary non-empty set. Define \( d \) by

\[
\begin{align*}
d(x, y) &= 0 \quad \text{if } x = y \\
      &= 1 \quad \text{if } x \neq y
\end{align*}
\]

\((X, d)\) is a metric space.
Example 1.8.2 Consider the real line \( R \) and the real function \( |x| \) defined on \( R \). Three elementary properties of this absolute function are

\[(i) \quad |x| \geq 0, \quad |x| = 0 \text{ iff } x = 0\]

\[(ii) \quad |-x| = |x|\]

\[(iii) \quad |x + y| \leq |x| + |y|\]

Define a metric on \( R \) by \( d(x, y) = |x - y| \) (usual metric on \( R \))

\[(i) \quad d(x, y) = 0 \text{ iff } x = y\]

\[(ii) \quad d(x, y) = |x - y|\]

\[= |-(y - x)|\]

\[= |y - x|\]

\[= d(y, x)\]

\[(iii) \quad d(x, y) = |x - y|\]

\[= |(x - z) + (z - y)|\]

\[\leq |x - z| + |z - y|\]

\[= d(x, z) + d(z, y)\]

Features

(1) The elements of each space can be added and subtracted in a natural way, and every element has a negative. Each space contains a special element, denoted by 0 and is called the origin or zero element.

(2) In each space there is defined
“a notion of distance from an arbitrary element to the origin” or

“notion of ‘size’ of an arbitrary element”.

The size of an element $x$ is a real number denoted as $||x||$ and called its norm. It satisfies

(a) $||x||\geq 0$ and $||x|| = 0$ iff $x = 0$
(b) $||-x|| = ||x||$
(c) $||x + y|| \leq ||x|| + ||y||$

(3) Each metric arises as norm of the difference between two elements.

$$d(x, y) = ||x - y||$$

**Definition 1.8.1** “The function $f$ is said to be bounded function if $\exists$ positive real number $K$ such that $|f(x)| \leq K$ for $x \in [0, 1]$.”

If $f$ and $g$ are two bounded functions defined on the closed unit interval $[0, 1]$ then

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(-f)(x) = -f(x)$$

The origin, denoted by 0, is the constant function which is identically zero.

$$0(x) = 0 \quad \forall x \in [0, 1]$$
Define the norm of a function $f$ by

$$
|| f || = \int_0^1 | f(x) | \, dx
$$

and the induced metric by

$$
d(f, g) = || f - g || = \int_0^1 | f(x) - g(x) | \, dx
$$

note that,

$$
|| f + g || \leq || f || + || g ||
$$

**Definition 1.8.2** The set of all bounded continuous real functions defined on closed unit interval, has another metric defined by

$$
|| f || = \sup \{ | f(x) | : x \in [0, 1] \}
$$

or $|| f || = \sup \{ | f(x) | \}$ and

$$
d(f, g) = || f - g || = \sup | f(x) - g(x) |$

### 1.9 Banach Space

Let $X$ be a non-empty set.

Assume that each pair $x, y \in X$ can be combined by a process called addition to yield an element $z$ in $X$ denoted by $z = x + y$

Assume that this operation $+$ satisfies

(i) $x + y = y + x$

(ii) $x + (y + z) = (x + y) + z$
(iii) ∃ unique element , denoted by 0 and called zero element or the origin such that
\[ x + 0 = x \text{ for every } x \]

(iv) To each element \( x \in X \) there corresponds a unique element in \( X \), denoted by \( -x \), and is called negative of \( x \) such that
\[ (x) + (-x) = 0 \]

Next, assume that each scalar \( \alpha \) and each element \( x \in X \) can be combined by the process called scalar multiplication such that to yield element \( y = \alpha x \) in \( X \) satisfying

(i) \( \alpha (x + y) = \alpha x + \alpha y \)

(ii) \((\alpha + \beta) x = \alpha x + \beta x \)

(iii) \((\alpha \beta) x = \alpha (\beta x) \)

(iv) \(1 \cdot x = x \)

The algebraic system \( X \) defined by these operations and axioms is called a linear Space.

**Example 1.9.1** Let \( R \) be the field of Real numbers. Let
\[ R^n = \{(\alpha_1, \alpha_2, \ldots \alpha_n) : \alpha_1, \alpha_2, \ldots \alpha_n \in R\} \]

\( R^n \) is a linear space over \( R \).

(i) Two elements \( x = (\alpha_1, \alpha_2, \ldots \alpha_n) \) and \( y = (\beta_1, \beta_2, \ldots \beta_n) \) of \( R^n \) are said to be equal if \( \alpha_i = \beta_i \) for every \( i = 1, 2 \ldots n \)
(ii) Define \( x + y = (\alpha_1 + \beta_1, \ldots , \alpha_n + \beta_n) \)
\[ x + y \in R^n \] since each \( \alpha_i + \beta_i \in R \)

(iii) For \( a \in R \) define \( a x = (a \alpha_1, \ldots , a \alpha_n) \), obviously
\[ a x \in R^n \]

The following properties are immediate.

(i) Addition is associative in \( R^n \)
\[ x + (y + z) = (x + y) + z \text{ for all } x, y, z \in R^n \]

(ii) Addition is commutative in \( R^n \)
\[ x + y = y + x \text{ for all } x, y \in R^n \]

(iii) Existence of additive identity in \( R^n \)
There exists \((0,0,\ldots,0)\in R^n\) such that
\[ x + 0 = 0 + x = x \text{ for all } x \in R^n \]

(iv) Existence of additive inverse of each element of \( R^n \)
For each \((\alpha_1, \alpha_2, \ldots \alpha_n)\in R^n\), \((-\alpha_1, -\alpha_2, \ldots -\alpha_n)\in R^n\)
is its additive inverse.

\( R^n \) is abelian group with respect to addition. Further,

(v) If \( a \in R \) and \( x, y \in R^n \), \( a(x + y) = ax + ay \)

(vi) If \( ab \in R \) and \( x \in R^n \), \((a + b)x = ax + bx \)

(vii) If \( ab \in R \) and \( x \in R^n \), \((ab)x = a(bx) \)
(viii) If \( x \in \mathbb{R}^n \) and 1 is unit element of \( \mathbb{R} \) then \( 1 \cdot x = x \)

Hence, \( \mathbb{R}^n \) is a linear space over \( \mathbb{R} \).

A normed linear space is a linear space on which a norm is defined.

**Definition 1.9.1** A *complete normed linear space* is called a *Banach Space*.

Let \((X, \| . \|)\) be a normed linear space, then

(i) a sequence \( < x_n > \) in \( X \) is called a Cauchy sequence if, “given \( \varepsilon > 0 \), \( \exists m_0 \in \mathbb{N} \) such that \( \| x_n - x_m \| < \varepsilon \) \( \forall m, n \geq m_0 \)”

(ii) a normed linear space \( X \) is called complete iff every Cauchy sequence in \( X \) is convergent

In a normed linear space, every convergent sequence is a Cauchy sequence.

**Example 1.9.2** The linear space \( \mathbb{R}^n \) of all \( n \)-tuples \( x = (x_1, x_2, \ldots, x_n) \) of real numbers is a Banach space under the norm

\[
\| x \| = \left( \sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}
\]

We have,

(i) \( \| x \| \geq 0 \), since each \( x_i \geq 0 \)
(ii) \(\|x\| = 0\) iff each \(x_i = 0\) iff \(x = 0\)

(iii)

\[
\|x + y\|^2 = \sum_{i=1}^{n} |x_i + y_i|^2
\]

\[
= \sum_{i=1}^{n} |x_i + y_i| |x_i + y_i|
\]

\[
\leq \sum_{i=1}^{n} |x_i + y_i| (|x_i| + |y_i|)
\]

\[
= \sum_{i=1}^{n} |x_i + y_i| |x_i| + \sum_{i=1}^{n} |x_i + y_i| |y_i|
\]

\[
\leq \|x + y\| (\|x\| + \|y\|)
\]

If \(\|x + y\| = 0\) then result is true. If \(\|x + y\| \neq 0\) then dividing both sides by \(\|x + y\|\) we get

\[
\|x + y\| \leq \|x\| + \|y\|
\]

Thus \(\mathbb{R}^n\) is a normed linear space.

Next, let \(\langle x_1, x_2, \ldots, x_n, \ldots \rangle\) be Cauchy sequence in \(\mathbb{R}^n\)

Let \(x_k = \langle x_1^k, x_2^k, \ldots, x_n^k, \ldots \rangle\) where \(x_i^k\) is \(i^{th}\) co-ordinate of \(x_k\).

Since \(\langle x_k \rangle\) is a Cauchy sequence, there exists positive integer \(n_0\) such that

\[
\text{For all } m, n \geq n_0 \text{ we have } \|x_n - x_m\| < \epsilon
\]

\[
\Rightarrow \sum_{i=1}^{n} |x_i^m - x_i^n|^2 < \epsilon^2
\]

\[
\Rightarrow |x_i^m - x_i^n| < \epsilon
\]

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\( \langle x_k \rangle \) is a Cauchy sequence in \( R \). \( R \) is complete. It follows that \( \langle x_1, x_2, \ldots x_n, \ldots \rangle \) converges to \( x \in R^n \) and consequently \( R^n \) is a Banach space.

Let \( X \) and \( Y \) be linear spaces.
A function \( A : X \to Y \) is said to be a linear operator from \( X \) to \( Y \) if

\[
A(x + y) = A(x) + A(y), \quad A(\alpha x) = \alpha A(x)
\]

for \( x, y \in X \) and for every \( \alpha \in R \).

\( L(X, Y) \) denotes set of all linear operators from a linear space \( X \) to linear space \( Y \).

Define addition and scalar multiplication pointwise on set \( L(X, Y) \)
for \( A, B \in L(X, Y) \) and \( \alpha \in R \),

\[
(A + B)(x) = Ax + Bx \\
(\alpha A)(x) = \alpha . Ax \quad \forall x \in X
\]

\( L(X, Y) \) is a linear space with its zero element as the zero operator \( O : X \to Y \) defined by

\[
O(x) = 0 \quad \forall x \in X
\]

and the additive inverse of \( A \in L(X,Y) \) is the operator \(-A : X \to Y \) defined by

\[
(-A)(x) = -Ax \quad \forall x \in X
\]

For more details see [21] and [26]
Bibliography


