Chapter 5

GRAVITATIONAL COLLAPSE WITH ESCAPING NEUTRINOS

1. Introduction

About two decades ago, a kind of strange object called quasi-stellar radio source (in short quasar), whose radio/optical power is around 10^{46} ergs/sec. was observed by astronomers. This is some 100 times the total energy out put rate of a gaint galaxy. Though, these radio sources have star like appearance on photographic plates, classical stellar structure theory precludes stars with masses greater than 100 solar masses. For such a star radiation pressure dominates and because of the thermodynamic properties of radiation, it will pulsate with large amplitudes. As a result energy will be ejected, thereby reducing the mass of the star. They cannot be considered as galaxies, because the light variation precludes this possibility (galaxies do not have light variations.) Hoyle and Fowler (1963) suggested that energies which lead to the formation of radio sources could be supplied through the gravitational collapse of a super star. Such an object, with a mass of one hundred million solar masses, would be located in the centre of the galaxy - whose gravitational collapse could supply the necessary energy if it were to shrink down close to the Schwarzschild radius. Hoyle and Fowler (1963 a,b) have examined the possibility that exceedingly massive stars, that is 10^6 to 10^8 solar masses, are formed at or near the galactic centres. When the central temperature of these bodies reaches the vicinity of 1 or 2 x 10^9 degree K, the neutrino emission becomes an important energy loss mechanism, and the central portion of the star will collapse. Michel (1963) considered the neutrino loss in massive stars and concluded that as the neutrino loss becomes appreciable, the pressure would start to fall. This results in failure of hydrostatic equilibrium and therefore, motion towards the centre. Since the neutrino energy loss rate increases roughly as T^9 at temperatures around 10^9 degrees, this approach to a new quasi-equilibrium state in fact leads to higher temperature and an even greater rate of energy loss, until finally free fall towards the centre would not be able to convert gravitational energy into neutrinos at the rate demanded by the central temperature. At this point there seems to be essentially nothing to halt the collapse until the matter becomes highly degenerate, at which time compression can increase the pressure without increasing the temperature. So the entire star might be expected to end up in the collapsed state with the central regions collapsing quite rapidly and eventually the outer layers falling in. Accordingly the rapid stage of collapse must occur in about the free fall time that is, about 10^3 sec. Fowler (1964) has also shown that in a nonrotating spherically symmetric, massive star, general relativistic considerations become important and gravitational collapse sets in at radius $R \sim 10^{18}$ Cm and central conditions $p_c \sim 4 \times 10^{-10}$ g Cm^{-3}, $T_c \sim 2.5 \times 10^5$ deg. collapse to the gravitational radius $R_g \sim 3 \times 10^{13}$ Cm occurs in a local time interval $\sim 10^3$ years for the outer regions and 1 year for the inner regions. Large redshift effects preclude the release of significant amounts of energy from such a rapidly collapsing system. In fact these discussions led several anothers especially Bondi (1964) Misner and Sharp (1964) May and White (1966) McVitte (1966) Vaidya (1968) and Krishna Rao (1972) to investigate the problem under different physical assumptions. In this Chapter we wish to prove that if a spherically symmetric star emits a non-neglegible fraction of its mass as neutrinos it cannot form a black-hole; since the radiation emitted by the star would modify the star's exterior geometry. Such a modified exterior geometry was first studied by Vaidya (1951, 1952, 1953) and has since been discussed by Raychaudhuri (1953), Isreal (1958), Papapetrou (1964), Lindquist, Schwarz and Misner (1965) and by Yu (1966). Vaidya's exterior geometry for a radiating spherical star is very similar to that of Schwarzschild. The only difference is that the mass M which characterises Vaidya's geometry is not a constant throughout the exterior of the star; rather it is an arbitrary function of retarded time u. That is, it is a constant on each out going radial, light like surface, but its value can decreases from one light like surface to the next. Vaidya calls this retarded time u as "Newtonian time" since the coordinate velocity of radially out going light is infinite in this coordinate system. However, the present approach in this problem is entirely different from that of the above mentioned authors. It is often thought that the material energy density within a massive star when it crosses the Schwarzschild radius ($r = 2M$) tends to infinity. In such an analysis the role of the free gravitational field (represented by the conformal tensor) is either ignored or has been neglected. In the present analysis we shall emphasize the role played by the energy density of the free gravitational field during collapse.

2. <u>The Energy Momentum Tensor</u>

We assume that the star is filled with a perfect fluid distribution of matter with neutrinos flowing in the radial direction and that they are neighter scattered nor absorbed by the surrounding matter. Thus the stress energy tensor can be written as

(5-2.1) $$T_a^b = M_a^b + N_a^b$$

Here

(5-2.2) $$M_a^b = (\rho + p)\, u_a u_b - p\, g_a^b$$

and

(5-2.3) $$N_a^b = q\, k_a\, k^b$$

where

(5-2.4) $$u^a\, u_a = 1,$$

(5-2.5) $$k_a\, k^a = 0 \cdot$$

The matter conservation law is expressed in the form of equation of continuity

(5-2.6) $$(n\, u^a)_{;a} = 0$$

where n is the baryon number density. Following Misner (1965) we introduce cooling rate per unit amount of matter $C(\mathcal{T}, n)$, $\mathcal{T}$ being the temperature. Thus,

(5-2.7) $$-u^a\, T_{a\,;\,b}^{b} = -nC = u^a\, N_{a\,;\,b}^{b}$$

Then nC gives the cooling rate per unit volume in the rest frame of the fluid. Also, Misner calls C as "energy generation rate" since this energy which disappears from the fluid will appear in N_a^b as radiation. Since the total energy momentum tensor satisfies the local energy-moment consergation laws

$$T^{b}_{a\,;\,b} = 0 \qquad (5\text{-}2.8)$$

the equations of motion take the form

$$(\rho + p)\, u_{a\,;\,b}\, u^{b} = (g^{b}_{a} - u_{a}\, u^{b})\, p_{,b} - N^{b}_{a\,;\,b} + n\, C\, u_{a} \qquad (5\text{-}2.9)$$

Also in view of

$$K_{a}\; N^{ab} = 0\,,\quad K_{a;b}\; N^{ab} = 0 \qquad (5\text{-}2.10)$$

we get the identify

$$K_{a}\; N^{ab}{}_{;b} = 0 \qquad (5\text{-}2.11)$$

Now, writting

$$\rho = n\,(1 + e) \qquad (5\text{-}2.12)$$

to define the specific internal energy e that does not include the rest mass energy, we can derive easily

$$e_{,a}\; u^{a} = -\,p\,\left(\frac{1}{n}\right)_{,a}\; u^{a} - C \qquad (5\text{-}2.13)$$

Further, the relation between temperature $\mathcal{T}$ entropy S and the cooling rate C takes simple form

$$\mathcal{T}\, S_{,a}\; u^{a} = -C \qquad (5\text{-}2.14)$$

which shows that in the absence of neutrino emission entropy is a constant for any time like observer. That is, changes in a perfect fluid are adiabatic.

3. The Metric and the Field Equations

As in the earlier work the metric is taken as

$$ds^2 = -e^{\lambda} dr^2 - R^2 (d\theta^2 + \sin^2\theta \, d\phi^2) + e^{\nu} dt^2 \tag{5-3.1}$$

where λ, R and ν are functions of r and t only. From (1-4.11) (or (3-3.4) we get

$$\Gamma = e^{-\lambda/2} R' = 4\pi n R^2 R^1 \tag{5-3.2}$$

and Einstein's field equations take the form

$$8\pi T^1_1 = -8\pi(p+q) = -\frac{1}{R e^{\lambda}}\left[R'^2/R + R'\nu'\right] + \frac{1}{R e^{\nu}}\left[2\ddot{R} + \dot{R}^2/R - \dot{R}\dot{\nu}\right] + \frac{1}{R^2} \tag{5-3.3}$$

$$8\pi T^2_2 = 8\pi T^3_3 = -8\pi p - 8\pi\epsilon - \frac{1}{R e^{\lambda}}\left[2R'' - R'^2/R - R'\lambda' + R'\nu'\right] + \frac{1}{R e^{\nu}}\left[2\ddot{R} - \dot{R}^2/R + \dot{R}\dot{\lambda} + \dot{R}\dot{\nu}\right] - \frac{1}{R^2} \tag{5-3.4}$$

$$8\pi T_4^4 = 8\pi(\rho + q) = -\frac{1}{R e^{\lambda}}\left[2R'' - R'^2/R - R'\lambda'\right]$$

(5-3.5)

$$+\frac{1}{R e^{\nu}}\left[\dot{R}^2/R + \dot{R}\dot{\lambda}\right] + \frac{1}{R^2}$$

(5-3.6) $$8\pi T_4^1 = 8\pi q\, e^{\frac{(\nu-\lambda)}{2}} = \frac{1}{R e^{\lambda}}\left[2\dot{R}' - R'\dot{\lambda} - \dot{R}\nu'\right]$$

where

$$8\pi\epsilon = -e^{-\lambda}\left[\frac{R''}{R} + \frac{R'^2}{R^2} - \nu''/2 - \frac{\nu'^2}{4} - \frac{R'\lambda'}{2R} + \frac{R'\nu'}{2R} + \frac{\lambda'\nu'}{4}\right]$$

(5-3.7)

$$+ e^{-\nu}\left[\ddot{\lambda}/2 - \dot{\lambda}^2/4 - \frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - \frac{\dot{R}\dot{\lambda}}{2R} + \frac{\dot{R}\dot{\nu}}{2R} - \frac{\dot{\lambda}\dot{\nu}}{4}\right] + 1/R^2$$

is the eigenvalue of the Weyl conformal tensor. Now the expression for $e^{-\lambda}$ is computed from the combination (5-3.3) + (5-3.5) - (5-3.4) . Thus,

(5-3.8) $$e^{-\lambda} = \left[1 + U^2 - \frac{8\pi}{3}(\rho + \epsilon)R^2\right](R')^{-2}$$

or

(5-3.9) $$\Gamma^2 = \left[1 + U^2 - \frac{8\pi}{3}(\rho + \epsilon)R^2\right]$$

From (5-3.6) we get

(5-3.10) $$D_t\lambda = \frac{2\,\partial U}{\partial R} - 8\pi q R\left[1 + U^2 - \frac{8\pi}{3}(\rho + \epsilon)R^2\right]^{-1/2}$$

where we have written

$$\frac{\partial}{\partial R} = (R')^{-1} \frac{\partial}{\partial r}$$

Now, the baryon conservation law (5-2.6), after making use of (5-3.10) to eliminate λ, takes the form

(5-3.11) $$D_t(n R^2) = - n R^2 \frac{\partial U}{\partial R} + 4\pi n R^3 \Gamma^{-1}$$

Again, from (5-3.3), (5-3.6) and (5-3.8) we get

(5-3.12) $$D_t U = - \frac{\Gamma^2}{\rho + p} \frac{\partial p}{\partial R} - \frac{4\pi}{3} R(\rho + \epsilon + 3p + 3q) - \frac{n \Gamma c}{\rho + p}$$

In writing the above equation we have made use of the radial component of the equations of motion given by (5-2.9). That is,

(5-3.13) $$\frac{1}{2}(\rho + p) D_r \nu = - D_r p - n \cdot c$$

Equations (5-2.10), (1-4.10) and (5-3.11) are termed as "dynamical equations." Eventhough, we have obtained explicit expression for λ, the same cannot be possible for ν. However, from equations (5-3.13) we can integrate for ν with suitable boundary conditions.

Another useful expression obtained by applying the aperator D_t to (5-3.8) and then using (5-3.3) and (5-3.6) is

$$(5\text{-}3.14) \qquad D_t\left[\frac{4\pi}{3}(\rho+\epsilon)R^3\right] = -4\pi p R^2 U - 4\pi R^2 q (U+\Gamma)$$

The above expression shows how the fluid pressure does work on a material sphere across its moving boundary. The second term on the right hand side of the above equation gives the neutrino flux [note that $(U+\Gamma)$ is the light velocity] over a two sphere of surface area $4\pi R^2$. Thus, (5-3.14) may be interpreted as the rate of loss of combined energy of the field and the material pressure during contraction $(U < 0)$ together with the neutrino emission.

4. Exterior Solution

The interior gravitational field of the sphere given in the previous section can be matched with exterior field given by Vaidya (1951, 1953) under suitable boundary conditions. The radiating Schwarzschild space-time given by Vaidya has one important property not possessed by the Schwarzschild space time which is important for the formation of spherical black-holes. The metric for Vaidya's radiating Schwarzschild space-time may be written as

$$(5\text{-}4.1) \quad ds^2 = \left[1 - \frac{2M(u)}{r}\right] du^2 - 2du\, dr - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

In case M is a constant in (5-4.1), we immediately obtain Schwarzschild space-time. Since the interior solution of the system has been matched to the Vaidya's radiating metric, and since the surface $r = 2M(u)$ is a space-like surface, the model can never reach black-hole stage. Thus, we conclude that neutrino emission prevents the formation of a spherical black-hole.

5. Conformally Flat Space-Times

It was shown by Krishna Rao (1973) that all spherically symmetric perfect fluid models with uniform density are conformally flat. The Schwarzschild interior solution with constant density being a particular case of the above models can be expressed in a conformally flat form. There are certain advantages of the conformally flat space-times. They are : (i) the light geometry inside a star is that of the flat Minkowski continuum, (ii) relativistic steller structure atleast in its kinematical aspect forms a link between special relativity and gravitation and (iii) the number of unknown functions in the metric tensor reduces to one and hence it is easy to handle the mathematical problem. Further, imposing the condition of spatial isotropy, the four dimensional

Riemannian manifold describing the interior geometry of a star is given by

$$(5\text{-}5.1) \qquad ds^2 = \alpha^{-2}(r', t')\,(dt'^2 - dr'^2 - r'^2\, d\Omega^2)$$

where r', θ, ϕ are the spherical polar coordinates of the Euclidean space.

It should be noted here that it is quite unnecessary to suppose that the star is spatially homogeneous in the sense that matter at every depth has the same thermodynamics functions. In fact the diffusion of neutrinos from the centre of the star towards the outer layers makes any assumption of homogeneity inappropriate.

The nonvanishing components of the energy momentum tensor T_a^b for the metric (5-5.1) obtained through Einstein's field equations are

$$(5\text{-}5.2) \qquad 8\pi T_1^1 = 4\,\frac{\alpha\alpha_1}{r'} - 3\alpha_1^2 - 2\alpha\alpha_{44} + 3\alpha_4^2$$

$$(5\text{-}5.3) \qquad 8\pi T_2^2 = 8\pi T_3^3 = 2\alpha(\alpha_{11} - \alpha_{44}) - 3(\alpha_1^2 - \alpha_4^2) + \frac{2\alpha\alpha_1}{r'}$$

$$(5\text{-}5.4) \qquad 8\pi T_4^4 = 2\alpha\alpha_{11} + \frac{4\alpha\alpha_1}{r'} - 3(\alpha_1^2 - \alpha_4^2)$$

$$(5\text{-}5.5) \qquad 8\pi T_4^1 = -8\pi T_1^4 = -2\alpha\alpha_{14}$$

Where the subscripts 1 and 4 after α denote differentation with respect to r' and t' respectively. By an appropriate modification of the scheme given in Section 2 and using the expressions (5-5.2) to (5-5.5) we get

(5-5.6) $$8\pi\rho = 3\left[\frac{2\alpha\alpha_1}{r'} - \alpha_1^2 + \alpha_4^2\right]$$

(5-5.7) $$8\pi p = 2\alpha(\alpha_{44} - \alpha_{11}) - 3(\alpha_4^2 - \alpha_1^2) - \frac{2\alpha\alpha_1}{r'}$$

(5-5.8) $$4\pi q = \frac{\alpha\left[\{\alpha_{11} - \alpha_1/r'\}\{\alpha_{44} + \frac{\alpha_1}{r'}\} - \alpha_{14}^2\right]}{\alpha_{11} + 2\alpha_{14} + \alpha_{44}}$$

(5-5.9) $$u = u^1/u^4 = -\frac{-\alpha_{11} - \alpha_1/r' + \alpha_{14}}{\alpha_{44} + \alpha_1/r' + \alpha_{14}}$$

The conservation laws discussed in Section 2 with appropriate modifications give the single equation of motion

(5-5.10) $$(\rho+p)\{u u_{,1} + u_{,4}\}\{(1-u^2)^{-1} - \alpha^{-1}(\alpha_1 + u\alpha_4)\} = -p_{,1} - u p_{,4} - \alpha^{-1} n c (1-u^2)^{\frac{1}{2}}$$

Also the equation for the cooling rate takes the form

(5-5.11) $$N^{1b}_{;b} = N^{4b}_{;b} = \alpha n c\left(\frac{1+u}{1-u}\right)^{1/2}$$

6. Slow Collapse and the Schwarzschild Interior Solution

For slow gravitational collapse the neutrino pro production vanishes and we write β in place of α and a suffix '0' for ρ, p, u in the foregoing analysis. The differential equation for β is readily obtained from q = 0 :

$$(5\text{-}6.1) \qquad \left(\beta_{11} - \beta_1/r'\right)\left(\beta_{44} + \beta_1/r'\right) - \beta_{14}^2 = 0$$

The Schwarzschild interior solution is a particular case of the above equation. In fact for the Schwarzschild interior solution β is given by (see Krishna Rao and Patel (1972))

$$(5\text{-}6.2) \qquad \frac{\beta(r;t')}{r'} = \left(\frac{2\pi\rho_0}{3}\right)^{1/2} \left[P + P^{-1} \right]$$

where

$$P = \frac{1}{r'}\left(\sqrt{1+r'^2} - 1\right) e^{\left(\sqrt{1+r'^2} + t'\right)}$$

and ρ_0 being the constant density in the Schwarzschild model. The Pressure is given by

$$(5\text{-}6.3) \qquad 8\pi p_0 = \beta\left(\frac{8\pi\rho_0}{3}\right)^{1/2} \frac{2}{\sqrt{1+r'^2}} \operatorname{Cosh}\left(\sqrt{1+r'^2} + t'\right) - 8\pi\rho_0$$

It is interesting to note that the velocity u_0 of the material particles in this coordinate system is given by

$$u_0 = \frac{-r'}{\sqrt{1+r'^2}}$$

Thus, it is obvious that the particle rushes towards $r' = 0$ with almost the speed of light and its velocity diminishes till it is instantaneously at rest with respect to the origin and then gains speed away from the origin asymptotically tending to the speed of light. The surfaces r = const. In the Schwarzschild coordinates are now given by P = Const. In the (r', t') - plane, P = Const. are hyperbolae since $\frac{1}{r'}(\sqrt{1 + r'^2} - 1)$ is very nearly equal to unity. Thus a conformal coordinate system bears a strong resemblance to the Kruskal coordinate system of the exterior Schwarzschild space-time.

7. Radiating Schwarzschild Interior Solution

Bondi (1964), Vaidya (1966), Misner (1965) and Misner and Hernandes (1966) have attempted to give a generalization of the Schwarzschild interior solution with escaping neutrinos. In this section we use a method given by Krishna Rao (1969, 1972) to generalise the static interior Schwarzschild solution into a non-static solution with escaping neutrinos. According to this method since the expression

$$(5\text{-}7.1) \qquad \alpha = (T_1^1 - T_1^4 - T_2^2) \;/\; (T_4^4 + T_1^4 - T_2^2)$$

does not involve q, without loss of generality, we assume that $u = u_0$. Therefore,

$$(5\text{-}7.2) \qquad \frac{\alpha_{11} - \alpha_1/_{r'} + \alpha_{14}}{\alpha_{44} + \alpha_1/_{r'} + \alpha_{14}} = \frac{\beta_{11} - \beta_1/_{r'} + \beta_{14}}{\beta_{44} + \beta_1/_{r'} + \beta_{14}}$$

where β satisfies the equation (5-6.1). From the equation (5-7.2) the relation between α and β is given by

$$(5\text{-}7.3) \qquad \alpha = F + r' F' + \beta$$

where F is an arbitrary function of the retarded time $(t' - r')$ and a prime for F denotes a differentiation with respect to the argument $(t' - r')$. Thus, substituting for β the expression given (5-6.2), we readily obtain the generalized interior Schwarzschild solution with escaping neutrinos. The density ρ and the pressure p of the fluid as well as the neutrino radiation density q are given by

$$(5\text{-}7.4) \qquad 8\pi\rho = 8\pi\rho_0 + \left(\frac{8\pi\rho_0}{3}\right)^{1/2}\Big[6\sinh\left(\sqrt{1+r'^2}+t'\right)\left(F + r'F'\right) + r'\left(P - P^{-1}\right)\left\{F' + \left(\sqrt{1+r'^2}+r'\right)F''\right\}\Big] + \left(F'^2 + 2FF''\right)$$

$$(5\text{-}7.5) \qquad 8\pi p = \alpha\left(\frac{8\pi\rho_0}{3}\right)^{1/2}\frac{2}{\sqrt{1+r'^2}}\cosh\left(\sqrt{1+r'^2}+t'\right) - 8\pi\rho$$

$$(5\text{-}7.6) \qquad 4\pi q = \alpha\, r' F'''$$

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