Chapter 4

INTERIOR SCHWARZSCHILD SURFACES

1. Introduction

Several authors including Thompson & Whitrow (1967), Bonner and Faulks (1967), McVittie (1967), Taub (1968), Bondi (1969) have investigated shear free collapse of gravitating spheres with uniform density. It was shown some time ago (Krishna Rao, 1973) that for all these uniform density models the energy density of the free gravitational field takes a simple form, viz. $\frac{4\pi\epsilon}{3} R^3$ = Const. and if ϵ vanishes, the solutions are conformally flat. For a given spherical configuration of matter Raychaudhuri's equation shows that it is the slowest collapse possible. One can easily obtain many inequivalent analytic solutions of the Einstein field equations for a shear free collapse of a spherical system since the number of unknown metric functions can be reduced from three to one. It is well known that shear free solution of Einstein equation could be generated by solving a single second order non-linear equation containing an arbitrary function of a radial coordinate. Here, in this Chapter, we shall show that for the case with shear the solutions are generated by solving a single second order non-linear differential equation containing an arbitrary function of the time-coordinate and that shear causes rapid collapse of the system. This fact can also be seen from Raychaudhuri's equation. Further, as mentioned in the previous Chapter the characterestic feature of the system with shear being the presence of gravitational radiation and since the solution is matching with that of Schwarzschild exterior we can say that the gravitational radiation emitted by this system is being trapped in its envelopes. Non-uniform models were discussed by Faulkes (1969) and recently Glass (1979) has investigated shear free collapse and gave some solutions.

In this Chapter we shall show that under some conditions gravitational collapse with shear will proceed not toward a final equilibrium but to a state of continuing collapse. So the matter will be enclosed by the Schwarzschild surface from the interior of which no light ray or other signal can escape. It is well known that in the Schwarzschild exterior metric, light rays emitted from the region $R < 2M$ remain confined to this region and cannot escape to infinity. This fact was clearly exhibited in the Kruskal (1960) diagram of the Schwarzschild metric where radial light rays follow 45° lines of the coordinate system. We know that the surface $R = 2M$ in the Schwarzschild metric is associated with an infinite red shift from objects falling in and infact this is not a physical singularity. So by the choice of an appropriate coordinate system this singularity can be avoided. For, if we choose an orthonormal frame and calculate the curvature components they are simply multiples of M/R^3 and therefore are finite at $R = 2M$. Since the Schwarzschild coordinate system covers only a part of

Kruskal's coordinate system the later can be regarded as more general in which we have two distinct singularities $r = 0$; one in the past and the other in future. There are two distinct $R = 2M$ surfaces: the $u = 0$ consists of outgoing radial light rays and lies in the future of a typical observer in the Schwarzschild R quadrant, while the other $V = 0$ consists of ingoing light rays and lies in the past. Also we can see in this coordinate system the world line $R = \text{Const} < 2M$ appears as a hyperbola, and the curve $R = \text{Const} < 2M$ is a spacelike curve. In fact, the upper half of the Kruskal diagram can be regarded as a black-hole while the lower half as a white hole, since any test-particle in the upper half plane will go to $r = 0$ while in the lower half plane it will rebound. And also one can infer from the Kruskal diagram (see Misner ; 1968) that not only freely falling particles but also test particles with acceleration, once they reach $R < 2M$ cannot escape hitting the $r = 0$ singularity. That is, in this region gravity is the strongest force than any other.

2. The Metric and the Field Equations

In this Chapter we discuss analogeous "Schwarzschild surfaces" in the interior of a spherecally symmetric moving fluids. So, we consider the expression for the interval in the form

$$ds^2 = -e^{\lambda} - R^2(d\Omega^2 - dt^2) \quad (4\text{-}2.1)$$

Now, Einstein's field equations give the following expressions for the non-vanishing components of the stress energy tensor :

$$8\pi T_1^1 = \frac{1}{R^2}\left[2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2}\right] - \frac{1}{2}e^{-\lambda}\left(6\frac{R'^2}{R^2}\right) + \frac{1}{R^2} \qquad (4\text{-}2.2)$$

$$8\pi T_2^2 = 8\pi T_3^3 = 8\pi\epsilon + \frac{1}{R^2}\left[2\frac{\ddot{R}}{R} - 2\frac{\dot{R}^2}{R^2} + \frac{\dot{\lambda}\dot{R}}{2R} - \frac{\dot{\lambda}\dot{R}}{2R} - 1 + \frac{\dot{\lambda}\dot{\lambda}}{4}\right] - e^{-\lambda}\left[2\frac{R''}{R} - 2\frac{R'^2}{R^2} + \frac{\dot{R}^2}{R^2} - \lambda'\frac{R'}{R}\right] \qquad (4\text{-}2.3)$$

$$8\pi T_4^4 = -\frac{1}{2}e^{-\lambda}\left[4\frac{R''}{R} - 4\frac{R'^2}{R^2} - 2\lambda'\frac{R'}{R} + 6\frac{R'^2}{R^2}\right] + \frac{1}{R^2}\left[\frac{\dot{R}^2}{R^2} + \frac{\dot{\lambda}\dot{R}}{R}\right] + \frac{1}{R^2} \qquad (4\text{-}2.4)$$

$$8\pi T_4^1 = \frac{1}{2}e^{-\lambda}\left[4\frac{\dot{R}'}{R} - 4\frac{\dot{R}R'}{R^2} - 2\frac{\dot{\lambda}R'}{R}\right] \qquad (4\text{-}2.5)$$

Assuming that pressure is anisotropic, we have in the case of the most general definite scheme, the energy momentum tensor

$$(4\text{-}2.6) \qquad T_a^b = \rho u_a u^b + \sum_{\alpha=1}^{3} p_{(\alpha)} \, v_a^{(\alpha)} \, v^{(\alpha)b}$$

where u_a is the unit velocity vector of the fluid particles, and ρ the proper material energy density p_α are the proper partial pressures and $v_a^{(\alpha)}$ are three space-like vectors (see Lichnerowicz : 1955). So, we have for our case

$$(4\text{-}2.7) \qquad T_1^1 = - p_r, \; T_2^2 = T_3^3 = - p_\theta = - p_\phi, \; T_4^4 = \rho \text{ and } T_4^1 = 0$$

Since $T_4^1 = 0$ the equation (4-2.5) gives immediately

$$(4\text{-}2.8) \qquad e^\lambda = \frac{4R'^2}{R^2}$$

Now, using (4-2.8), the field equations (4-2.2) to (4-2.4) with the help of (4-2.7) turn out to be three independent equations :

$$(4\text{-}2.9) \qquad 8\pi p_r = - 8\pi T_1^1 = - \frac{1}{R^2}\left[2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} + 1\right] + \frac{3}{4}$$

$$8\pi p_\theta = 8\pi p_\phi = -8\pi T_2^2 = -8\pi T_3^3$$

(4-2.10)

$$= \frac{1}{R^2}\left[\frac{2\dot{R}\dot{R}'}{RR'} - \frac{\ddot{R}'}{R'} - \frac{\dot{R}^2}{R^2}\right] + \frac{3}{4}$$

$$8\pi \rho = 8\pi T_4^4$$

(4-2.11)

$$= \frac{1}{R^2}\left[2\frac{\dot{R}\dot{R}'}{RR'} - \frac{\dot{R}^2}{R^2} + 1\right] - \frac{3}{4}$$

3. The Solution with Equation of State $\rho = p_\theta = p_\phi$

Now, we assume that the transverse pressure p_θ and the material energy density ρ are equal giving the differential equation

(4-3.1) $$\ddot{R}' + \left(1 - \frac{3}{2}R^2\right) R' = 0$$

Integrating (4-3.1) with respect to r, we get

(4-3.2) $$\ddot{R} - \frac{R^3}{2} + R = F(t)$$

$F(t)$ being arbitrary. Let us consider the case with $F(t) = 0$. That is,

(4-3.3) $$\ddot{R} - \frac{R^3}{2} + R = 0$$

which can be regarded as the ordinary differential equation for a soft spring oscillator. The first ingegral of (4-3.3) is given by

$$(4\text{-}3.4) \qquad \frac{U^2}{2} - \frac{E(r)}{R^2} = \frac{1}{2} \left({R^2}/{4} - 1\right)$$

This equation expresses the energy conservation law with $\frac{1}{2}\left({R^2}/{4} - 1\right)$ as the total energy of the system, $\frac{U^2}{2}$ $\left(=\frac{\dot{R}^2}{2R^2}\right)$ being the kinetic energy and ${E}/{R^2}$ the potential energy.

From the field equations we immediately obtain the value for $e^{-\lambda}$ as

$$(4\text{-}3.5) \qquad e^{-\lambda} = \left[1 + U^2 - \frac{8\pi}{3}(\rho + \epsilon)\, R^2\right] (R')^{-2}$$

Since from (4-2.11) $e^{-\lambda} = \frac{R^2}{4R'^2}$, we can obtain the expression for the potential energy of the system

$$(4\text{-}3.6) \qquad \frac{E(r)}{R^2} = \frac{4\pi}{3} (\rho + \epsilon)\, R^2 = {M}/{R}$$

where

$$(4\text{-}3.7) \qquad M = \frac{4\pi}{3} (\rho + \epsilon)\, R^3$$

Rewriting the equation (4-3.4)

$$(4\text{-}3.8) \qquad \frac{1}{2}\,\frac{\dot{R}^2}{R^2} - \frac{M}{R} = \frac{1}{2}\,(\Gamma^2 - 1)$$

we get an analogous equation for R as a function of t whose solutions are well known for a fixed shell of matter A, for which the mass M and Γ are constants in the pressure free case. So we find that for anisotropic pressure case also, as in the pressure free case, each shell of matter moves independently of the other shells inside or outside of it. Its motion, described by giving its circumsference as a function of its proper time, is exactly the same as that of a test particle in the Schwarzschild field of a mass M(A), equal to the mass enclosed by the shell in question. If initial conditions are specified for all shells, equation (4-3.7) can be solved for each A.

4. General Properties of the System

To find an implicit solution of equation (4-3.4) we write it in the integral form

$$(4\text{-}4.1) \qquad \int \frac{dR}{\left[2E - R^2 - R^4/_4\right]^{1/2}} = \pm\, t$$

We now plot a phase portrait taking $\dot{R}$ on the y-axis and R on the X-axis for all constant $E \geq 0$. Then the xy-plane

is called the phase space for the equation (4-3.4).
To make a phase portrait, first we have to find a constant solution of the equation (4-3.4). If we put $R = C$ then $\dot{R} = 0$, $\ddot{R} = 0$ the resulting algebraic equation $2C - C^3 = 0$ shows that $R \equiv 0$, $R \equiv \sqrt{2}$, $R \equiv -\sqrt{2}$ are constant solutions in the phase plane $y = \dot{R}$. Graphically, one can depict the constant solutions by plotting the points $(0,0)$, $(\sqrt{2},0)$ $(-\sqrt{2},0)$. These points are called critical points. The next step in the construction of the phase portiant is to plot each one of the curves

$$\dot{R}^2 = -R^2 + \frac{R^4}{4} + 2E, \qquad |R| \leq \sqrt{2}$$

that has a critical point on it. If $R = 0$, then $\dot{R} = 0$ then $E = 0$ and we must plot $\dot{R}^2 = -R^2(1-R^2/4)$, $|R| \leq \sqrt{2}$. The only point satisfying these conditions is the critical point $(0,0)$. If $R = \pm\sqrt{2}$ and $\dot{R} = 0$ then $E = 1/2$ and we must plot $\dot{R}^2 = (\sqrt{2} - \frac{R}{\sqrt{2}})^2$ for $|R| \leq \sqrt{2}$. Suppose the system is put to motion with initial condition $R = R_o$ and $\dot{R} = \dot{R}_o$ which satisfy

$$\dot{R}_o^2 + R_o^2 - \frac{R_o^4}{4} = 1.$$

and let $R = \phi(t)$, $\dot{R} = \dot{\phi}(t)$ denote corresponding solutions of the equation (4-3.3). If we plot this solution parametrically, its trace will be a closed curve which corresponds to a periodic motion of the system. The period of motion can be found with the aid of equation (4-4.1) and we can see that the period of oscillation depends on the total energy of the system and hence on the amplitude. Such dependence is characterstic of nonlinear operators. In linear oscillators amplitude and period are completely independent. Since the system is in adiabatic motion, we can say convective and pulsational instabilities do not occur (see Moore and Spiegel: 1966). The energy density of the free gravitational field for this system takes the form

$$(4\text{-}4.3) \qquad 8\pi\epsilon = \frac{1}{R^2}\left[1 + \dot{R}'/R' - \frac{2\ddot{R}}{R} + 6\frac{\dot{R}^2}{R^2} - 4\frac{\dot{R}\,\dot{R}'}{R\,R'}\right]$$

and the shear invariant σ is given by

$$\sigma = \frac{1}{R}\left[\dot{R}'/R' - 2\frac{\dot{R}}{R}\right]$$

so that the relation connecting the shear invariant σ and the energy density of the free gravitational field ϵ is given by

$$(4\text{-}4.4) \qquad 8\pi\epsilon = \frac{1}{R^2}\left(1 + \dot{R}\sigma\right) + \sigma^2 + \frac{\dot{\sigma}}{R}$$

i.e. if the system is shear free then $8\pi\epsilon = \frac{1}{R^2}$

an analogus expression for an isothermal gas sphere with the eqation of state $p = \rho$. For this system the expression Θ and acceleration A are given by

$$(4\text{-}4.5) \qquad \Theta = \frac{1}{R}\left[\dot{R}'/_{R'} + \dot{R}/_{R} \right]$$

$$(4\text{-}4.6) \qquad A = -e^{-\lambda/2}\,\nu'/2 = -\frac{R}{4R'} \times \frac{R'}{2R} = -\frac{1}{2}$$

5. <u>The Exterior Solution</u>

We shall now assume that for $r > r_o$ the stress energy tensor vanished that is $\rho = p_r = p_\theta = p_\phi = 0$ and construct a solution of the field equations in a coordinate system which is an extension of the coordinate system used here. Since the density and pressure are both required to vanish it follows that M must be a constant say M_b . Then from $p_r(r_o, t) = 0$ we have

$$(4\text{-}5.1) \qquad \frac{M^2}{E^2} = \frac{M_b^2}{E^2} = \frac{3}{4}$$

We may verify that the constant M_b which enters into equation $\dot{R}^2 + R^2 - \frac{R^4}{4} = 2E(r)$ as $R = \frac{E}{M_b}$ is the gravitational mass as measured by an external observer by showing that how the line element given by the equation $ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - R^2 d\Omega^2$ may be transfered into Schwarzschild one. In fact our interior line element $ds^2 = - e^{\lambda} dr^2 - R^2 (d\Omega^2 - dt^2)$ can be transfored into Schwarzschild exterior metric

$$(4\text{-}5.2) \qquad ds = \left(1 - \frac{2M_b}{R}\right) d\tau^2 - \frac{dR^2}{1 - \frac{2M_b}{R}} - R^2 \, d\Omega^2$$

where

$$d\tau = \frac{U e^{\lambda/2} \, dr + R^2/2 \, dt}{1 - \frac{2M_b}{R}}$$

$$U^2 = -\left(1 - R^2/4 - \frac{2M_b}{R}\right) = \dot{R}^2/R^2$$

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