Chapter 10

A Study of Thermal Stresses Induced by a Point Heat Source in an Annular Disc by Quasi-Static Approach

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Chapter 10. *Thermal Stresses Induced by a Point Heat in an Annular Disc* 144

10.1 Introduction

Nowaki in [1] discuss the temperature distribution on upper face of circular plate, by assuming zero temperature on the lower surface and circular edge thermally insulated. Roy Choudhary [2] invented the quasi-static thermal stresses due to transient temperature along the circumference of a circle over the upper face with lower face at zero temperature and the fix circular edge of the plate thermally insulated. In [3] Phythian studied the cylindrical heat flow with arbitrary heating rates at the outer surface and zero heat flux at the internal boundary. Deshmukh et al. [4] studied the thermal stresses induced by a point heat source in a circular plate by quasi-static approach. Recently Gaikwad and Ghadle [5] and [6] determine temperature, displacement and thermal stresses in an inverse transient thick annular disc and a non-homogeneous thick rectangular plate due to internal heat generation. An inverse transient temperature, displacement and thermal stresses in an annular disc are determined by Khobragade et al. [7].

In this paper we consider a non-homogeneous annular disc and determine the temperature, displacement and quasi-static thermal stresses under an arbitrary time dependent heat flux at the outer surface and zero heat flux in the internal surface with the help of integral transform technique as in Ozisik [8]. The results are obtained in a series form in terms of Bessel’s functions. The mathematical model has
been constructed for copper material and solved by using computational mathematical software Math-Cad 2007 and thermal stresses are discussed graphically using Origin software.

To the authors knowledge, no literature on non-homogeneous thermoelastic problem in an annular disc due to internal heat generation has been published. The results presented here will be more useful in engineering problem particularly, in the determination of the state of strain in an annular disc.

10.2 Formulation of the Problem

Consider a thin annular disc occupying the space D defined by $a \leq r \leq b$. Initially disc is at arbitrary temperature distribution $F(r)$, for time $t > 0$ heat is generated within the solid at a rate of $g(r, t)/k$. The inner circular boundary surface at $r = a$ is insulated, while the arbitrary time dependent heat flux $Q(t)$ is applied at outer circular boundary $r = b$. The arbitrary time dependent heat flux means radial heat flux prescribed on boundary surface and converges rapidly for large value of $t$ often a rapidly convergent series is required for small value of time. Under this conditions the displacement and thermal stresses in the disc due to internal heat generation are to be determined.
Chapter 10. Thermal Stresses Induced by a Point Heat in an Annular Disc

The differential equation governing the displacement potential function $\Psi(r, t)$ as in Khobragade et.al \[7\] is given by

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} = (1 + \nu) a_t T \quad (10.2.1)$$

The stress functions $\sigma_{rr}$ and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = -\frac{2\mu}{r} \frac{\partial \Psi}{\partial r} \quad (10.2.2)$$

$$\sigma_{\theta\theta} = -\frac{2\mu}{r^2} \frac{\partial^2 \Psi}{\partial r^2} \quad (10.2.3)$$

while in each case the stress functions $\sigma_{zz}$, $\sigma_{zr}$ and $\sigma_{\theta z}$ are zero with the disc in the plane state of stress. The temperature of the annular disc satisfies the following heat conduction equation as in Deshmukh et.al \[4\] as

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{g(r, t)}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}, \quad a \leq r \leq b, \quad t > 0 \quad (10.2.4)$$

Subject to the boundary conditions

$$k \frac{\partial T}{\partial r} = 0 \text{ at } r = a, \quad t > 0 \quad (10.2.5)$$

$$k \frac{\partial T}{\partial r} = Q(t) \text{ at } r = b, \quad t > 0 \quad (10.2.6)$$

and initial condition

$$T(r, t) = F(r) \text{ when } t = 0, \quad a \leq r \leq b \quad (10.2.7)$$
Chapter 10. *Thermal Stresses Induced by a Point Heat in an Annular Disc*  

10.3 Solution of the Problem

10.3.1 Determination of $T(r, t)$

To obtain the expression for temperature $T(r, t)$ we introduce the finite Hankel transform over the variable $r$ and its inverse transform defined by Ozisik \[8\] as

$$
\overline{T}(\lambda_n, t) = \int_{r'=a}^{b} r'(K_0(\lambda_n, r'))T(r', t)dr' \quad (10.3.1)
$$

$$
T(r, t) = \sum_{n=1}^{\infty} \overline{T}(\lambda_n, t)(K_0(\lambda_n, r)) \quad (10.3.2)
$$

where

$$
K_0(\lambda_n, r) = \frac{\pi \lambda_m J'_0(\lambda_m b)Y'_0(\lambda_m b)}{\sqrt{2} \sqrt{1 - \frac{J'^2_0(\lambda_m b)}{J'^2_0(\lambda_m a)}}}
$$

$$
\times \left[ \frac{J_0(\lambda_m r)}{J'_0(\lambda_m b)} - \frac{Y_0(\lambda_m r)}{Y'_0(\lambda_m b)} \right] \quad (10.3.3)
$$

and $\lambda_1, \lambda_2, \lambda_3, \ldots$, are the positive roots of the transcendental equation

$$
\frac{J'_0(\lambda_m a)}{J_0(\lambda_m b)} - \frac{Y'_0(\lambda_m a)}{Y_0(\lambda_m b)} = 0 \quad (10.3.4)
$$

where $k$ and $\alpha$ are the thermal conductivity and thermal diffusivity of the material of the annular disc respectively. Equations (10.2.1) to (10.2.7) constitute the mathematical formulation of the heat conduction problem in a annular disc.
where $J_n(X)$ and $Y_n(X)$ are the Bessel’s function of first and second kind of order $n$ respectively. Also this transform defined in (10.3.1) satisfies the relation

$$H \left[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right] = -\lambda_n^2 T(\lambda_n, t) \quad (10.3.5)$$

On applying the finite Hankel transform defined in (10.3.1) and its inverse transform defined in (10.3.2) to the equation (10.2.4), one obtain the expression for temperature as

$$T(r, t) = \sum_{m=1}^{\infty} K_0(\lambda_m, r) e^{-\alpha \lambda_m^2 t}$$

$$\times \left\{ \int_a^b r' K_0(\lambda_m, r') F(r') dr' + \frac{\alpha}{k} \int_{t'=0}^t e^{\alpha \lambda_m^2 t'} \right\}$$

$$\times \left[ \int_a^b r'(K_0(\lambda_m, r')) g(r', t) dr' + bK_0(\lambda_m, b)Q(t') \right] dt' \quad (10.3.6)$$

10.3.2 Determination of Displacement function $\Psi$

Substituting the equation (10.3.6) in equation (10.2.1), one obtain the displacement function $\Psi$ as

$$\Psi = -(1 + \nu)at \sum_{m=1}^{\infty} \frac{1}{\lambda_m^2} K_0(\lambda_m, r) e^{-\alpha \lambda_m^2 t}$$

$$\times \left\{ \int_a^b r' K_0(\lambda_m, r') F(r') dr' + \frac{\alpha}{k} \int_{t'=0}^t e^{\alpha \lambda_m^2 t'} \right\}$$

$$\times \left[ \int_a^b r'(K_0(\lambda_m, r')) g(r', t) dr' + bK_0(\lambda_m, b)Q(t') \right] dt' \quad (10.3.7)$$
10.3.3 Determination of Thermal Stresses $\sigma_{rr}$ and $\sigma_{\theta\theta}$

Substituting the equation (10.3.7) into equations (10.2.2) and (10.2.3), one obtains the thermal stresses $\sigma_{rr}$ and $\sigma_{\theta\theta}$ as

$$
\sigma_{rr} = \sqrt{2}(1 + \nu)\pi \mu a t \sum_{m=1}^{\infty} \frac{J_0'(\lambda_m b)Y_0'(\lambda_m r)}{\sqrt{1 - \frac{J_0^2(\lambda_m b)}{J_0^2(\lambda_m a)}}} \times \left[ \frac{J_0'(\lambda_m r)}{J_0'(\lambda_m b)} - \frac{Y_0'(\lambda_m r)}{Y_0'(\lambda_m b)} \right] e^{-\alpha \lambda_m t}
\times \left\{ \int_a^b r'K_0(\lambda_m, r')F(r')dr' + \frac{\alpha}{k} \int_{t'=0}^t e^{\alpha \lambda_m^2 t'} \right\}
\times \left[ \int_a^b r'(K_0(\lambda_m, r'))g(r', t)dr' + bK_0(\lambda_m, b)Q(t') \right] dt'
$$

(10.3.8)

and

$$
\sigma_{\theta\theta} = \sqrt{2}(1 + \nu)\pi \mu a t \sum_{m=1}^{\infty} \frac{\lambda_m J_0'(\lambda_m b)Y_0'(\lambda_m r)}{\sqrt{1 - \frac{J_0^2(\lambda_m b)}{J_0^2(\lambda_m a)}}} \times \left[ \frac{J_0''(\lambda_m r)}{J_0'(\lambda_m b)} - \frac{Y_0''(\lambda_m r)}{Y_0'(\lambda_m b)} \right] e^{-\alpha \lambda_m t}
\times \left\{ \int_a^b r'K_0(\lambda_m, r')F(r')dr' + \frac{\alpha}{k} \int_{t'=0}^t e^{\alpha \lambda_m^2 t'} \right\}
\times \left[ \int_a^b r'(K_0(\lambda_m, r'))g(r', t)dr' + bK_0(\lambda_m, b)Q(t') \right] dt'
$$

(10.3.9)

10.4 Special Case and Numerical Results

Setting $F(r) = r^2$

$g(r, t) = g_i \delta(r - r_1) \delta(t - \tau)$

$Q(t) = e^{-\omega t}, \alpha > 0$

where $r$ is the radius measured in meter, $\delta$ is the Dirac-delta function,
\( \omega > 0 \). The heat sources \( g(r, t) \) is an instantaneous point source of strength \( g_{pi} = 50 J/m \) situated at certain circle along the radial direction of the disc and releasing its heat instantaneously at the time of \( t \to \tau = 4 \).

- Inner radius of annular disc \( a = 1 \) m
- Outer radius of annular disc \( b = 2 \) m
- Circular path of annular disc \( r_1 = 1.5 \) m

The numerical calculation has been carried out for a Copper (pure) annular disc with the material properties defined as

- Thermal diffusivity \( \alpha = 112.34 \times 10^{-6} \ m^2s^{-1} \).
- Thermal conductivity \( k = 38610^{-6} \ w/mk \).
- Density \( \rho = 8954 \ Kg/m^3 \).
- Specific heat \( c_p = 383 \ J/kgK \).
- Poisson ratio \( \nu = 0.35 \).
- Coefficient of linear thermal expansion \( a_t = 16.5 \times 10^{-6} /K \).
- Lame constant \( \mu = 26.67 \).
- \( \lambda_1 = 3.1665, \lambda_2 = 6.3123, \lambda_3 = 9.4445, \lambda_4 = 12.5812, \) and \( \lambda_5 = 15.7199 \) are the positive roots of transcendental equation:

\[
\frac{J_0''(\lambda_m a)}{J_0''(\lambda_m b)} - \frac{Y_0''(\lambda_m a)}{Y_0''(\lambda_m b)} = 0
\]
We set for convenience
\[ A = \frac{\pi}{100\sqrt{2}}, \quad B = \frac{\pi(1 - \nu)a_t}{100\sqrt{2}}, \quad C = \frac{\pi(1 - \nu)\mu a_t}{100} \]

here \( A, B \) and \( C \) are constants. The numerical calculation has been carried out with the help of computational mathematical software Math-Cad 2007 and the graphs are plotted with the help of Origin software.

### 10.5 Conclusion

- From Fig. 10.1, it is observed that temperature reaches maximum at the center and then decreases towards the outer circular edge. After some time \( t = 4 \), due to internal heat generation, the temperature of the copper plate increases proportionally.

- From Fig. 10.2, it is observed that there is displacement occurring around the outer circular edge which is proportional to the temperature due to point heat source of strength \( g_{pi} \).

- From Fig. 10.3, it is observed that the radial stress function \( \sigma_{rr} \) is zero at the outer circular boundary. As the heat source \( g(r; t) \) is an instantaneous point heat source situated at certain circle within the central part \( r_1 = 1.5 \) of the annular disc, we can observe the radial stress developing with compressive stresses around the circle of radius \( r_1 = 1.5 \) and the radial stress decreases towards the outer circular edge.
Figure 10.1: Temperature distribution $T/A$

Figure 10.2: Displacement distribution $\Psi/B$

Figure 10.3: Radial stress distribution $\sigma_{rr}/C$
In Fig. 10.4, the angular stress function $\sigma_{\theta\theta}$ increases from the inner face to the outer circular edge. It reaches maximum in the region $1.4 \leq r \leq 2$ and develops a compressive stresses in the central part of $1 \leq r \leq 1.4$ with a tensile stresses in the annular region of $1.4 \leq r \leq 2$.

In this chapter, we extend the work of [4] to one dimensional non-homogeneous boundary value problem of heat conduction in an annular disc and determined the expressions of temperature, displacement and stresses due to internal heat generation. As a special case, a mathematical model is constructed for copper (pure) annular disc with specified material properties. The heat source is an instantaneous point heat source of strength $g_{pi}$ situated at certain circle along the radial and axial direction of the plate, and releases its heat instantaneously at the time of $t = \tau$. Due to heat generation within the annular disc, the radial stress develops as compressive stresses,
whereas the angular stress develops with compressive stresses around the center and tensile stresses around the outer circular edge. Also it can be observed from the figures of temperature reaches maximum at central part and displacement increases from inner surface to outer surface of an annular disc.

The results obtained here are useful for engineering problems, particularly in the determination of the state of stress in thin circular plates. Also any particular case of special interest can be derived by assigning suitable values to the parameters and functions in (10.3.6)-(10.3.9).
Bibliography


