Chapter 3

Study of an Inverse Transient
Thermoelastic Hollow Sphere

Chapter 3. *An Inverse Transient Thermoelastic Hollow Sphere*

### 3.1 Introduction

An inverse thermoelastic problem consists of determination of temperature of the heating medium, heat flux on the boundary surface of the solids, when the condition of the displacement and stress are known at some points of the solid under considerations. This inverse problem is very important in view of it’s relevant to various industrial machinery such as main shaft of lathe, turbine and roll of rolling mills under the heating. Ghonge and Ghadle [1] studied the inverse transient thermoelastic problems of solid sphere. Grysa and Cialkowski [2] studied certain inverse problems of temperature and stress field. Grysa and Koalowski [3] studied the one-dimensional transient thermoelastic problems and derived the heating temperature and heat flux on the surface of an isotropic infinite slab. Further Noda [4] investigate some inverse problem of coupled thermal stress fields in a thick plate. Tikhe and Deshmukh [5] discuss the thermal deflection of inverse heat conduction problem of thin circular plate. Deshmukh and Warbhe [6] solved inverse heat conduction problem in a semi-infinite circular plate and derived its thermal deflection by quasi-static approach. The present paper deals with determination of temperature, radial displacement and stress function on outer curve surface of a hollow sphere when temperature is known at interior under consideration and initially is kept at zero. The governing heat conduction equation has been solved by using Laplace transform
technique. The results are obtained in a series form of trigonometric functions. The result’s for temperature, radial displacement and stresses have been computed numerically and illustrated graphically.

3.2 Formulation of the Problem

Consider a hollow sphere of isotropic and homogeneous material occupying the space D: $a \leq r \leq b, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq 2\pi$. The differential equation governing the Radial Displacement function $U(r,t)$ as in [7]

$$\frac{\partial^2 U}{\partial r^2} + \frac{2}{r} \frac{\partial U}{\partial r} - \frac{2U}{r^2} = \frac{(1 + v)}{(1 - v)^\alpha} \frac{\partial T}{\partial t}$$  \hspace{1cm} (3.2.1)

where $v$ and $\alpha$ are the poisson’s ratio and linear coefficient of thermal expansion of the material of the hollow sphere respectively and $T$ is the temperature of the hollow sphere satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} = \frac{1}{t} \frac{\partial T}{\partial t}$$  \hspace{1cm} (3.2.2)

with initial condition

$$T(r, t)|_{t=0} = 0$$  \hspace{1cm} (3.2.3)

boundary conditions

$$T(a, t) = 0$$  \hspace{1cm} (3.2.4)

$$T(b, t) = g(t)$$  \hspace{1cm} (3.2.5)
and interior condition

\[ T(\xi, t) = f(t), \quad a \leq \xi \leq b. \quad (3.2.6) \]

here function \( f(t) \) is known function of \( t \), where as \( g(t) \) is unknown function of \( t \), and the stress functions are given by,

\[ \sigma_{rr}(r, t) = \frac{2\alpha E}{1-v} \left[ \frac{2(r^3 - a^3)}{r^3(b^3 - a^3)} \int_a^b T(r')r'^2 dr' - \frac{2}{r^3} \int_a^r T(r')r'^2 dr' \right] \]

\[ \sigma_{\theta\theta}(r, t) = \sigma_{\phi\phi}(r, t) = \frac{\alpha E}{1-v} \left[ \frac{2r^3 + a^3}{r^3(b^3 - a^3)} \int_a^b T(r')r'^2 dr' + \frac{1}{r^3} \int_a^r T(r')r'^2 dr' - T(r) \right] \quad (3.2.8) \]

where \( k \) and \( E \) are the thermal diffusivity and Young’s modulus of the material of the hollow sphere respectively.

The equation’s (3.2.1) to (3.2.8) constitute the mathematical formulation of the problem under consideration.

### 3.3 Solution of the Problem

#### 3.3.1 Determination of Temperature Function

Applying Laplace integral transform as in [8] to the equation’s (3.2.2), (3.2.4)-(3.2.6) and using (3.2.3), one obtain

\[ \frac{d^2T}{dr^2} + \frac{2dT}{rdr} - \frac{sT}{k} = 0 \quad (3.3.1) \]
\[ \mathbf{T}(a, s) = 0 \] (3.3.2)
\[ \mathbf{T}(b, s) = \mathbf{g}(s) \] (3.3.3)
\[ \mathbf{T}(\xi, s) = \mathbf{f}(s), a \leq \xi \leq b \] (3.3.4)

where \( \mathbf{T} \) denote the Laplace transform of \( T \) and \( s \) is the Laplace transform parameter.

Now, we obtain the temperature in transform domain by using method of variation of parameters

\[ \mathbf{T}(r, s) = A \frac{1}{r} \sinh(qr) + B \frac{1}{r} \cosh(qr) \] (3.3.5)

where \( q = \sqrt{\frac{s}{k}} \)

using conditions (3.3.2), (3.3.4) in equation (3.3.5)

\[ A \frac{1}{a} \sinh(qa) + B \frac{1}{a} \cosh(qa) = 0 \] (3.3.6)
\[ A \frac{1}{\xi} \sinh(q\xi) + B \frac{1}{\xi} \cosh(q\xi) = \mathbf{f}(s) \] (3.3.7)

solving equations (3.3.6) and (3.3.7) for \( A \) and \( B \), once we get,

\[ A = \frac{\xi \cosh(qa) \mathbf{f}(s)}{\sinh(q\xi) \cosh(qa) - \cosh(q\xi) \sinh(qa)} \] (3.3.8)
\[ B = \frac{-\xi \sinh(qa) \mathbf{f}(s)}{\sinh(q\xi) \cosh(qa) - \cosh(q\xi) \sinh(qa)} \] (3.3.9)

Now, substituting the values of \( A \) and \( B \) from equations (3.3.8) and (3.3.9) in equation (3.3.5), finally we get temperature function in
Laplace transform domain as

\[ T(r, s) = \frac{\xi}{r} \left[ \frac{\sinh[(r - a)q]}{\sinh[(\xi - a)q]} \right] \mathcal{F}(s) \] (3.3.10)

Applying inverse laplace transform as in [8] to equation (3.3.10)

\[ T(r, t) = \frac{\xi}{r} L^{-1} \left[ \frac{\sinh[(r - a)q]}{\sinh[(\xi - a)q]} \mathcal{F}(s) \right] \] (3.3.11)

Let

\[ \overline{G}(s) = \frac{\sinh[(r - a)q]}{\sinh[(\xi - a)q]} \]

Then

\[ G(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \frac{\sinh[(r - a)q]}{\sinh[(\xi - a)q]} ds \] (3.3.12)

Now to evaluate this integral we use Cauchy Residue theorem.

\[ s_m = -k \frac{m^2 \pi^2}{(\xi - a)^2}, \quad m = 1, 2, 3, ..., \] are the simple poles of integrand in (3.3.12).

Residue at \( s = s_m \)

\[ = s_m = \lim_{s \to s_m} \left[ s - s_m \right] \frac{\sinh((\xi - a)\sqrt{\frac{s}{k}})}{\sinh((\xi - a)\sqrt{\frac{s}{k}})} \]

\[ \times \lim_{s \to s_m} \left[ e^{st} \sinh\left( (r - a)\sqrt{\frac{s}{k}} \right) \right] \]

\[ = \lim_{s \to s_m} \left[ \frac{\sqrt{s}}{k} \right] \frac{(\xi - a) \cosh((\xi - a)\sqrt{\frac{s}{k}})}{(\xi - a) \cosh((\xi - a)\sqrt{\frac{s}{k}})} \]

\[ \times \left[ e^{-kt\left( \frac{m^2 \pi^2}{\xi^2} \right)} \sinh\left( \frac{im \pi r}{(\xi - a)} \right) \right] \]
\[
G(t) = \frac{2k\pi}{(\xi - a)^2} \sum_{m=1}^{\infty} (-1)^{m+1} m \sin \left( \frac{m\pi r}{(\xi - a)} \right) e^{-kt\left(\frac{m^2\pi^2}{(\xi - a)^2}\right)}
\]

Therefore value of \(G(t)\) is given by,

\[
G(t) = \frac{2k\pi}{(\xi - a)^2} \sum_{m=1}^{\infty} (-1)^{m+1} m \sin \left( \frac{m\pi r}{(\xi - a)} \right) e^{-kt\left(\frac{m^2\pi^2}{(\xi - a)^2}\right)}
\]

Now the convolution theorem is defined as

\[
L^{-1} \left[ \mathcal{G}(s) \mathcal{F}(s) \right] = \int_{0}^{t} f(t')G(t-t')dt'
\]

\[
T(r, t) = \frac{\xi}{r} L^{-1} \left[ \mathcal{F}(s) \mathcal{G}(s) \right] = \frac{\xi}{r} L^{-1} \left[ \mathcal{F}(s) \frac{\sinh[(r-a)q]}{\sinh[(\xi - a)q]} \right]
\]

\[
= \frac{2k\pi\xi}{r(\xi - a)^2} \sum_{m=1}^{\infty} (-1)^{m+1} m \sin \left( \frac{m\pi r}{(\xi - a)} \right) \int_{0}^{t} f(t')e^{-k\left(\frac{m^2\pi^2}{(\xi - a)^2}\right)(t-t')}dt'
\]

Now using (3.3.15) in (3.2.4) once obtain the unknown temperature function on outer curved surface as

\[
g(t) = \frac{2k\pi\xi}{b(\xi - a)^2} \sum_{m=1}^{\infty} (-1)^{m+1} m \sin \left( \frac{m\pi b}{(\xi - a)} \right) \int_{0}^{t} f(t')e^{-k\left(\frac{m^2\pi^2}{(\xi - a)^2}\right)(t-t')}dt'
\]
3.3.2 Determination of Radial Displacement Function

To determine the radial displacement function, first we obtain the necessary integral

\[
\int T r^2 dr = \frac{2k\pi \xi}{(\xi - a)^2} \sum_{m=1}^{\infty} (-1)^{m+1} m \left[ \frac{\sin \left( \frac{m\pi r}{\xi - a} \right)}{\left( \frac{m\pi}{\xi - a} \right)^2} - \frac{r \cos \left( \frac{m\pi r}{\xi - a} \right)}{\left( \frac{m\pi}{\xi - a} \right)} \right] 
\times \int_0^t f(t') e^{-k \left[ \frac{m^2 \pi^2}{(\xi - a)^2} \right] (t-t')} dt'
\]

Solving equation (3.2.1) for \( U(r, t) \) and using equation (3.3.17) one obtain the radial displacement function

\[
U(r, t) = \frac{2k\pi \xi \alpha (1 + \nu)}{(\xi - a)^2 (1 - \nu)} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left\{ \frac{1}{r^2} \left[ \frac{\sin \left( \frac{m\pi b}{\xi - a} \right)}{\left( \frac{m\pi}{\xi - a} \right)^2} - \frac{b \cos \left( \frac{m\pi b}{\xi - a} \right)}{\left( \frac{m\pi}{\xi - a} \right)} \right] 
\times \left[ \frac{1}{r^2} \left( \frac{a^3}{r^2 (1 + \nu) (b^3 - a^3)} + \frac{a^3}{r^2 (b^3 - a^3)} \right) 
\times \left[ \frac{\sin \left( \frac{m\pi a}{\xi - a} \right)}{\left( \frac{m\pi}{\xi - a} \right)^2} - \frac{\sin \left( \frac{m\pi a}{\xi - a} \right)}{\left( \frac{m\pi}{\xi - a} \right)^2} \right] \right] \right\}
\times \int_0^t f(t') e^{-k \left[ \frac{m^2 \pi^2}{(\xi - a)^2} \right] (t-t')} dt'
\]
3.3.3 Determination of Stress Functions

Using the series expansion of $T(r,t)$ and equation (3.3.15) in equation (3.2.7) and (3.2.8) one obtain

\[
\sigma_{rr} = \frac{2k\xi\alpha E}{\pi(1-v)} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left\{ \frac{2}{a^3} \left[ \sin \left( \frac{m\pi a}{\xi} \right) - \frac{m\pi a}{\xi} \cos \left( \frac{m\pi a}{\xi} \right) \right] - \frac{2}{r^3} \left[ \sin \left( \frac{m\pi r}{\xi} \right) - \frac{m\pi r}{\xi} \cos \left( \frac{m\pi r}{\xi} \right) \right] \right\} \times \int_0^t f(t') e^{-k \left[ \frac{m^2\pi^2}{\xi^2} \right] (t-t')} \, dt'
\]

(3.3.19)

\[
\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{2k\xi\alpha E}{\pi(1-v)} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left[ \frac{2}{a^3} \sin \left( \frac{m\pi a}{\xi} \right) - \frac{m\pi a}{\xi} \cos \left( \frac{m\pi a}{\xi} \right) \right] - \frac{1}{r^3} \left[ \sin \left( \frac{m\pi r}{\xi} \right) - \frac{m\pi r}{\xi} \cos \left( \frac{m\pi r}{\xi} \right) \right] - \left[ \frac{\pi^2 m^2}{r^2 \xi^2} \sin \left( \frac{m\pi r}{\xi} \right) \right] \times \int_0^t f(t') e^{-k \left[ \frac{m^2\pi^2}{\xi^2} \right] (t-t')} \, dt'
\]

(3.3.20)

3.4 Special Case and Numerical Result

For special case we consider $f(t) = (1 - e^{At})(r - a)$, where $A > 0$ and evaluate the necessary integral

\[
\int_0^t (1 - e^{t'}) e^{-4km^2\pi^2(t-t')} \, dt' = \left[ \frac{(1 - e^{-4km^2\pi^2 t})}{4km^2\pi^2 t} - \frac{e^t}{(1 + 4km^2\pi^2 t)} \right]
\]

(3.4.1)

Now, any particular case can be carried out by using equation’s
In this chapter numerical calculations are carried out for a steel hollow sphere (SN 50C) for which material constants are as follows: $a = 1\,m$, $\xi = 0.5\,m$, $\pi = 3.14$, $k = 15.9 \times 10^{-6}(m^2s^{-1})$, $\alpha = 11.6 \times 10^{-6}(K^{-1})$, $E = 215(GPa)$, $v = 0.281$. The results here obtained are depicted graphically by using Matlab software as shown.
3.5 Conclusion

The temperature, radial displacement and thermal stresses on the outer curved surface of a one-dimensional hollow sphere have been
obtained, when the interior temperature and other initial and boundary conditions are known, with the aid of Laplace transform technique. The results are obtained in terms of trigonometric series which converges for special case. which are calculated numerically and illustrated graphically by using computational programming as in fig.(3.1)-(3.4). From figures it is clearly concludes that the given temperature yields to increase the displacement and its analytically decreases as temperature spread on either sides, Also thermal stresses increases depending on temperature flow through the width of hollow sphere. This paper contains new and novel contribution of thermal displacement and thermal stresses in hollow sphere of the isotropic and homogeneous material. The results presented here will be more useful in engineering problem.
Bibliography


