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INTRODUCTION
Chapter 1

In this chapter we give certain basic definitions of some algebraic concepts which are indispensable for our work. We give a brief survey of the work done by M. Chacron, H.E. Bell, H. Abu-Khuzam, A. Yaqub, J. Grosen, M. Ashraf and M.A. Khan in periodic rings.

**Associative ring:** An associative ring $R$, some times called a ring in shot, is an algebraic system with two binary operations addition $`+`$ and multiplication $`*`$ such that

(i) the elements of $R$ form an abelian group under addition and a semigroup under multiplication,

(ii) multiplication $`*`$ is distributive on the right as well as on the left over addition $`+`$

i.e., $(x + y)z = xz + yz$, $z(x + y) = zx + zy$ for all $x, y, z$ in $R$.

**Commutator:** The commutator $[x, y]$ is defined by $[x, y] = xy - yx$ for all $x, y$ in a ring. This can be considered to be a measure of noncommutativity of a ring.
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**Center:** In a ring $R$, the center denoted by $C$ is the set of all elements $x \in R$ such that $xy = yx$ for all $y \in R$.

**Centralizer:** The centralizer of an element $a$ of a group $G$ is the set of elements of $G$ which commute with $a$.

**Commutative ring:** If the multiplication in a ring $R$ is such that $xy = yx$ for all $x, y$ in $R$, then we call $R$ as a commutative ring.

**Nilpotent element:** An element $x$ of a ring $R$ is called nilpotent if there exists some positive integer $n$ such that $x^n = 0$.

**Idempotent element:** An element $x$ of a ring is called an idempotent element if $x^2 = x$.

**Orthogonal idempotents:** A set $\{e_1, e_2, \ldots, e_n\}$ of idempotents is called a complete set of orthogonal idempotents if $e_i e_j = e_j e_i = 0$ whenever $i \neq j$ and $1 = e_1 + e_2 + \ldots + e_n$.

**Potent element:** An element $x$ is potent if there exists $n > 1$ such that $x^n = x$.

**Zero divisor:** A nonzero element $r$ of $R$ is a zero divisor if there exists a nonzero element $s$ in $R$ such that $sr = 0$ or $rs = 0$.

**Periodic element:** An element $x \in R$ is called a periodic element if there exist distinct positive integers $m, n$ such that $x^n = x^m$. 
**Periodic ring:** A ring $R$ is called periodic if for every $x$ in $R$, there exist distinct positive integers $m, n$ such that $x^n = x^m$.

**Weakly periodic ring:** A ring $R$ is called a weakly periodic ring if every element of $R$ is expressible as a sum of a nilpotent element and a potent element of $R$. That is $R = N + P$, where $N$ is the set of nilpotent elements of $R$ and $P$ is the set of potent elements of $R$.

**Weakly periodic-like ring:** A ring $R$ with center $C$ is called a weakly periodic-like ring if every element $x$ in $R\setminus C$ can be written in the form $x = a + b$, $a$ in $N$, $b$ potent.

**Quasi-periodic ring:** A ring $R$ is called a quasi-periodic ring if for each $x$ in $R$ there exist integers $k, n, m$ all depending on $x$ such that $n > m > 0$ and $x^n = kx^m$.

**Generalized periodic ring:** A ring $R$ is called a generalized periodic ring if for every $x$ in $R$ such that $x \not\in N \cup C$, we have $x^n - x^m \in N \cap C$, for some positive integers $m, n$ of opposite parity.

**D-ring:** A ring $R$ such that every zero divisor is nilpotent is called a D-ring.

**D*-ring:** A ring $R$ is called a D*-ring, if every zero divisor $x$ in $R$ can be written as $x = a + b$, where $a$ is a nilpotent element and $b$ is potent.
**p-ring:** A ring $R$ is called a p-ring, if $x$ is any element of $R$ such that $x^p = x$ and $px = 0$ (p is prime).

**Generalized p-ring:** A generalized p-ring (p prime) is a ring of prime characteristic $p$ such that $x^p y - xy^p \in \mathbb{N}$ for all $x, y \in R \setminus (N \cup J \cup C)$.

**$N_0$-ring:** Suppose $R$ is a ring, $N$ the set of nilpotents, $C$ the center, and $J$ the Jacobson radical of $R$. We say that $R$ is an $N_0$-ring if for all $x, y$ which are not in $(N \cup J \cup C)$, there exists an integer $n = n(x, y) > 1$ such that $x^n y - xy^n \in N$, $(x, y \in R \setminus (N \cup J \cup C)) : (n > 1)$.

**Division ring:** A ring $R$ is said to be a division ring if its nonzero elements form a group with respect to multiplication.

**n-torsion free ring:** A ring $R$ is n-torsion free if $nx = 0$ implies $x = 0$ for all $x \in R$.

**Prime ring:** A ring $R$ is called a prime ring if for any $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$.

**Simple ring:** A ring $R$ is said to be simple if whenever $A$ is an ideal of $R$, then either $A = R$ or $A = 0$.

**Semisimple ring:** A ring is semisimple in case the radical (i.e., the maximal ideal consisting of all nilpotent elements) is the zero ideal.
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**Primitive ring:** A ring $R$ is defined as primitive incase it possess a regular maximal right ideal, which contain no two sided ideal of the ring other than the zero ideal.

**Primary ring:** A Primary ring $R$ is a ring with identity which contains atmost one prime ideal $\neq R$.

**Near ring:** A ring $R$ is called a near ring with two binary operations $+$ and $*$ such that (i) $(R, +)$ is a group, (not necessary abelian) (ii) $(R, *)$ is a semigroup (iii) $a * (b + c) = a * b + a * c$ for all $a, b, c$ in $R$.

**D-near ring:** A near ring $R$ is called a D-near ring if every nonzero homomorphic image $T$ of $R$ satisfies the following conditions:

(i) $T$ has a nonzero right distributive element.

(ii) The additive group $(T, +)$ of $T$ is abelian implies that $T$ is a ring.

**Zero-commutative:** A near ring $R$ is called zero-commutative if $xy = 0$ implies $yx = 0$ for all $x, y \in R$.

**Zero-Symmetric:** A near ring $R$ is called zero-symmetric if $0x = 0$ for all $x \in R$ (we may recall that left distributivity in $R$ yields $x0 = 0$).

**C-ring:** A ring $R$ is a C-ring if for each $x \in R$, there exists an integer $n > 1$ and an integer $k$ such that $x^n = kx$. If $R$ is a C-ring the same $n$ works for all $x \in R$, then $R$ is a $C_n$-ring.
**Boolean ring:** A ring in which every element is idempotent is called a Boolean ring.

**Local ring:** A ring with a unique maximal left ideal is a local ring. These rings also have a unique maximal right ideal, and left and right unique maximal ideal coincide.

**Matrix ring:** The matrix ring, denoted by $M_n(R)$ is the set of $n \times n$ matrices over an arbitrary ring $R$.

**J-ring:** A J-ring is a ring $R$ with the property that for every $x$ in $R$ there exists an integer $n > 1$ such that $x^n = x$.

**m-PE ring:** A ring $R$ called $m$-$PE$ ring ($m^{th}$-power endomorphism ring) if $f_m$ is an additive endomorphism of $R$ - that is, if $R$ satisfies the polynomial identity $(x + y)^m = x^m + y^m$.

**Jacobson radical:** The Jacobson radical of a ring $R$ is the intersection of all the maximal left ideals of $R$.

**Prime radical:** The prime radical of $R$ is the intersection of all prime ideals of $R$.

**Subdirectly irreducible:** A ring $R$ is said to be subdirectly irreducible if for every subdirect product representation of $R$ is trivial.
**Galois field:** A finite field is called a Galois field. A Galois field with $p^n$ elements is usually written as $GF(p^n)$.

**Nil ideal:** An ideal $A$ of a ring $R$ is said to be a nil ideal if it contains only nilpotent elements.

**Annihilator:** If $S$ is a subset of a ring $R$, then $A(S) = \{x \in R/ \forall s \in S, xs = 0\}$ is called the annihilator of $S$ in $R$.

**Nilpotent ring:** A ring is called nilpotent if there is a fixed positive integer ‘$r$’ such that every product involving ‘$r$’ elements is zero.

**Quasi-regular:** Let $R$ be ring (with unity) and let $r$ be an element of $R$. Then $r$ is said to be quasi-regular, if $1-r$ is a unity in $R$; that is, invertible under multiplication. The notions of right or left quasi-regularity correspond to the situations where $1-r$ has a right or left inverse, respectively.

**Right quasi-inverse:** An element $x$ of a ring is said to be right quasi-inverse if there is $y$ such that $xy + x + y = 0$.

In 1969 [29, 30] Chacron defined the periodic rings and quasi-periodic rings. Also he has proved that co-algebraic rings are necessarily periodic, that is, if $R$ is a ring such that $x^n = x^{n+1}p(x)$, for any $x \in R$, $n$ a positive integer and $p(x)$ a polynomial in $x$. This is called as Chacron’s criterion.
In 1976 [16] Bell proved that for periodic $R$ with commuting nilpotent elements, the commutator ideal is nil and nilpotent elements form an ideal. In 1977 [17] Bell introduced a set of criterions on words and shown that if for a periodic ring $R$ there is one type of word (out of nine possible types) such that for each $x, y$ in $R$ there exists a word of the type with length at least three with $xy = w(x, y)$, then $R$ is commutative. In 1980 [19] Bell presented a variety of restrictions on nilpotent elements which imply some measure of commutativity in periodic rings.

In [5, 6] Abu-Khuzam and Yaqub established the periodic rings with commuting nilpotents and structure of certain periodic rings.

In 1988, Bell [21] extended the results of Abu-Khuzam [5] to the $k^{th}$ commutator. Bell [22] proved that a periodic ring with only finitely many noncentral zero divisors must be either finite or commutative.

In 1990 Grosen et. al., [34] introduced the concept of weakly periodic rings and proved that if $R$ is a weakly periodic ring which satisfies $[x - x^n, y - y^n] = 0$ and $[x^n, y^n] \in C$, for $x, y \in R$, $n$ a positive integer, then $R$ is commutative.

The above studies have opened many avenues for further work. There have been several interesting studies of other periodic rings.

In periodic rings we have several important identities. So we are interested to study some properties of periodic rings in this work. In the next chapter we discuss certain properties of rings with additional conditions and prove that these rings are periodic.