ABSTRACT

The study of associative rings has yielded many interesting results in algebra. During the last 5 decades there have been many results concerning conditions that force a ring to be commutative. In particular, the results on periodic rings have applications in other branches of algebra.

Many mathematicians of recent years have studied the periodic rings with keen interest and their investigations throw light on the study of other properties of periodic rings. Among these mathematicians M.Chacron, H.E.Bell, Abu-Khuzam, A.Yaqub, M.A. Khan, H. Tominaga, M.Ashraf, and Asma Ali are the ones whose contributions to this field are outstanding.

Throughout this work a ring $R$ is a synonym for an associative ring. $N$, $C$ and $J$ denote the set of nilpotent elements of $R$, the center of $R$ and the Jacobson radical of $R$ respectively.

In this work we present some properties of prime rings and near rings with certain identities. Using these properties we prove that these rings are periodic. We establish the structure of periodic rings. We prove some commutativity results on periodic rings, weakly periodic rings, quasi-periodic rings, generalized periodic rings, D-rings and D*-rings.

The commutator $[x,y]$ is defined as $[x,y] = xy - yx$. A ring $R$ is called periodic, if for each $x$ in $R$, there exist distinct positive integers $m$ and $n$ such
that \( x^n = x^m \). An element \( x \) is potent if there exists \( n = n(x) > 1 \) such that \( x^n = x \). A ring \( R \) is called weakly periodic, if every \( x \) in \( R \) can be written in the form \( x = a + b \) for some nilpotent element \( a \) and some potent element \( b \) in \( R \).

A ring \( R \) with center \( C \) is called weakly periodic-like ring if every element \( x \) in \( R \setminus C \) can be written in the form \( x = a + b \), \( a \in N \), \( b \) potent. \( R \) is called a quasi-periodic ring, if for each \( x \) in \( R \) there exist integers \( k, n, m \) all depending on \( x \) such that \( n > m > 0 \) and \( x^n = k \, x^m \). A ring \( R \) is called generalized periodic if for every \( x \) in \( R \) such that \( x \not\in (N \cup C) \), we have \( x^n - x^m \in (N \cap C) \), for some positive integers \( m, n \) of opposite parity. A ring \( R \) such that every zero divisor is nilpotent is called a \( D \)-ring. A ring \( R \) is called a \( D^* \)-ring, if every zero divisor \( x \) in \( R \) can be written as \( x = a + b \), where \( a \) is a nilpotent element and \( b \) is a potent element. A ring \( R \) is called a p-ring, if \( x \) is any element of \( R \) such that \( x^p = x \) and \( px = 0 \) (\( p \) is prime). A generalized p-ring (\( p \) prime) is a ring \( R \) of prime characteristic \( p \) such that \( x^py - xy^p \in N \) for all \( x, y \) in \( R \setminus (N \cup J \cup C) \).

A ring \( R \) is called a prime ring if for any \( a, b \in R \), \( a R b = (0) \) implies that either \( a = 0 \) or \( b = 0 \). A ring \( R \) is \( n \)-torsion free if \( nx = 0 \) implies \( x = 0 \) for \( x \in R \). A ring \( R \) is called a near ring with two binary operations \( + \) and \( * \) such that (i) \((R, +)\) is a group, not necessary abelian (ii) \((R, *)\) is a semi group (iii) \( a * (b + c) = a * b + a * c \) for all \( a, b, c \) in \( R \).
The first chapter is devoted to present the necessary background. We give a brief survey of the work done by M.Chacron, H.E.Bell, Abu-Khuzam, A.Yaqub, Grosen, Rosin, M.Ashraf and M.A.Khan.

In chapter 2, we present some prime rings, and near rings, which are periodic. Chacron proved that if a ring $R$ satisfies the identity $x^n = x^{n+1} p(x)$ for every $x \in R$ and $p(x)$ is a some polynomial in $x$, then $R$ is periodic. In section 2.1, we show that if $R$ is a prime ring satisfying $[x^2, y] - [x, y^2] \in C$, then $xy + yx \in C$, for all $x, y$ in $R$. Using this we prove that if $R$ is a 2-torsion free noncommutative prime ring satisfying $[x^n, y] - [x, y^n]$ and $[x^{n+1}, y] - [x, y^{n+1}]$ in the center $C$, then $R$ is periodic. In section 2.2, we consider a ring $R$ satisfying the condition $x^k y^l = y^m x^n$, where $x$ and $y$ are any elements of $R$, $k$, $l$, $m$ and $n$ are positive integers with either $k \neq m$ or $l \neq n$, depending on $x$ and $y$. First we show that if $R$ contains no nonzero idempotents, then $R$ is nil. We use this to prove that $R$ is periodic. In section 2.3, we present some properties of near rings. Using these properties, it is shown that if $R$ is a near ring with 1 satisfying $xy = y^m x^n p(x)$, where $x, y \in R$ and $n, m$ are positive integers, then $R$ is periodic.

In chapter 3, we establish the structure of periodic rings. Section 3.1 contains some elementary properties of periodic rings. In Section 3.2, we see that if $R$ is a periodic ring satisfying $[x^n, [x^n, a]] = 0$ and $[x^{n+1}, [x^{n+1}, a]] = 0$ for any $x \in R$, $a \in N$, $n$ a positive integer and $N$ is commutative, then $R$ is
commutative. Using this it is proved that a periodic ring $R$ satisfying $xy^n - yx^n$ and $xy^{n+1} - yx^{n+1}$ in the center of $R$ is commutative. In Section 3.3, we discuss some finiteness conditions of periodic rings. We prove the decomposition theorem for finite periodic rings whose prime radicals are in the center. That is, if $R$ is a finite periodic ring and the radical is in the center of $R$, then $R$ is a direct sum of a finite commutative ring and finitely many matrix rings over finite fields.

Chapter 4 deals with some properties of weakly periodic quasi-periodic and generalized periodic rings. In section 4.1, using some properties of weakly periodic rings proved by Abu-Khugam et. al., we prove that if $R$ is a weakly periodic ring satisfying $[x a - x a^2 x, x] = 0$ for all $x \in R$, $a \in N$, then $R$ is commutative. Also it is proved that in a weakly periodic ring $R$ with $[a, b]$ potent for all $a \in N$, $b \in N$, if there exists a word $w = w(x, y)$ such that $w[[x, y], xy] = 0$, then $R$ is commutative. In section 4.2, we use the properties of quasi-periodic rings proved by Bell to prove that an $n$-torsion free quasi-periodic ring $R$ with identity 1 satisfying $x^n y^n = y^n x^n$ and $(xy)^{n+1} - x^{n+1} y^{n+1}$ in the center $C$ of $R$, is commutative. In section 4.3, we prove that every non-zero idempotent element in a 2-torsion free generalized periodic ring is central. Using this we prove that if $R$ is a generalized periodic ring with idempotent element which is not a zero divisor, then $R$ is commutative. We also prove that if $[a, b] = ab - ba$ is potent $a, b \in N$, then the generalized periodic ring $R$ is commutative.
Chapter 5 is devoted to the discussion of further possible developments of some results which we wish to study in future. In section 5.1 we discuss some results on D-ring and D*-ring. We prove that if $R$ is a D-ring satisfying $xy = (xy)^2 p(x,y)$, for all $x, y \in R$ and $p(x,y)$ is a polynomial in two non-commuting indeterminates $x$ and $y$, then $R$ is either a zero ring or a periodic field and also we prove that any normal D*-ring is either periodic or D-ring. We wish to try for some more properties of D-rings and D*-rings. In section 5.2, we show that if $R$ is a weakly periodic-like ring satisfying $(xy)^n - y^n x^n$ in the center of $R$, then $R$ is commutative. We wish to study results concerning commutativity of weakly periodic-like rings with other identities in the center. In section 5.3, we discuss certain properties of p-rings and generalized p-rings. Using these properties, we prove that if $R$ is a generalized p-ring with central idempotents and $J$ is commutative, then $R$ is commutative. We wish to try for further properties of generalized p-rings.

These are a few ideas which arise in the course of our study. We do hope several others will emerge as we proceed.

Some of the results of chapters 2 and 3 are published in the journal “International Review on Pure and Applied Mathematics” Vol. 6-2, (2010), 217-220. Some results of chapter 4 are accepted for publication in the “Proceedings of the International Conference on Groupoids, Semigroups and Automata” and “Proceedings of the International Conference on Advances in Mathematics and Computational Methods”.