Chapter – 5

PROBLEMS FOR FURTHER STUDY
This chapter is devoted to the discussion on further possible developments of some results which we plan to study in our future work.

In chapter 2 we have proved that near rings are periodic with some additional conditions. We wish to study other rings which are periodic. In chapters 3 and 4 we have seen some properties of periodic, weakly periodic, quasi-periodic and generalized periodic rings. We would like to investigate other identities and properties of these rings.

In section 5.1, we discuss some results on D-rings and D*-rings. We prove that if \( R \) is a D-ring satisfying \( xy = (xy)^2 p(x,y) \), for all \( x, y \in R \) and \( p(x,y) \) is a polynomial in two noncommuting indeterminates \( x \) and \( y \), then \( R \) is either a zero ring or a periodic field. Also we prove that any normal D*-ring is either periodic or D-ring. In section 5.2, we show that if \( R \) is a weakly periodic-like ring satisfying \((xy)'' - y''x'' \) in the center of \( R \), then \( R \) is commutative. In section 5.3, we discuss certain properties of p-rings and generalized p-rings. Using these properties, we prove that if \( R \) is a generalized p-ring with central idempotents and \( J \) is commutative, then \( R \) is commutative.
5.1. Structure of D-rings and D*-rings


In this section, we discuss some results on D-rings and D*-rings. We prove that if \( R \) is a D-ring satisfying \( xy = (xy)^2 p(x,y) \) for all \( x, y \in R \) and \( p(x,y) \) is a polynomial in two non-commuting indeterminates \( x \) and \( y \), then \( R \) is either a zero ring or a periodic field. Also we prove that any normal D*-ring is either periodic or D-ring.

We know that a ring \( R \) is a D-ring, if its every zero divisor is nilpotent. A ring \( R \) is a D*-ring, if every zero divisor \( x \) in \( R \) can be written as \( x = a + b \), where \( a \in N, b \in P \) and \( ab = ba \). Nil ring, every domain of \( R \) and the ring of integers (mod \( p^k \)), \( p \) prime are examples of D-rings. Clearly every D-ring is a D*-ring, but converse is not true. A Boolean ring is a D*-ring but not a D-ring.

We start with the following properties of D-rings.

**Lemma 5.1.1:** Let \( R \) be a D-ring. Then \( aR \) is a nil right ideal for all \( a \in N \).

**Proof:** Since \( a^k = 0 \), \( a^{k-1} \neq 0 \) implies \( a^{k-1} (ax) = 0 \), and thus \( ax \in N \). Hence \( aR \) is a nil right ideal of \( R \). \( \square \)
Lemma 5.1.2: Let $R$ be a D-ring. If $e$ is an idempotent element of $R$, then $e = 0$ or $e = 1$.

Proof: Suppose $e^2 = e \neq 0$, and $x \in R$.

Then $e(ex-x) = 0$ and hence $ex - x = 0$.

Otherwise, $e$ will be nilpotent, since $R$ is a D-ring. Thus $e = 0$.

Similarly, $xe - x = 0$ for all $x$ in $R$, and thus $e = 1$. □

Theorem 5.1.3: Let $R$ be a D-ring such that $N$ is an ideal of $R$. Then, either $R = N$ or $R/N$ is a domain.

Proof: Suppose that $R \neq N$.

Let $\bar{x} = x + N$ and $\bar{y} = y + N$ be two elements in $R/N$ such that $xy = 0$.

Then $xy \in N$.

This implies that $(xy)^m = 0$ and $(xy)^{m-1} \neq 0$ for some positive integer $m$.

Hence $(xy)^{m-1} (xy) = 0$.

This implies that $y$ is a zero divisor or $(xy)^{m-1} x = 0$.

Therefore, $y$ is a zero divisor or $x$ is a zero divisor, since $(xy)^{m-1} \neq 0$.

Hence $y \in N$ or $x \in N$, so $\bar{x} = \bar{0}$ or $\bar{y} = \bar{0}$.

Thus $R/N$ is a domain. □

Corollary 5.1.4: Let $R$ be a D-ring with $N$ commutative. Then either $R = N$ or $N$ is an ideal and $R/N$ is a domain.
**Proof:** If $N$ is commutative, $N$ is an additive subgroup of $R$, hence an ideal by Lemma 5.1.1.

This corollary follows from the above theorem. 

**Theorem 5.1.5:** If $R$ is a periodic D-ring, then $R$ is either nil or local.

Further, if $R$ has an identity element, then $N$ is an ideal and $R/N$ is a field.

**Proof:** Since $R$ is periodic, for each $x \in R$, there exists a positive integer $k = k(x)$ such that $x^k$ is idempotent [19]. Using Lemma 5.1.2, $x^k = 0$ or $x^k = 1$, and hence $x$ is either nilpotent or invertible.

Therefore, $R$ is nil or local.

If $R$ has an identity element, then $R$ is local and hence $N$ is an ideal.

Thus $R/N$ is a periodic division ring and hence $R/N$ is a field. 

**Theorem 5.1.6:** If $R$ is a periodic D-ring, then $C(R)$ is nil.

**Proof:** If $R$ is nil, there is nothing to prove.

Suppose $R \neq N$, and let $x \in R/N$.

Then $x^n = x^m$ for some integers $n > m \geq 1$.

It is easily verified that $x^{m(n-m)}$ is a nonzero idempotent, and hence by Lemma 5.1.2, $1 \in R$.

By theorem 5.1.5, $R$ is local, $N$ is an ideal, and $R/N$ is a field.

Thus, $C(R)$ is nil.
Now we prove that a D-ring with a condition is either a zero ring or a periodic field.

**Theorem 5.1.7:** Let $R$ be a D-ring such that for each $x, y \in R$ there exists a polynomial $p(x,y)$ in two noncommuting indeterminates, with integer coefficient, for which

$$xy = (xy)^2 p(x,y).$$ \hspace{1cm} 5.1.1

Then $R$ is either a zero ring or a periodic field.

**Proof:** Theorem 1 of [26] states that any ring $R$ satisfying 5.1.1 is a direct sum of a J-ring and a zero ring. In view of theorem 12 [9], a D-ring with 5.1.1 must be either a J-ring or a zero ring. By Lemma 5.1.2, D-rings which are also J-rings must be periodic division rings; and J-rings are commutative by Jacobson's famous "$d^n = a$ Theorem". \hfill $\Box$

Now we present a result on a ring $R$ in which every zero divisor is potent.

**Theorem 5.1.8:** Let $R$ be a ring in which every zero divisor is potent. Then $N = \{0\}$ and $R$ is normal. Moreover, if $R$ is not a domain, then $J = \{0\}$.

**Proof:** If $a \in N$, then $a$ is a zero divisor and hence potent by hypothesis. So $a^n = a$ for some positive integer $n$, and since $a \in N$, there exist a positive integer $k$ such that $0 = a^k = a$. So $N = \{0\}$.

Let $e$ be any idempotent element of $R$ and $x$ is any element of $R$.

Then $ex - exe \in N$, and hence $ex - exe = 0$. 
Similarly, \( xe = exe \). So \( ex = xe \) and \( R \) is normal.

Let \( x \) be a nonzero divisor of zero.

Then \( xJ \) consists of zero divisors, which are potent.

Therefore \( xJ = \{0\} \).

But then \( J \) consists of zero divisors, hence potent elements, and therefore \( J = \{0\} \).

Next we prove some results on \( D^* \)-rings.

**Theorem 5.1.9:** A ring \( R \) is a \( D^* \)-ring if and only if every zero divisor of \( R \) is periodic.

**Proof:** We assume \( R \) is a \( D^* \)-ring and let \( x \) be any zero divisor. Then

\[ x = a + b, \ a \in N, \ b \in P, \ ab = ba. \]

So \( (x - a) = b = b^n = (x - a)^n \).

This implies, since \( x \) commutes with \( a \), that \( (x - a) = (x - a)^n = x^n + \text{sum of pairwise commuting nilpotent elements} \).

Hence \( x - x^n \in N \) for every zero divisor \( x \).

Since each such \( x \) is included in a subring of zero divisors, which is periodic by Chacron's theorem, \( x \) is periodic.

Suppose, conversely, that each zero divisor is periodic.

Then by the proof of Lemma 1 [17], \( R \) is a \( D^* \)-ring.

**Theorem 5.1.10:** If \( R \) is any normal \( D^* \)-ring, then either \( R \) is periodic or \( R \) is a \( D \)-ring. Moreover, \( aR \subseteq N \) for each \( a \in N \).
**Proof:** If $R$ is a normal $D^*$-ring which is not a $D$-ring, then $R$ has a central idempotent zero divisor $e$.

Then $R = eR \oplus A(e)$, where $eR$ and $A(e)$ both consist of zero divisors of $R$, hence (in view of Theorem 5.1.10) are periodic. Therefore $R$ is periodic.

Now consider $a \in N$ and $x \in R$.

Since $ax$ is a zero divisor, hence a periodic element, $(ax)^j = e$ is a central idempotent for some $j$.

Thus $(ax)^{j+1} = (ax)\cdot ax = a^2y$ for some $y \in R$.

By repeating this argument we see that for each positive integer $k$, there exists $m$ such that $(ax)^m = a^2w$ for some $w \in R$.

Therefore $aR \subseteq N$. □

**Corollary 5.1.11:** Let $R$ be a $D^*$-ring which is not a $D$-ring.

If $N \subseteq C$, then $R$ is commutative.

**Proof:** Since $N \subseteq C$, $R$ is normal. Therefore commutativity follows from theorem 5.1.10 and a theorem of Herstein. □

Using the above results, we wish to try for some more properties of $D$-ring and $D^*$-rings.
5.2 Weakly periodic-like rings


In this section, we show that if $R$ is a weakly periodic-like ring satisfying $(xy)^n - y^n x^n$ in the center of $R$, then $R$ is commutative.

We know that a ring $R$ is weakly periodic-like if every $x$ in $R \setminus C$ can be written in the form $x = a + b$, $a \in N$, $b$ potent ($b^k = b$ for some $k > 1$).

Throughout this section $R$ denotes an associative ring, $N, C, C(R)$ and $J$ denote the set of nilpotents, the center of $R$, the commutator ideal of $R$ and the Jacobson radical of $R$ respectively.

From [63, 10] we have the following properties of weakly periodic-like rings.

**Theorem 5.2.1**: [63] Let $R$ be a weakly periodic-like ring.

(a) If $N$ is an ideal, then for each $x$ in $R$, either $x \in C$ or $x - x^n \in N$ for some integer $n = n(x) > 1$.

(b) Every ideal $I$ of $R$ is weakly periodic-like.

(c) $J \subseteq N$ or $J \subseteq C$.

(d) If $C(R) \subseteq J$, then $N$ is an ideal and $R/N$ is commutative.
**Theorem 5.2.2:** [63] If \( R \) is a weakly periodic-like ring with \( N \subseteq C \), then \( R \) is commutative.

**Theorem 5.2.3:** [63] Let \( R \) be a weakly periodic-like ring with \( N \) commutative. Then \( N \) is an ideal and \( R/N \) is commutative.

**Theorem 5.2.4:** [10] Let \( R \) be a weakly periodic-like ring such that each non-central element is uniquely expressible as a sum of a potent element and a nilpotent element. If \( N \) is commutative, then \( R \) is commutative.

**Lemma 5.2.5:** Let \( R \) be a weakly periodic-like ring with the set of nilpotents commutative and with idempotents central. If \( R \) has a property which implies commutativity in weakly periodic-like rings with identity and which is inherited by ideals, then \( R \) is commutative.

**Proof:** First we prove that the set \( P \) of potent elements is central.

Suppose \( a \in P \) with \( a^n = a, \ n > 1 \). 5.2.1

Let \( e = a^{-1} \). Then, since \( e \) is central idempotent, \( eR \) is a ring with identity.

Hence \( eR \) is weakly periodic-like which, in fact, is an ideal of \( R \).

The hypothesis of the Lemma,

\[ eR \text{ is commutative.} \] 5.2.2

Therefore,

\[ (e)(x) = (x)(e) \] for all \( x \) in \( R \). 5.2.3

Since \( e = a^{-1} \) is a central idempotent element of \( R \).
Hence 5.2.3 implies that

\[ eax = exa = xae. \]

That is \( a^{n-1} x = x a^{n-1} \) or \( a^n x = xa^n \).  

Thus, by 5.2.2, \( ax = xa \) for all \( x \) in \( R \).

This proves that the set \( P \) of potent elements of \( R \) is central.  

To complete the proof, suppose \( x, y \in R \).

If \( x \in C \) or \( y \in C \), then clearly \( [x, y] = 0 \).

So suppose \( x \notin C \) and \( y \notin C \).

Then by definition,

\[ x = a + b, y = a' + b', a, a' \text{ nilpotent and } b, b' \text{ potent} \]

By 5.2.5, \( b \) and \( b' \) are central.

Hence \( [x, y] = [a + b, a' + b'] = [a, a'] = 0 \), since \( N \) is commutative.

This proves the lemma. \( \square \)

**Theorem 5.2.6:** Let \( R \) be a weakly periodic-like ring, and let \( n \) be a fixed positive integer. Suppose \( R \) is \( n(n+1) \) torsion free and, for all \( x, y \in R \), \( (xy)^n - y^n x^n \in C \). Suppose, further that the set \( N \) of nilpotents is commutative. Then \( R \) is commutative.

**Proof:** Let \( \mathcal{P} \) be the ring property \( (xy)^n - y^n x^n \) is always central.

Clearly, this property is satisfied by all subrings and all homomorphic images of any subring of \( R \).
Moreover, this property \( \mathcal{P} \) is not satisfied by any complete matrix ring \( D_n \) of \( n \times n \) matrices over any division ring \( D \), where \( n > 1 \), as can be seen by taking \( x \) and \( y \) in \( D_n \) to be

\[
x = E_{11}, \quad y = E_{11} + E_{12}, \quad E_{11}, E_{12} \in D_n.
\]

Hence, by theorem 2 iii [10], we have

\[
\text{if } x \in R \setminus C, \text{ then } x - x^m \in N \text{ for some integer } m > 1. \quad 5.2.6
\]

Moreover, by theorem 2(ii) [10], \( N \) is an ideal which is commutative.

So \( N \) is a commutative ideal and hence \( N^2 \subseteq C \). \( 5.2.7 \)

We now distinguish two cases

**Case 1:** \( 1 \in R \). Suppose \( a \in N, b \in R \).

Then, by hypothesis,

\[
[(a + 1)b]^n - b^n (a + 1)^n \in C \text{ and } [b(a + 1)]^n = (a + 1)^n b^n \in C.
\]

By subtracting and using \( N^2 \subseteq C \), we get

\[
(n + 1) [a, b^n] \subseteq C,
\]

and since \( R \) is \( n(n + 1) \)-torsion free, we conclude that

\[
[a, b^n] \subseteq C \text{ for all } a \in N, b \in R. \quad 5.2.8
\]

Since \( N \) is commutative, 5.2.8 implies

\[
[a, b^n] \subseteq N \text{ for all } a \in N, b \in R. \quad 5.2.9
\]

Now, suppose \( x_1, x_2, \ldots, x_k \in R \). Since \( R/C(R) \) is commutative,

\[
(x_1 \ldots x_k)^n - x_1^n \ldots x_k^n \subseteq C(R) \subseteq N, \text{ by } 5.2.7.
\]

But \( N \) is commutative, and hence
By combining 5.2.9 and 5.2.10, we conclude that

\[[a, x_1^{n} \ldots x_k^{n}] = [a, x_1^{n} \ldots x_k^{n}], a \in N. \]

Let $S$ be the subring of $R$ generated by the $n$-th powers of elements of $R$.

Then by 5.2.11

\[[a, x] \in C(S) \text{ for all } a \in N(S), x \in S, \]

where $N(S)$ and $C(S)$ denote the set of nilpotents of $S$ and the center of $S$, respectively.

Since $S$ is periodic, $N(S)$ is commutative, and from 5.2.12, Lemma 3.2.3 shows that $S$ is commutative, and hence

\[[x^n, y^n] = 0 \text{ for all } x, y \in R. \]

We continue to assume that $a \in N, b \in R$.

Then by hypothesis and the facts that $R/C(R)$ is commutative and $C(R) \subseteq N$, we have

\[[((1 + a)b)^n - b^n (1 + a)^n = c \in C \]

and \[[b(1 + a))^n - (1 + a)^n b^n = c' \in C. \]

5.2.14 and 5.2.15 imply the following:

\[b [(1 + a)b]^n (1 + a) - b^{n+1} (1 + a)^{n+1} = bc (1 + a), \]

\[(1 + a) [b(1 + a))^n b - (1 + a)^{n+1} b^{n+1} = (1 + a) c' b. \]

We note that $c \in N$, and $N^2 \subseteq C$.

Hence $bca \in C$. 

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Similarly $ac'b \in C$.

In view of this, and the facts that $c \in C$ and $c' \in C$,

we see that

$$[bc(1 + a), b] = 0 \text{ and } [(1 + a)c'b, b] = 0.$$ 

Hence by 5.2.16 and 5.2.17, we conclude that

$$\{b(1 + a)\}^{n+1} - b^{n+1} (1 + a)^{n+1} \text{ commutes with } b,$$ 
and $$\{(1 + a)b\}^{n+1} - (1 + a)^{n+1} b^{n+1} \text{ commutes with } b \text{ also.}$$

Recalling that $N^2 \subseteq C$, and subtracting 5.2.18 from 5.2.19, we obtain

$$n[b^{n+1}, a] \text{ commutes with } b \text{ for all } a \in N, b \in R.$$ 

Since $N$ is commutative and $R$ is $n(n + 1)$ - torsion free, 5.2.20, implies,

$$[a, b^{n+1}] \text{ commutes with } b \text{ for all } a \in N, b \in R.$$ 

By 5.2.13, $[x^n, y^n] = 0$ for all $x, y \in R, 1 \in R$, and $R$ is $n$-torsion free.

Hence, we see that

$$[a, b^n] = 0 \text{ for all } a \in N, b \in R.$$ 

Now, 5.2.21 implies that

$$ab^{n+2} - b^{n+1} ab = ba b^{n+1} - b^{n+2} a \text{ and by 5.2.22, we get}$$

$$b^n ab^2 - b^{n+1} ab = b^{n+1} ab - b^{n+2} a.$$ 

Hence $b^n \{[a, b] \} = b^n [b[a, b]]$.

Thus $b^n [[a, b], b] = 0$ for all $a \in N, b \in R$.

We replacing $b$ by $(b + 1)$ in this last equation give us

$$(b + 1)^n [[a, b], ab] = 0 \text{ for all } a \in N, b \in R.$$
By a Lemma in [50], 5.2.22 and 5.2.23 imply that
\[ [[a, b], b] = 0 \text{ for all } a \in N, b \in R. \] 5.2.24

Since \( R \) is periodic, \( N \) is commutative, and 5.2.24 holds, we see that \( R \) is commutative by Lemma 3.2.3.

**Case 2:** \( R \) does not have an identity. In this case, we first prove the following:

**Claim 1:** The idempotents of \( R \) are central.

Let \( e^2 = e \in R, r \in R \). By hypothesis,
\[ [e (e + er - ere)]^n - (e + er - ere)^n e^n \in C. \]

Hence \( er - ere \in C \). Therefore
\[ er - ere = e (er - ere) = (er - ere)e = 0, \]
and thus \( er - ere \). Similarly \( re - ere \).

Thus \( e \) is central.

**Claim 2:** Under the hypothesis of theorem, if \( \sigma : R \rightarrow S \) is a homomorphism of \( R \) onto \( S \), then the nilpotents of \( S \) coincide with \( \sigma (N) \), where \( N \) is the set of nilpotent of \( R \).

This claim has been stated in Lemma 4 [7].

To complete the proof of the theorem, first me recall that \( R \) is isomorphic to a subdirect sum of subdirectly irreducible rings \( R_i \). Suppose that \( \sigma_i : R \rightarrow R_i \) is the natural homomorphism of \( R \) onto \( R_i \).
Let \( x_i \in R_i \) and let \( \sigma_i(x) = x_{i, x} \in R_i \).

Since \( R \) is periodic, \( x^s = x^r \) for some integers \( s > r > 0 \),

Hence \( e = x^{(s-r)r} \) is idempotent. 5.2.25

By claim 1, \( e \) is central in \( R_i \), and hence \( \sigma_i(e) \) is a central idempotent of \( R_i \).

Since \( R_i \) is subdirectly irreducible, \( \sigma_i(e) = 0 \) or \( \sigma_i(e) = 1 \) if \( 1_i \in R_i \).

**Case A:** \( R_i \) does not have an identity.

In this case \( \sigma_i(e) = 0 \) and hence, \( x_i^{(s-r)r} = 0 \).

This \( R_i \) is nil and hence, by claim 2, \( R_i = \sigma_i(N) \).

By hypothesis, \( N \) is commutative. Therefore \( R_i \) is commutative.

**Case B:** \( R_i \) has an identity \( 1_i \).

We note that \( R_i \) need not be \( n(n+1) \)-torsion free.

So let \( \sigma_i(e_0) = 1_i \), \( e_0 \in R_i \), and we choose integers \( s > r > 0 \) such that \( e_0 = e_0^r \).

Let \( e = e_0^{(s-r)r} \).

Then \( e \) is idempotent and, moreover, \( \sigma_i(e) = 1_i^{(s-r)r} = 1_i \).

Also, \( e \) is central and hence \( e \) is a nonzero central idempotent element of \( R_i \).

Thus, \( eR_i \) is a ring with identity \( e \).

Because \( eR_i \) inherits all the hypotheses of the ground ring \( R \), it follows by the first part of the proof that \( eR_i \) is commutative and hence \([ex, ey] = 0 \) for all \( x, y \in R_i \).
This implies

\[ [\sigma_i(x), \sigma_j(y)] = 0 \text{ for all } x, y \in R. \]

Thus \( R_i = \sigma_i(R) \) is again commutative. Hence the ground ring \( R \) is commutative, and the theorem is proved. □

We wish to study results concerning commutativity of weakly periodic-like rings with other identities in the center.

### 5.3 Generalized p-rings with central idempotents

The concept of p-rings, (p prime) was defined by McCoy and Montgomery [48]. They proved that every p-ring is a direct product of field \( F_p \), and every p-ring is isomorphic to a subring of a direct product of fields \( F_p \). Moisil [49] developed the concept of unitary p-rings, where \( p = 3 \). In [46] Loustau shown that if \( R \) is a p-ring with \( p^2 \), then \( R \) is isomorphic to a subdirect sum of copies of the Galois field with \( p \) elements. Yaqub [62] introduced the concept of generalized p-rings and proved that generalized p-ring is commutative under some restrictions.

In this section, we discuss certain properties of p-rings and generalized p-rings. Using these properties, it is proved that if \( R \) is a generalized p-ring with central idempotents and \( J \) is commutative, then \( R \) is commutative.
We know that a ring $R$ is a p-ring (p-prime) in which $x^p = x$ and $px = 0$ for all $x$ in $R$. A ring $R$ is a generalized p-ring of prime characteristic $p$ if $x^p y - xy^p \in N$ for all $x, y$ in $R\setminus(N \cup J \cup C)$.

The Boolean rings are simple 2-rings and

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| 0 \leq 1 \in GF(2) \right\}$$

is a generalized 2-ring but not a 2-ring. In this section $F_p$ denotes field of integers reduced modulo $p$, $N$, $C$ and $J$ denote the set of nilpotents, the center of $R$ and Jacobson radical of $R$ respectively.

We have the following properties of p-rings, proved by McCoy and Montgomery [48] and Loustau [46].

**Theorem 5.3.1:** [48] A p-ring $R$ may be imbedded in a p-ring $R^*$ which contains a unit element.

**Theorem 5.3.2:** [48] Every finite p-ring contains a unit element and is a direct sum of field $F_p$.

**Theorem 5.3.3:** [48] If $R$ is any p-ring containing a unit element $e$ and if $a$ is any nonzero element of $R$, there exists a homomorphism $h$ of $R$ into $F_p$ such that $h(a) \neq 0$.

**Theorem 5.3.4:** [48] If $R$ is any p-ring, it is isomorphic to a subring of a direct sum of rings $F_p$. 


Theorem 5.3.5: [46] Let $R$ be a p-ring with $p \neq 2$, then $R$ is associative and commutative. Thus, $R$ is a subdirect sum of copies of $GF(p)$.

Now we present some properties of generalized p-rings.

Theorem 5.3.6: Suppose $R$ is a generalized p-ring ($p$ prime) with identity and with central idempotents. Suppose, further, that $J$ is commutative. Then $R$ is commutative.

Proof: First we prove the following result:

If $b \in R$ satisfies the equation $b^p = b$, then $b$ is central. 5.3.1

Let $r \in R$. Since $b^{p-1}$ is idempotent, $b^{p-1}$ is central, and hence

$$b^{p-1} (rb - br) = rb^p - b^p r = rb - br.$$ 

This implies that

$$(b^{p-1} - 1) (rb - br) = 0 \text{ for all } r \in R. \quad 5.3.2$$

Since $R$ is of prime characteristic $p$, an elementary number theoretic result shown that 5.3.2 is equivalent to

$$(b + 1) (b + 2) \ldots \ldots (b + (b + 1)) (rb - br) = 0, \text{ } r \in R. \quad 5.3.3$$

Furthermore, since $R$ is of prime characteristic $p$, we have

$$b^p = b \text{ implies } (b + 1)^p = (b + 1).$$

Hence the above argument can be repeated with $b$ replaced by $b + 1$ throughout.

Thus 5.3.3 now yields

$$(b + 2) (b + 3) \ldots \ldots (b + (p - 1)) (b + p) (r + (b + 1) - (b + 1)r) = 0,$$
and hence
\[ b(b + 2)(b + 3) \ldots \ldots \ldots (b + (p - 1))(rb - br) = 0. \] 5.3.4

Subtracting 5.3.3 from 5.3.4, we obtain
\[ 1 \cdot (b + 2)(b + 3) \ldots \ldots \ldots (b + (p - 1))(rb - br) = 0. \] 5.3.5

Repeating this argument, where \( b \) is replaced by \( b + 1 \) again throughout, we see that 5.3.5 now yields
\[ 1 \cdot (b + 3)(b + 4) \ldots \ldots \ldots (b + (p - 1))(b + p)(rb - br) = 0. \]

Hence
\[ 1 \cdot b(b + 3)(b + 4) \ldots \ldots \ldots (b + (p - 1))(rb - br) = 0. \] 5.3.6

Subtracting 5.3.6 from 5.3.5, we obtain
\[ 1 \cdot 2 \cdot (b + 3)(b + 4) \ldots \ldots \ldots (b + (p - 1))(rb - br) = 0. \]

By continuing this process, we eventually obtain
\[ (p - 1)! (rb - br) = 0 \text{ for all } r \in R. \] 5.3.7

Since \( (p - 1)! \) is relatively prime to the prime characteristic \( p \) of \( R \), 5.3.7 yields \( rb - br = 0 \) for all \( r \) in \( R \), and hence \( b \) is central, which proves 5.3.1.

Because \( R \) is an \( N_0 \)-ring with central idempotents, by Lemma 2.4 [62], \( N \subseteq J \).

If we combine this with \( x^p y - xy^p \in N \) yields \( x^p y - xy^p \in N \) for all \( x, y \) in \( R(J \cup C) \), since \( N \subseteq J \). 5.3.8

We now prove that
\[ J \subseteq C. \] 5.3.9
Suppose not. Then there exists an element \( j \in J \notin C \), and thus for some \( x \in R \), \([x, j] \neq 0\), where \([x, j] = xj - jx\).

Since, by hypothesis, \( J \) is commutative, \( x \notin J \).

Hence \( x \notin (J \cup C) \).

Since, \([x + 1, j] = [x, j] \neq 0\), we see that \( x + 1 \notin J \cup C \) either.

Hence, by 5.3.8, \( x^p(x + 1) - x(x + 1)^p \in N \), which implies that \( x^p - x \in N \), since \( R \) is of prime characteristic \( p \).

Hence \((x - x^p), (x - x^p)^p, (x - x^p)^{p^2}, \ldots, (x - x^p)^{p^{k+1}}\) are pairwise commuting nilpotent elements of \( R \), which implies that

\[
(x - x^p) + (x - x^p)^p + (x - x^p)^{p^2} + \ldots + (x - x^p)^{p^{k+1}} \in N. \tag{5.3.10}
\]

Since \( R \) is of prime characteristic \( p \), 5.3.10 implies that

\[
x - x^p + x^p - x^{p^2} + x^{p^2} - x^{p^3} + \ldots + x^{p^{k+1}} - x^{p^k} \in N,
\]

which simplifies to

\[
x - x^{p^k} \in N \text{ for all positive integer } k. \tag{5.3.11}
\]

Since, in particular, \( x - x^p \in N \), we have \((x - x^p)^{p^{k_0}} = 0\) for some positive integer \( k_0 \).

Hence

\[
x^{p^{k_0}} = (x^{p^{k_0}})^p. \tag{5.3.12}
\]

Therefore, by 5.3.1 and 5.3.12, we conclude that

\[
x^{p^{k_0}} \in C. \tag{5.3.13}
\]

Hence \( x = (x - x^{p^{k_0}}) + x^{p^{k_0}}; (x - x^{p^{k_0}}) \in N \) by 5.3.11,
\[ x^{p^{n}} \in C, \quad 5.3.14 \]

by 5.3.13.

Since \( N \subseteq J \), 5.3.14 implies that

\[ x = a + b, \quad a = x - x^{p^{n}} \in J, \quad b = x^{p^{n}} \in C. \quad 5.3.15 \]

Hence \([x, j] = [a + b, j] = [a, j] + [b, j] = 0\), since \( J \) is commutative.

Thus \([x, j] = 0\), a contradiction.

This contradiction proves 5.3.9 and hence \( J \subseteq C \). Since \( N \subseteq J \), \( N \subseteq C \), and thus 5.3.8 implies that

\[ x^{p} - x \in N \text{ for all } x, y \in R \setminus C, \quad 5.3.16 \]

Now, suppose \( x \not\in C \), and hence \( x + 1 \not\in C \).

By 5.3.16, \( x^{p}(x + 1) - x(x + 1)^{p} \in N \subseteq C \).

Hence \( x^{p} - x \in C \), since \( R \) is prime characteristic \( p \). This implies that \( x - x^{p} \in C \) for all \( x \) in \( R \).

Hence by Lemma 2.3 [62], \( R \) is commutative, and the theorem is proved. \( \square \)

Now we prove the characterization of generalized p-rings.

**Theorem 5.3.7:** Suppose \( R \) is a generalized p-ring (p prime) and \( R \) is weakly periodic-like. Suppose, Further, that the set \( N \) of nilpotents is commutative, and all idempotents are central. Then \( R \) is commutative.

**Proof:** By Lemma 2.5 [62], \( J \subseteq N \cup C \), and hence \( J \) is commutative, since \( N \) is commutative.
We Now distinguish two cases.

**Case 1:** \( 1 \in R \). Then, by theorem 5.3.6, \( R \) is commutative, since \( J \) is commutative.

**Case 2:** \( 1 \not\in R \). If zero is the only potent element of \( R \), then by definition of a weakly periodic-like ring, \( R = N \cup \{0\} \).

Hence \( R \) is commutative, since \( N \) is commutative. Thus we assume that \( R \) has a nonzero potent element.

Let \( a \) be any nonzero potent element of \( R \).

Let \( a^n = a \) with \( n > 1 \), and let \( e = a^{n-1} \).

Then \( e \) is a nonzero idempotent \( \text{ hypothesis, must be central.} \)

Hence \( eR \) is a ring with identity. Moreover, \( eR \) is weakly periodic-like.

Also, \( eR \) is a generalized p-ring because \( eR \) satisfies the condition imposed on \( R \), since the Jacobson radical of \( eR \) is \( eJ \), where \( J \) is the Jacobson radical of \( R \). Thus, \( eR \) is a ring with identity which satisfies all of the hypotheses of the theorem.

Hence by Case 1, \( eR \) is commutative, which implies that \( e[x, y] = 0 \) for all \( x, y \) in \( R \).

In particular, \( e[a, y] = 0 \) for all \( y \) in \( R \) and thus \( a^{n-1}[a, y] = 0 \) for all \( y \) in \( R \), since \( e = a^{n-1} \).

Since \( a^n = a \) and \( a^{n-1} \) is central, we conclude that \( ay = ya \) for all \( y \) in \( R \).

If follows that
all potent elements of $R$ are central.\hfill 5.3.17

To complete the proof, let $x, y \in R$ and suppose $x \notin C$ and $y \notin C$.

Then, by definition of a weakly periodic-like ring, $x = a + b$, $y = a' + b'$; $a, a'$ nilpotent; $b, b'$ potent.

Hence, using 5.3.17 and the hypothesis that $N$ is commutative, we see that

$$[x, y] = [a + b, a' + b'] = [a, a'] = 0.$$  

Thus $R$ is commutative, and the theorem is proved.\hfill $\square$

We wish to try for further properties of generalized $p$-rings.