Chapter - 4

COMMUTATIVITY OF CERTAIN PERIODIC RINGS
In 1990 Grosen et. al., [34] introduced the concept of weakly periodic ring and proved that if R is weakly periodic with 1, the set of nilpotent elements N is commutative and every zero divisor is equal to the sum of idempotent and nilpotent, then N is an ideal of R. Quasi-periodic rings were introduced by Chacron [29]. Bell [20] studied the common properties of quasi-periodic rings and periodic rings. Bell and Yaqub [28] established the concept of generalized periodic rings.

This chapter deals with some properties of weakly periodic, quasi-periodic and generalized periodic rings. In section 4.1, using some properties of weakly periodic rings proved by Abu-Khuzam et. al., we prove that if R is a weakly periodic ring satisfying \([xa - xa^2x, x] = 0\) for all \(x \in R, a \in N\), then R is commutative. Also it is proved that in a weakly periodic ring R with \([a, b]\) potent for all \(a \in N, b \in N\), if there exists a word \(w = w(x,y)\) such that \(w[[x,y], xy] = 0\), then R is commutative. In section 4.2, we use the properties of quasi-periodic rings proved by Bell to prove that an \(n\)-torsion free quasi-periodic ring R with identity 1 satisfying \(x^n y^n = y^n x^n\)
and \((xy)^{n+1} - x^{n+1} y^{n+1}\) in the center \(C\) of \(R\), is commutative. In section 4.3, we prove that every nonzero idempotent element in a 2-torsion free generalized periodic ring is central. Using this we prove that if \(R\) is a generalized periodic ring with idempotent element which is not a zero divisor, then \(R\) is commutative. We also prove that if \([a,b] = ab - ba\) is potent \(a, b \in N\), then the generalized periodic ring \(R\) is commutative.

**4.1 A commutativity study for weakly periodic rings**

Bell and Klein [24] established sufficient conditions for finiteness, commutativity or periodicity of weakly periodic rings. In [11] Abu-Khuzam et. al., using a word \(w = w(x, y)\), studied weakly periodic rings with conditions on commutators. Yaqub [61] shown that a weakly periodic ring \(R\) in which certain extended commutators are potent must have a nil commutator ideal and the set \(N\) of nilpotents form an ideal, which coincides with the Jacobson radical \(J\) of \(R\). Recently Rosin and Yaqub [53] studied the structure of weakly periodic rings with a particular emphasis on conditions which imply that such rings are commutative or have a nil commutator ideal.

In this section, using some properties of weakly periodic rings proved by Abu-Khuzam et. al., we prove that if \(R\) is a weakly periodic ring satisfying \([xa - xa^2x, x] = 0\) for all \(x \in R, a \in N\), then \(R\) is commutative.
Also it is proved that in a weakly periodic ring $R$ with $[a, b]$ potent for all $a \in N$, $b \in N$, if there exists a word $w = w(x, y)$ such that $w[[x, y], xy] = 0 = [[x, y], xy]w'$, then $R$ is commutative.

A ring $R$ is called weakly periodic if every element of $R$ is expressible as a sum of a nilpotent element and a potent element of $R$. It is well known that if $R$ is periodic then it is weakly periodic. For $x, y \in R$, $[x, y]_1 = xy - yx$ is the usual commutator, and for every positive integer $k > 1$, we define inductively $[x, y]_k = [[x, y]_{k-1}, y]$. A word $w(x, y)$ is a product in which each factor is $x$ or $y$. The empty word is defined to be 1.

We begin with the following results for an arbitrary ring $R$, which are useful to prove main results.

**Lemma 4.1.1:** Suppose that $R$ is a ring with identity 1. If $x^n[x, y] = 0$ and $(x+1)^n[x, y] = 0$ for some $x, y$ in $R$ and some integer $n > 0$, then $[x, y] = 0$.

A similar statement holds if we assume $[x, y] x^n = 0$ and $[x, y] (x+1)^n = 0$.

**Proof:** We multiply the equation $(x +1)^n[x, y] = 0$ by $x^{n-1}$ on the left and expand the binomial to get

$$0 = \sum_{k=0}^{n} \binom{n}{k} x^{k+n-1}[x, y] = x^{n-1}[x, y], \text{ using } x^n[x, y] = 0.$$

Similarly, we write $x^n[x, y] = 0$ in the form $(-1 + (x +1))^n[x, y] = 0$.

We expand this and multiply by $(x+1)^{n-1}$ to get

$$0 = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (x+1)^{k+n-1}[x, y] = (-1)^n (x+1)^{n-1}[x, y].$$
Hence we have shown \( x^{n-1}[x, y] = 0 = (x + 1)^{n-1}[x, y] \).

By continuing this process we obtain \( x[x, y] = 0 = (x + 1)[x, y] \).

Hence \( [x, y] = 0 \). \( \square \)

**Theorem 4.1.2:** If \( P \) is the set of potent elements, \( N' = \{ x \in N | x^2 = 0 \} \) is commutative and \( N \) multiplicatively closed, then \( PN \subseteq N \). In particular, if \( N \) is commutative and \( P \cup N \) generates \( R \), then \( N \) forms an ideal.

**Proof:** First, we claim that \( EN \subseteq N \), where \( E \) is the set of idempotent elements of \( R \).

Let \( e \in E \), and \( a^{2n} = 0 \).

Since both \( ea - eae \) and \( ae - eae \) belong to \( N' \),

\[
ed^2e - (ea)^2 = (ea - eae)(ae - eae) = e(ae - eae)(ea - eae) = 0,
\]

i.e., \( (ea)^2 = ea^2e \).

By repeating this procedure, we see that \( (ea)^{2n} = ea^{2n}e = 0 \). So \( (ea)^{2n+1} = 0 \).

Hence \( EN \subseteq N \).

Now, let \( x \in P \). Then, by the above claim, there exists a positive integer \( n \) such that \( x^nN \subseteq N \).

Let \( a \) be an arbitrary element in \( N \), and suppose that \( n > 1 \).

It is easy to see that \( x^i ax^{n-i} \in N \) (\( 0 \leq i \leq n \)).

If \( n = 2m \) then \( x^m ax^m \in N \) and hence \( x^ma \in N \).

Next, if \( n = 2m + 1 \), then \( b = xax^{2m+1}a \).

Therefore \( (x^{2m+1}a)^n = x^nb \in N \).
Hence \( x^{m+1}N \subseteq N \), as for the case \( n = 2m \).

We have thus seen that in either case there exists a positive integer \( n' < n \) such that \( x^n N \subseteq N \).

\textit{Hence} \( xN \subseteq N \).

\textbf{Lemma 4.1.3:} Suppose that \( R \) satisfies the following condition: for each \( x, y \in R \) there exist \( f(x), g(x) \) in \( x^2 \mathbb{Z}[x] \) such that \([x - f(x), y - g(y)] = 0\).

If for each \( a \in N^* \) and \( x \in R \), there exists a positive integer \( k \) such that \([a, x]_k (=[ [a, x]_{k-1}, x]) = 0\), then \( R \) is commutative.

\textbf{Proof:} By theorem C and Lemma 1[44], we see that \([a, x] = 0\) for all \( a \in N^* \) and \( x \in R \).

Hence Lemma 2 [44], shows that \( R = C(N^*) \) is commutative. \( \square \)

\textbf{Lemma 4.1.4:} Let \( R \) be a subdirectly irreducible ring. Then the only central idempotent elements of \( R \) are 0 and 1.

\textbf{Proof:} Suppose \( e \) is a central idempotent element of \( R \).

Let \( I = \{ex/ x \in R \} \) and \( I' = \{x - ex/ x \in R \} \).

It is readily verified that \( I \) and \( I' \) are ideals in \( R \), and \( I \cap I' = \{0\} \).

Since \( R \) is subdirectly irreducible, \( I = \{0\} \) or \( I' = \{0\} \).

If \( I = \{0\} \), then \( e = 0 \).

On the other hand, if \( I' = \{0\} \), then \( e = 1 \). \( \square \)

Now we present certain important properties of weakly periodic rings.
Lemma 4.1.5: Let $R$ be a weakly periodic ring. Then the Jacobson radical $J$ of $R$ is nil. If, furthermore, $xR \subseteq N$ for all $x \in N$, then $N = J$ and $R$ is periodic.

Proof: Let $x$ be an arbitrary element of $J$, and we write $x = b - a$, where $b^n = b$ ($n > 1$) and $a \in N$.

Then $x + a = b = b^n = (x + a)^n$.

Since $x$ is in $J$, we see that $a - a^n \in J$.

Hence $a \in J$.

This proves that $b^n = b = x + a \in J$.

Since $b^{n-1}$ is an idempotent with $b = b^{n-1}b$, we get $b = 0$,

so that $x = -a \in N$.

The latter assertion is clear. □

Lemma 4.1.6: Let $R$ be an arbitrary ring (not necessarily weakly periodic) which satisfies $w[[x, y], xy] = 0 =[[x, y], xy] w'$, then the idempotents of $R$ are central.

Proof: Suppose $e^2 = e \in R, x \in R, f = e + ex - exe$.

Then $f^2 = f, ef = f, fe = e$ and hence $[e, f] = (ex - exe)$.

Now, by hypothesis, there exists a word $w = w(e, f)$ such that $w[[e, f], ef] = 0$.

Since $ef = f$ and $fe = e$ we have $w = w(e, f) = e$ or $w = f$. 

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Hence, we have $0 = w[[e, f], ef] = e[[e, f], ef]$ or $f[[e, f], ef] = -(ex - exe)$. Thus $ex = exe$ for all $x$ in $R$.

Now, by taking $f' = e + x - exe$, and by hypothesis, an argument similar to the above shows that $[[e, f'], ef'] w = 0$; $w = e$ or $w = f'$.

It is easily seen that $[[e, f'], ef'] = -[f', e] = exe - xe$.

Thus $xe = exe$ for all $x$ in $R$.

Hence the idempotents of $R$ are central. □

Now we are able to prove the main results of this section.

**Theorem 4.1.7:** Let $R$ be a weakly periodic ring. Suppose that (i) for each $x \in R$ there exists $f(x) \in x^2Z[x]$ such that $x - f(x) \in C(N)$ and (ii) for each $x \in N + P$ and $a \in N$, $[xa - xa^2x, x] = 0$. Then $R$ is commutative.

**Proof:** By (i), we can easily see that $N$ is commutative.

Hence, by Theorem 4.1.2, $N$ is a commutative ideal.

Now, let $x \in N + P$ and $a \in N$.

Then $x - x^n \in N$ and $x^2 - x^{n+1} \in N$.

In particular, $R$ satisfies the condition $[x - f(x), y - g(y)] = 0$, for each $x, y \in R$ and $f(x), g(x)$ in $x^2Z[x]$.

Further, since $N^2 \subseteq C$, we have $[xa - xa^2x, x] = 0$.

Then $[xa - a^2x, x] = 0$.

$\text{i.e., } (xa - a^2x)x - x(xa - a^2x) = 0$,

$\text{i.e., } xax - x^2a = 0.$
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So \( x[a, x] = 0 \).

We substitute \( x+1 \) for \( x \). Then \((x + 1)[a, (x + 1)] = 0\).

By Lemma 4.1.1 \([a, x] = 0\), for all \( x \in R \) and \( a \in N \).

Hence by Lemma 4.1.3, \( R \) is commutative. \( \square \)

**Theorem 4.1.8**: Let \( R \) be a weakly periodic ring such that for all \( x, y \in R \), there exists a word \( w = w(x, y) \) and a positive integer \( k = k(x, y) \) for which \( w[[x, y], xy] = 0 \). Then the commutator ideal of \( R \) is nil, the set \( N \) of nilpotents is an ideal of \( R \) and \( R \) is periodic.

**Proof**: First, we observe that all the hypotheses are inherited by any homomorphic image of the ground ring \( R \). Moreover, any division ring \( D \) which is weakly periodic must satisfy the condition "\( x^{n(x)} = x, n(x) > 1 \), for all \( x \) in \( D \)" and hence by a well known theorem of Jacobson, \( D \) is commutative.

Thus the theorem is true for division rings.

Next, suppose that \( R \) is a primitive ring which satisfies the hypothesis of the theorem and suppose \( R \) is not a division ring. Since the hypothesis involving the existence of the word \( w = w(x, y) \) is inherited by all subrings and all homomorphic images of \( R \), therefore some complete matrix ring \( D_m \) over some division ring \( D \), with \( m > 1 \), satisfies the word hypothesis. This, however, is false, as can be seen by taking

\[
x = E_{11}, \ y = E_{11} + E_{12}; \ x, y \in D_m.
\]
In this case, \([x, y], xy\] = ± En for all positive integer \(k\) and, moreover any word \(w = w(x, y)\) must be equal to \(x\) or \(y\) (since \(x^2 = x, y^2 = y, xy = y, yx = x\)).

Hence \(w[[x, y], xy] = ± xE_{12} = ± E_{12}\) or \(w[[x, y], xy] = ± yE_{12} = ± E_{12}\).

In either case we obtain \(w[[x, y], xy] = ± E_{12} ≠ 0\).

In other words, the "word" hypothesis is not satisfied, a contradiction.

This contradiction shows that any primitive ring which satisfies the hypotheses must be a division ring, and hence must be commutative as remarked above. Therefore, theorem is also true for all semi simple rings, which implies \(R/J\) is commutative, and thus

\[C(R) ⊆ J.\]

Combining this with lemma 4.1.5, we see that

\[C(R) ⊆ J ⊆ N.\]

Hence the commutator ideal \(C(R)\) of \(R\) is nil.

This, as is readily verified, implies that \(N\) is an ideal of \(R\).

To prove \(R\) is periodic, let \(x ∈ R\).

Then by the definition of weakly periodic, \(x = a + b\), for some \(a ∈ N\), \(b\) potent \((b^n = b, n > 1)\).

Thus, \(x - a = b = b^n = (x - a)^n (a ∈ N)\).

Since \(N\) is an ideal, this implies that \(x - x^n ∈ N\), and thus, \(x^α = x^{α+1}f(α)\) for some integer \(α ≥ 1\) and some polynomial \(f(α)\) with integer coefficients.

Hence \(R\) is periodic, by chacron's Theorem.

This completes the proof of theorem. \(\Box\)
Theorem 4.1.9: Let $R$ be a weakly periodic ring such that

(i) $[a, b]$ is potent for all nilpotent elements $a, b$ in $R$,

(ii) for all $x, y$ in $R$, there exist words $w = w(x, y), w' = w'(x, y)$ such that $w[[x, y], xy] = 0 = [[x, y], xy] w'$, then $R$ is commutative.

Proof: By theorem 4.1.8 we have

$R$ is periodic and the set $N$ of nilpotents is an ideal of $R$. 4.1.1

Now, by Hypothesis (i), $[a, b]$ is potent for all $a \in N, b \in N$. 4.1.2

Thus $[a, b]^n = [a, b]$ for some $n > 1$ $(a \in N, b \in N)$. 4.1.3

By re-iterating in 4.1.2, we obtain

$[a, b]^{\lambda(n-1)+1} = [a, b]$ for all positive integers $\lambda$. 4.1.4

Since $N$ is an ideal and $a \in N, b \in N$, we have $[a, b] \in N$.

Hence by 4.1.3, we conclude that

$[a, b] = 0$ for all $a \in N, b \in N$ 4.1.5

i.e, $N$ is commutative.

Moreover by lemma 4.1.6,

the set $E$ of idempotents of $R$ is contained in the center $C$. 4.1.6

As is well known, we have

$R \cong a$ subdirect sum of subdirectly irreducible rings $R_i$. 4.1.7

Let $\sigma: R \rightarrow R_1$ be the natural homomorphism of $R$ onto $R_i$. In view of 4.1.1 $R_i$ must be periodic also. Hence by lemma 2 [11], we have

the nilpotents of $R_i = \sigma(N)$. 4.1.8
But $N$ is commutative. Hence by 4.1.7,

the set $N_i$ of nilpotents of $R_i$ is commutative. \hspace{1cm} 4.1.8

We now distinguish two cases

Case 1: $1 \not\in R_i$.

Let $x_i \in R_i$ and let $\sigma: x \rightarrow x_i$. Since $R$ is periodic, let $x^r = x^s$, $r > s \geq 1$.

Let $e = x_i^{s(r-s)}$. Then $e^2 = e$ and hence $e \in C$. Thus,

$e_i = x_i^{s(r-s)}$ is a central idempotent of $R_i$. \hspace{1cm} 4.1.9

Since we are assuming that $R_i$ does not have an identity, the central idempotent element $e_i$ of the subdirectly irreducible ring $R_i$ must be equal to zero.

Hence $x_i^{s(r-s)} = 0$ for all $x_i \in R_i$.

Thus $R_i = N_i$ is commutative, by 4.1.8.

Case 2: $1 \in R_i$. The above argument in Case 1 shows that $x_i^{s(r-s)}$ is a central idempotent in the subdirectly irreducible ring $R_i$, and hence

$x_i^{s(r-s)} = 0$, or $x_i^{s(r-s)} = 1_i$, for all $x_i \in R_i$. Thus,

every element of $R_i$ is nilpotent or is a unit in $R_i$. \hspace{1cm} 4.1.10

Moreover, $R_i$ (as a homomorphic image of $R$) satisfies all the hypotheses of theorem 4.1.8.

Hence, the set $N_i$ of nilpotents of $R_i$ forms an ideal of $R_i$.

This ideal $N_i$ is also commutative, by 4.1.8.

We have thus shown that

the set $N_i$ of nilpotents of $R_i$ is a commutative ideal of $R_i$. \hspace{1cm} 4.1.11
We claim that
\[ N_i \] is contained in the center \( C_i \) of \( R_t \).

4.1.12

Suppose not. Let \( a_i \in N_p, x_i \in R, \) be such that
\[ [a_i, x_i] \neq 0, a_i \in N_p, x_i \in R, \]

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By 4.1.11, \( x_i \notin N_i \), and hence by 4.1.10, \( x_i \) is a unit in \( R_t \).

Let \( u_i = 1_i + a_i \). Then \( u_i \) is a unit in \( R_t \) (since \( a_i \in N_i \)).

By hypothesis (ii), there exists a word \( w = w(u_i, x_i) \) and a positive integer
\[ k_i = k_i(u_i, x_i) \] such that
\[ 0 = w[[u_i, x_i], (u_i - 1_i) x_i] = w(u_i - 1_i)[[u_i, x_i], x_i]. \]

Therefore \( w[a_i[[u_i, x_i], x_i] = 0. \)

Hence \( w[u_i, x_i]_2 = 0. \)

By theorem 2 [11] it follows that \( R \) is commutative.

4.2 Commutativity of quasi-periodic rings

Chacron [29] introduced the structure of quasi-periodic rings. Later Bell [20] studied the common properties of quasi-periodic rings and periodic rings. In this section, we use the properties of quasi-periodic rings proved by Bell to prove that an n-torsion free quasi-periodic ring \( R \) with identity 1 satisfying \( x^n y^n = y^n x^n \) and \( (xy)^{n+1} - x^{n+1} y^{n+1} \) in the center \( C \) of \( R \), is commutative.
A ring $R$ is called quasi-periodic if for each $x \in R$ there exist integers $k, n, m$, all depending on $x$, such that $n > m > 0$ and $x^n = kx^m$. A ring $R$ is a $C$-ring if for each $x \in R$, there exists an integer $n > 1$ and an integer $k$ such that $x^n = kx$. If in a $C$-ring the same $n$ works for all $x \in R$, then $R$ is called $C_n$-ring. The direct sum of the ring of integers and torsion free nil ring provides a simple example of a quasi-periodic ring which is neither a periodic ring nor a $C$-ring.

Throughout the section $Z$ will denote the set of ring of integers, $Z^+$ the set of positive integers, $J$ the Jacobson radical, $C(R)$ the commutator ideal, and $T(R)$ the ideal of torsion elements.

Now we present some basic properties of quasi-periodic rings. The following Lemma 4.2.1 and Lemma 4.2.2, the proofs which use standard elementary computations, extend some basic facts about periodic rings to quasi-periodic rings. These are very useful to prove the results of this section.

**Lemma 4.2.1**: (i) If $x \in R$ satisfies $x^n = kx^m$ for $n > m > 0$ and $k \in Z$, then for each $j \in Z^+$ and each $s \geq m$, we have $x^{s+j(n-m)} = k' x^s$.

(ii) If $R$ is a quasi-periodic ring, and $\{x_1, x_2, \ldots, x_s\}$ is any finite subset of $R$, then there exist $n, m, k_1, k_2, \ldots, k_s \in Z$ such that $n > m > 0$ and $x_i^n = k_i x_i^m$, $i = 1, 2, \ldots, s$. 
(iii) If $R$ is quasi-periodic and $x \in R$, there exists $m \in \mathbb{Z}^+$ and $k \in \mathbb{Z}$ such that $y = x^m$ satisfies the equality $y^2 = ky$.

(iv) In any ring $R$, if $x$ satisfies $x^n = kx^m$ for $n > m > 0$ and $k \in \mathbb{Z}$, then $x^{n-m+1} - kx \in N$. In particular, a quasi-periodic ring with no nonzero nilpotent elements is a C-ring, and a periodic ring with no nonzero nilpotent elements is a J-ring.

**Lemma 4.2.2:** If $R$ is a quasi-periodic ring and $x \in T(R)$, then there exist distinct $n, m \in \mathbb{Z}^+$ with $x^n = x^m$. In particular, if $(R, +)$ is a torsion group, $R$ must be periodic.

**Lemma 4.2.3:** If $R$ is a quasi-periodic ring with $(R, +)$ a group, and if there exists an element $y$ such that every power of $y$ is of infinite additive order, then $T(R) \subseteq N$.

**Proof:** Let $y$ be an element of $R$, which has an infinite additive order.

Let $x \in T(R)$. We choose $s \in \mathbb{Z}^+$ such that $sx = 0$.

By Lemma 4.2.1 (ii) we get $n, m, a, b, c \in \mathbb{Z}$ with $n > m > 0$, such that

$$x^n = ax^m, \quad y^n = by^m, \quad (x+sy)^n = c(x+sy)^m.$$  \hspace{1cm} 4.2.1

Since $x(sy) = (sy)x = 0$, we have $x^n + s^ny^n = c(x^m + s^my^m)$.

Then $ax^m + s^by^m = cx^m + cs^my^m$, or $(s^n b - cs^m)y^m = (c-a)x^m$. \hspace{1cm} 4.2.2

Since the element on the right side of this equation is in $T(R)$, we have $s^n b - cs^m = 0$. Hence $c = s^{n-m}b$. 

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Recalling that \( sx = 0 \), we have \( cx'' = 0 \). From 4.2.2, \( ax'' = 0 \). This together with 4.2.1, implies \( x'' = 0 \).

Hence \( T(R) \subseteq N \).

An obvious modification of the proof yields a result for C-rings, which extends theorem 2.6 of [32].

**Corollary 4.2.4:** If \( R \) is a C-ring and \( (R, +) \) is a group, then \( T(R) \) is a nil ideal.

**Theorem 4.2.5:** If \( R \) is any quasi-periodic ring, then at least one of the following must hold:

(i) \( R \) is periodic:

(ii) \( T(R) \) is a nil ideal.

**Proof:** In view of Lemma 4.2.2, we consider the case of \( (R, +) \) is a group.

By Lemma 4.2.3, if there exists an element \( y \) of infinite order with all its powers of infinite order, then \( T(R) \) is nil.

On the other hand, if there exists no such \( y \), then for each \( y \in T(R) \) some power of \( y \) has finite additive order. By Lemma 4.2.2 \( R \) is periodic. □

**Corollary 4.2.6:** If \( R \) is a quasi-periodic ring with no non-trivial nil ideals, then either \( R \) is periodic or \( (R, +) \) is torsion-free.

We extend the following results on periodic rings to quasi-periodic rings.

(i) If \( R \) is any periodic ring \( J \) is nil.
(ii) If $R$ is a periodic ring with all nilpotent elements central, then $R$ is commutative.

**Theorem 4.2.7:** If $R$ is any quasi-periodic ring, then $J$ is nil.

**Proof:** Let $x \in J$. If $x$ or any power of $x$ is in $T(R)$, then by Lemma 4.2.2 we get $n, m \in \mathbb{Z}^+$, $n > m$, with $x^n = x^m$.

It follows that some power of $x$ is idempotent and since $J$ has no non-trivial idempotents. Hence $x$ must be nilpotent.

Therefore, we assume $J \cap N \neq 0$. By Lemma 4.2.1 (ii), $x \in J \cap N$ and $k \in \mathbb{Z}$ such that $x^2 = kx \neq 0$ and $x^j \not\in T(R)$ for all $j \in \mathbb{Z}^+$.

Let $y$ be the quasi-inverse of $x$, so that

$$xy - x - y = 0. \quad 4.2.3$$

By multiplying left this equation with $x$ gives $x^2y - x^2 - xy = 0$.

Then $kxy - kx - xy = 0. \quad 4.2.4$

We multiplying 4.2.3 by $k$ and comparing with 4.2.4 yields

$$xy = ky. \quad 4.2.5$$

By substituting 4.2.5 into 4.2.3 gives

$$(k - 1)y = x. \quad 4.2.6$$

By multiplying right this equation with $y$ and recalling 4.2.5 now yields

$$(k - 1)y^2 = ky. \quad 4.2.7$$

We choose $n, m, j \in \mathbb{Z}$ with $n > m > 0$, such that

$$y^n = jy^m. \quad 4.2.8$$
By raising both sides of 4.2.7 to the \( n \)-th power, obtaining \((k-1)^ny^2n = k^n y^n\).

From 4.2.8, we obtain
\[
(k-1)^nj^2y^{2m} = k^nj^my^m.
\]

This equation can be rewritten as \((k-1)^nj^2(k-1)y^{2m} = k^nj^my^m\), which in view of 4.2.7, becomes \((k-1)^nj^2k^my^m = k^nj^my^m\).

Thus, we have
\[
((k-1)^njk^m)y^m = 0. \tag{4.2.9}
\]

Now by 4.2.6 and our restrictions on \( x \), we cannot have any power of \( y \) in 
\( T(R) \). Thus \( j \neq 0 \) follows from 4.2.8, and from 4.2.9 we get \((k-1)^n-jk^m = k^n-m\),
which can hold only if \( k = 2 \).

Thus, by 4.2.6 we get \( y = x \) and from 4.2.3 it follows that \( x^2 = 2x \).

We note that \((2x)^j \not\in T(R)\) for every \( j \in \mathbb{Z}^+ \).

Thus, if we choose \( m \in \mathbb{Z}^+ \) and \( t \in \mathbb{Z} \) such that \((2x)^m = t(2x)^m\), \((2x)^m \) can
play the role of \( x \) in the argument above, and \( t = 2 \).

Consequently, \((2x)^2m = 2(2x)^m\).

Hence \(2^{2m}x^{2m} = 2^{m+1}x^m\).

Substituting \( 2x \) for \( x^2 \) gives \(2^{2m}2^{m+1}x^m = 2^{m+1}x^m \) or \((2^{2m} - 2^{m+1})x^m = 0\).

But this implies that \( x^m \in T(R) \). This is a contradiction.

Hence we conclude that \( J \) is nil. \( \Box \)

**Lemma 4.2.8:** If \( R \) is any quasi-periodic ring with \( N \) contained in the center, then \( R \) is commutative.

**Proof:** By the hypothesis, \( N \) is an ideal of \( R \). 

\section*{Chapter 4}
We consider $\bar{R} = R/N$, which has no non-zero nilpotent elements.

By theorem 2.1 [20], $R$ is either a $J$-ring or is isomorphic to a subring of $Z$.

In the former case, $R$ is periodic by Lemma 2.1.3 and hence commutative by Lemma 3.2.1.

In the latter case, $(\bar{R}, +)$ is cyclic with generator $g + N$.

Thus, each element of $R$ is of the form $kg + u$, where $k \in Z$ and $u \in N$, and it is immediate that any two elements commute.

The following theorem extend theorem 3.1.4.

**Theorem 4.2.9:** Let $R$ be a quasi-periodic ring in which nilpotent elements commute with each other. Then $C(R)$ is nil, and $N$ is an ideal.

**Proof:** In the view of theorem 4.2.7, it is sufficient to show that $R/J$ is commutative.

Now Lemma 4.2.1 part (i) and (iv), implies that if $\bar{R} = R/I$ is any factor ring of $R$ and $u$ is any nilpotent element of $\bar{R}$, then $u$ is the canonical image of a nilpotent element of $R$ [16].

Thus, our hypotheses are inherited by homomorphic images of $R$, and we need only establish commutativity under the assumption that $R$ is primitive.

But division rings are commutative as in the proof of theorem 4.1 [20], and $2 \times 2$ total matrix rings do not have all nilpotent elements commuting, so by the Jacobson structure theory the theorem follows.
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Theorem 4.2.10: Let $R$ be a quasi-periodic ring with 1 which is not periodic. Then $C(R)$ is nil.

Proof: Since $R$ is not periodic, Lemma 4.2.2 guarantees that $1 \not\in T(R)$.

Thus $\bar{R} = R/T(R)$ has 1 and $(\bar{R}, +)$ is torsion-free.

By theorem 4.2.5, $T(R)$ must be nil.

So we complete the proof by establishing commutativity of any quasi-periodic ring $S$ with 1 having $(S, +)$ torsion-free.

If $x \in S$ and $x^2 = 0$, then $1 + x$ is invertible.

Hence there exists $k \in \mathbb{Z}$ and $n > 1$ such that $(1 + x)^n = k(1 + x)$.

But this implies $1 + nx = k + kx$ or $(n - k)x = k - 1$.

This is impossible unless $k = 1$ and $x = 0$.

Thus $S$ has no nonzero nilpotent elements and is commutative by theorem 2.1 [20]. □

Theorem 4.2.11: Let $n > m > 0$ and $k \in \mathbb{Z}$. If $R$ is any ring satisfying the identity $x^n = kx^m$, then $R$ must be periodic. If $m = 1$, then $R$ is a direct sum of a $J$-ring and a nil ring.

Proof: By theorem 4.2.5, we assume that $T(R)$ is a nil ideal and $k \neq 0$.

For any $x \in R$ we have $x^n = kx^m$ and $(2x)^n = k(2x)^m$.

Hence $2^n x^n = 2^m kx^m$ and finally $(2^n - 2^m)kx^m = 0$.

Thus $x^m \in T(R)$ for all $x \in R$, so $R$ is a nil ring.
Now consider the special case \( m = 1 \).

For each \( x, y \in R \), we have \( x^n = kx \) and \( y^n = ky \) and therefore \( x^n y = xy^n \).

It follows from the result in [47] that \( R \) is a direct sum of \( J \)-ring and a nil ring.

Now we prove that a quasi-periodic ring is commutative under some conditions, before that we need the following lemma.

**Lemma 4.2.12:** Let \( R \) be an \( n \)-torsion free ring with identity 1 such that 
\[
[x^n, y^n] = 0 \quad \text{for all} \quad x, y \in R.
\]
Let \( N \) denote the set of nilpotent elements of \( R \). Then (i) \( a \in N, x \in R \) imply \([a, x^n] = 0 \) (ii) \( a \in N, b \in N \) imply \([a, b] = 0 \).

**Proof:** Since \( a \) is nilpotent, there exists a positive integer \( m \) such that 
\[
[a^k, x^n] = 0 \quad \text{for all} \quad x \in R \quad \text{and all integers} \quad k \geq m.
\]
Suppose that \( m_0 \) is the least such integer, that is, 
\[
[a^k, x^n] = 0, \quad \text{for all} \quad x \in R \quad \text{and all integers} \quad k \geq m_0, \quad m_0 \text{ minimal (} m_0 \geq 1 \). 4.2.10

Claim that \( m_0 = 1 \).

Suppose not. Then \( m_0 \geq 2 \).

Also, since \([x^n, y^n] = 0 \) for all \( x, y \in R \).
\[
[(1 + a^{m_0 - 1})^n, x^n] = 0, \quad x \in R. \quad 4.2.11
\]

Combining 4.2.10 and 4.2.11, we see that \( n[a^{m_0 - 1}, x^n] = 0 \) and hence 
\[
[a^{m_0 - 1}, x^n] = 0, \quad \text{since} \quad R \text{ is } n\text{-torsion free}.\]

Combining this fact with 4.2.10, we conclude that

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\[ [a^k, x^n] = 0 \text{ for all integers } k \geq m_0 - 1, \ (m_0 - 1 \geq 1). \quad (4.2.12) \]

Clearly 4.2.12 contradicts the minimality of \( m_0 \) in 4.2.10.

This contradiction shows that \( m_0 = 1 \), and hence by 4.2.10, \( [a, x^n] = 0 \), which proves (i).

To prove (ii), we argue in a similar manner.

We observe that, since \( b \) is nilpotent, there exists a positive integer \( p_0 \) such that

\[ [a, b^k] = 0 \text{ for all integers } k \geq p_0, \ (p_0 \text{ minimal, } p_0 \geq 1). \quad (4.2.13) \]

Suppose \( p_0 > 1 \).

By part (i), since \( a \in N, [a, (1 + b^{n-1})^n] = 0. \quad (4.2.14) \]

Combining 4.2.13 and 4.2.14, and arguing as in part (i), we obtain a similar contradiction.

Hence \( p_0 = 1 \).

Thus by 4.2.13, \( [a, b] = 0. \)

Which proves part (ii). \( \square \)

**Theorem 4.2.13**: Let \( R \) be a quasi-periodic ring with identity 1 and suppose that \( R \) is \( n \)-torsion free, and that for all \( x, y \in R, x^ny^n = y^nx^n \) and

\( (xy)^{n+1} - x^{n+1}y^{n+1} \in C, \) then \( R \) is commutative.

**Proof**: By hypothesis, \( [x^n, y^n] = 0 \) for all \( x, y \in R. \)

From Lemma 4.2.12 it follows that the nilpotent elements commute with each other.
From Lemma 4.2.9 this implies that the set of nilpotent elements \( N \) forms an ideal.

Hence \( N \) is a commutator ideal.

This implies that \( N^2 \subseteq C \).

Let \( a \in N, b \in R \). Then by hypothesis,

\[
((a+1)b)^{n+1} - (a+1)^{n+1}b^{n+1} \in C, \text{ and}
\]

\[
b(a+1)^{n+1} - b^{n+1}(a+1)^{n+1} \in C.
\]

Subtracting 4.2.17 from 4.2.16, and using the fact that \( N^2 \subseteq C \), we get

\[
ab^{n+1} - b^{n+1}a - (n+1)ab^{n+1} + (n+1)b^{n+1}a \in C.
\]

Thus \( n[a, b^{n+1}] \in C \).

Hence, since \( R \) is \( n \)-torsion free, we get

\[
[a, b^{n+1}] \in C, \ (a \in N, \ b \in R).
\]

Therefore, \( [a, b^{n+1}] = [a, b]b^n + b[a, b^n] \in C \) by 4.2.18.

But, by Lemma 4.2.12 (i), \( [a, b^n] = 0 \).

Hence by 4.2.19, \( [a, b]b^n \in C, \ (a \in N, \ b \in R) \).

Thus,

\[
[[a, b]b^n, b] = 0 = [[a, b], b]b^n.
\]

Replacing \( b \) by \( b+1 \) in the above argument and using lemma 4.2.12, we see that

\[
[[a, b], b] = 0, \ (a \in N, \ b \in R).
\]

Using Lemma 4.2.12(i), 4.2.21 and Lemma 3.1.1, we get

\[
0 = [a, b^n] = nb^{n-1}[a, b].
\]

Since \( R \) is \( n \)-torsion free, we conclude that \( b^{n-1}[a, b] = 0 \).
By taking $b + 1$ instead of $b$, and using Lemma 4.2.8, we get

$$[a, b] = 0 \ (a \in \mathbb{N}, b \in \mathcal{R}).$$

Thus, the nilpotent elements are central.

Hence by Lemma 4.2.8, $\mathcal{R}$ is commutative.

This completes the proof. □

4.3 Commutativity results for generalized periodic rings

Bell and Yaqub [28] studied the generalized periodic rings and proved the commutativity of generalized periodic rings with some conditions.

In this section, we prove that every nonzero idempotent element in a 2-torsion free generalized periodic ring is central. Using this we prove that if $\mathcal{R}$ is a generalized periodic ring with idempotent element which is not a zero divisor, then $\mathcal{R}$ is commutative. We also prove that if $[a, b] = ab - ba$ is potent $a, b \in \mathbb{N}$, then the generalized periodic ring $\mathcal{R}$ is commutative.

We know that $\mathcal{R}$ is a generalized periodic ring if for every $x$ in $\mathcal{R}$, $x \in \mathbb{N} \cup \mathcal{C}$, we have

$$x^n - x^m \in \mathbb{N} \cap \mathcal{C}, \quad 4.3.1$$

for some positive integer $m, n$ of opposite parity. Throughout this section $\mathcal{R}$ denotes a generalized periodic ring. The set of nilpotents, the center, the
Jacobson radical and the commutator ideal of $R$ are denoted by $N$, $C$, $J$, and $C(R)$ respectively.

We start with some basic properties of generalized periodic rings which are proved by Bell and Yaqub.

**Lemma 4.3.1:** In a generalized periodic ring $R$, we have

(i) $C(R) \subseteq J$

(ii) $J \subseteq N \cup C$

(iii) $N \subseteq J$

**Proof:**

(i) By a well known theorem of Herstein [37], if $R$ is a division ring which satisfies 4.3.1, then $R$ is commutative.

Next, suppose that $R$ is a primitive ring which satisfies 4.3.1.

Since 4.3.1 is inherited by all subrings of $R$ and by all homomorphic images of $R$, it follows, by Jacobson's Density Theorem, that if $R$ is not a division ring, then some complete matrix ring $D_m$ with $m > 1$, over a division ring $D$ satisfies 4.3.1.

This is false, as can be seen by taking $x = E_{12} + E_{21}$ in $D_m$.

Hence a primitive ring which satisfies 4.3.1 is necessarily a division ring, and hence is commutative by Herstein's Theorem.

Therefore, any semisimple ring which satisfies 4.3.1 is commutative, which proves (i).

(ii). Suppose $x \in J, x \notin N, x \notin C$.

Then, by 4.3.1, $x^n - x^m \in N, n \neq m$, and thus for some $q \in Z^+, g(\lambda) \in Z[\lambda]$,
$x^g = x^{g+1}g(x)$, \(g(\lambda) \in \mathbb{Z}[\lambda])$.  \hspace{1cm} 4.3.2

It is readily verified that \(e = [xg(x)]^q\) is an idempotent element in \(J\) (since \(x \in J\)), and hence \(e = [xg(x)]^q = 0\).

But, by 4.3.2, \(x^g = x^g e = 0\), and hence \(x \notin N\), contradiction.

This contradiction proves (ii).

(iii) First, we prove that \(ax \in N\) for all \(a \in N\) and all \(x \in R\).  \hspace{1cm} 4.3.3

To show this, first note that by (i) and (ii),

\[C(R) \subseteq N \cup C.\] \hspace{1cm} 4.3.4

Suppose 4.3.3 is false, and let \(a \in N\), \(x \in R\), \(ax \notin N\) (for some \(a\) and \(x\)).

Let \(\tilde{R} = R/C(R)\) and let \(\tilde{x} = x + C(R)\) be an arbitrary element of \(\tilde{R}\).

Since \(\tilde{R}\) is commutative, 4.3.3 implies that \(\tilde{ax}\) is nilpotent, and hence \((ax)^r \in C(R)\) for some positive integer \(r\).

Thus, by 4.3.4 \((ax)^r \in N\) or \((ax)^r \in C\). By hypothesis, \(ax \notin N\).

So \((ax)^r \in C\) for some positive integer \(r\).

Since \(a \in N\), let \(a^r = 0\). Note that, since \((ax)^r \in C\),

\[(ax)^r = a(ax)^r (xa)^{r-1} x = a^2 xt_1 (some t_1 \in R).\]

Continuing this process, we see that \([(ax)^r]^k = a^kt_{k-1} (some t_{k-1} \in R), for all \(k \in \mathbb{Z}^+.\)

In particular, \([(ax)^r]^0 = a^kt_{0-1} = 0\) (since \(a^0 = 0\)), and hence \(ax \in N\), contradiction. This contradiction proves 4.3.2.
To complete the proof of (iii), let $a \in N$, $x \in R$.

Then, by 4.3.3, $ax \in N$ and hence $ax$ is right quasi-regular for all $x$ in $R$, which implies $a \in J$.

This completes the proof of the lemma. □

**Theorem 4.3.2:** The set $N$ of nilpotents of a generalized periodic ring $R$ is an ideal of $R$.

Proof: By Lemma 4.3.1 (iii), (ii), we have

$$N \subseteq J \subseteq N \cup C.$$  \hspace{1cm} 4.3.5

Let $a \in N$, $b \in N$.

Then $a \in J$, $b \in J$, $a - b \in J$ and hence $a - b \in N$ or $a - b \in C$.

If $a - b \in C$, then $ab = ba$ and hence $a - b \in N$.

So, in any case, $a - b \in N$ for all $a \in N$, $b \in N$.

We have already established that $ax \in N$ for all $a \in N$, $x \in R$, and a similar argument yields $xa \in N$. Therefore, $N$ is an ideal. □

**Lemma 4.3.3:** Let $R$ be a generalized periodic ring. Then $R/N$ is commutative and hence $C(R) \subseteq N$.

Proof: By Theorem 4.3.2, $N$ is an ideal, and hence we consider $R/N$.

Let $x \in R$, $x \notin C$. Then, by 4.3.1,

$$x^n - x^m \in N,$$  \hspace{1cm} 4.3.6

for some $n > m$, say.

It is easy to verified that
\[(x_{n-m+1}-x)^m = (x_{n-m+1}-x)x^{m-1}g(x), \text{ some } g(\lambda) \in Z[\lambda],\]
\[= (x^n-x^m)g(x).\]

Hence \(x_{n-m+1}-x \in N\), since \(x^n = x^m \in N\).

Therefore, for all \(x \in R\), we have
\[x - x_{n-m+1} \in N \text{ or } x \in C, n > m, (x \in R).\]
**4.3.7**

Hence, \(R/N\) has the property that for each \(x \in R/N\), there exists \(k > 1\) for which \(x - x^k\) is central.

By a well known theorem of Herstein [37], it follows that \(R/N\) is commutative, and thus \(C(R) \subseteq N\). \(\square\)

Since \(N\) is an ideal of \(R\), \(N \subseteq J\).

Combining this with \(C(R) \subseteq N\) Lemma 4.3.1 (ii) we obtain

**Lemma 4.3.4:** Let \(R\) be a generalized periodic ring. Then
\[C(R) \subseteq N \subseteq J \subseteq N \cup C.\]
**4.3.8**

Next, we consider a ring which is both weakly periodic and generalized periodic.

**Theorem 4.3.5:** If a ring \(R\) is both generalized periodic and weakly periodic, then \(R\) is periodic.

**Proof:** Let \(x \in R\). Since \(R\) is weakly periodic, we have
\[x = a+b \text{ for some } a \in N, \text{ b potent } (b^q = b, q > 1).\]
**4.3.9**

Thus, \(x - a = (x - a)^q\) and since \(N\) is an ideal, we have \(x - x^q \in N\).
By a well known theorem of Chacron [30], it follows that $R$ is periodic. □

The following theorem extends theorem 3.1.4.

**Theorem 4.3.6:** Let $R$ be a generalized periodic ring and suppose $N \subseteq C$. Then $R$ is commutative.

**Proof:** Let 4.3.7, for each $x \in R$, either $x \in C$ or $x - x^k \in N$ for some $k > 1$.

Since $N \subseteq C$, for every $x \in R$, $x - x^k \in C$ for some $k > 1$.

Therefore, by Herstein's Theorem [37], $R$ is commutative. □

**Corollary 4.3.7:** Let $R$ be a generalized periodic ring, and suppose $J \subseteq C$. Then $R$ is commutative.

**Proof:** By Lemma 4.3.4, $N \subseteq J$, and hence $N \subseteq C$.

Thus, $R$ is commutative, by Theorem 4.3.6 □

**Corollary 4.3.8:** Let $R$ be a generalized periodic ring with Jacobson radical $J$. Then $J = N$ or $R$ is commutative.

**Proof:** By Lemma (4.3.1(ii)), it follows that

$$J = (J \cap N) \cup (J \cap C).$$  \hspace{1cm} 4.3.10

From this we conclude that

$$J = J \cap N \text{ or } J = J \cap C.$$  \hspace{1cm} 4.3.11

This implies that

$$J \subseteq N \text{ or } J \subseteq C.$$  \hspace{1cm} 4.3.12
If $J \subseteq N$, then $J = N$. On the other hand, if $J \subseteq C$, then $R$ is commutative, by Corollary 4.3.7.

Now we prove that the idempotents of a 2-Torsion free generalized periodic-ring $R$ are central.

**Lemma 4.3.9:** If $R$ is a 2-torsion free generalized periodic ring then every nonzero idempotent is central.

**Proof:** Suppose the idempotent $e$ of $R$ is not central.

Since $R$ is a generalized periodic ring, $e \notin (N \cup C)$ and $-e \notin (N \cup C)$.

Hence $(-e)^n - (-e)^m \in (N \cup C)$, where $m, n$ are opposite parity.

So $(-e)^n - (-e)^m \in C$.

If $n, m$ are either even or odd positive integers, then $0 \in C$.

Otherwise $-2e$ or $2e \in C$.

That is, $[2e, x] = 0$ or $2[e, x] = 0$, for every $x \in R$.

Since $R$ is a 2-torsion free generalized periodic ring, $[e, x] = 0$.

So $e \in C$, a contradiction.

This contradiction proves that nonzero idempotents are central.

**Lemma 4.3.10:** Let $R$ be a generalized periodic ring. If $e$ is any nonzero idempotent in $R$ and $\alpha \in N$, then $ea \in C$.

**Proof:** The proof is by contradiction.

Suppose the lemma is false.
Let \( a_0 \in N \) and \( ea_0 \notin C \).

4.3.13

Since \( e \in C \) (by lemma 4.3.9) and \( a_0 \in N \), we have \( ea_0 \) is nilpotent.

Let \((ea_0)^\alpha \in C, \forall \alpha \geq a_0, a_0 \) minimal.

4.3.14

Since \( ea_0 \notin C \) we have \( \alpha_0 > 1 \). Let \( a = (ea_0)^{\alpha-1} \).

Then \( a = (ea_0)^{\alpha-1} \in N, a \notin C \) (by the minimality of \( a_0 \)),

\[ a^k \in C, \forall k \geq 2. \]

Since \( e \in C, \) and \( e^2 = e \neq 0, e \notin N. \)

4.3.15

Equation 4.3.15 implies that \( e + a \notin C \) and \( e + a \notin N \) and hence

\[ (e+a)^{\mu'} - (e+a)^{\mu''} \in C, \text{ where } \mu', \mu'' \text{ are of opposite parity}. \]

4.3.16

Combining 4.3.15 and 4.3.16 we see that

\[ (\mu' - \nu')ea \in C \text{ where } \mu' - \nu' \text{ is an odd integer}. \]

4.3.17

Equation 4.3.15 also implies that \( (-e + a) \) is not in \( N \cup C. \)

So \( (e + a)^{\mu''} - (e + a)^{\mu'''} \in N, \text{ where } \mu'', \mu''' \text{ are of opposite parity}. \)

4.3.18

Combining 4.3.15 and 4.3.18, we see that \( (-e)^{\nu''} - (-e)^{\nu'''} \in N, \)

4.3.19

and hence \( 2e \notin N, \text{ since } \nu'', \mu''' \text{ are of opposite parity}. \)

Therefore \( (2e)^r = 0, r \in \mathbb{Z}^+ \) and thus

\[ 2'e = 0, \text{ which implies that } 2'e \in C, r \in \mathbb{Z}^+. \]

4.3.20

Now combining 4.3.17 and 4.3.20 and since \( (2', \mu' - \nu') = 1, \)

we see that \( ea \in C. \)

Hence by 4.3.15, \( a = ea \in C, \) which contradicts 4.3.15.

This contradiction proves the lemma. 

\[ \square \]
**Theorem 4.3.11:** Suppose $R$ is a generalized periodic ring containing an idempotent element which is not a zero divisor. Then $R$ is commutative.

**Proof:** Let $e$ be an idempotent element in $R$.

Let $a \in N$. By lemma 4.3.10, we have $ea \in C$ and hence $[ea, x] = 0$ for all $x \in R$. This implies that $eax - xea = 0$.

Since idempotents are central, so that $eax - exa = 0$.

Therefore $e[a, x] = 0$. 4.3.21

Since $e$ is not a zero divisor, $[a, x] = 0$, $\forall x \in R$, $\forall a \in N$.

By lemma 4.3.15 and a well-known theorem of Herstein [39], it follows that $R$ is commutative and the theorem is proved. □

**Theorem 4.3.12:** Suppose $R$ is a generalized periodic ring, $N$ the set of nilpotents, and $E$ the set of idempotents of $R$. Suppose every commutator $[a, b] = ab - ba$ with $a \in N$ and $b \in N$ is potent. Then $R$ is commutative.

**Proof:** By lemma 4.3.3, $C(R) \subseteq N$ and hence $[a, b] \in N$.

By hypothesis $[a, b] = [a, b]^{\lambda} = [a, b]^{1+\lambda(q+1)}$ for all positive integer $\lambda$.

Hence $[a, b] = 0$, Since $[a, b] \in N$. 4.3.22

Thus $[a, b] = 0$ for all $a, b \in N$ that is $N$ is commutative.

We also have $x - x^{n+1} \in N$. We proved that for every $x$ in $R$, we have $x - x^k \in N$, for some $k > 1$ or $x \in C (x \in R)$. 4.3.23

Combining equation 4.3.22, 4.3.23 we see that for all $x, y$ in $R$,

$[x - x^k, y - y^r] = 0$ for some $k > 1, r > 1$. 4.3.24
As is well known

\[ R \cong \text{a subdirect sum of subdirectly irreducible rings } R_i. \]  

4.3.25

We now see the structure of each of this subdirect summands \( R_i \), to prove their commutativity.

**Case 1:** \( R_i \), does not have an identity.

Let \( \sigma: R \to R_i \) be the natural homomorphism of \( R \) onto \( R_i \) and let \( \sigma: x \to x_i. \)

Let \( N_i \) and \( C_i \) denote the set of nilpotents and the center of \( R_i \) respectively.

We claim that

\[ R_i \subseteq N_i \cup C_i. \]  

4.3.26

Suppose not, let \( x \in R, x_i \notin N_i, x_i \notin C_i \) and let \( \sigma: x \to x_i \) (\( x \in R \)).

Then clearly \( x \notin N \) and \( x \notin C \) and hence \( x^n - x^m \in N_i \)

for some positive integers \( n \) and \( m, n \neq m \).

This implies that \( x^q = x^q e \) for some positive integer \( q \) and some idempotent \( e \) in \( R \). In a generalized periodic ring idempotents are central and hence \( x^q = x^q e, e^2 = e \in C_i. \)

This implies in \( R_i \) that

\[ x_i^q = x_i^q e_i, e_i^2 = e_i \in C_i. \]  

4.3.27

Since \( e_i \) is a central idempotent in the subdirectly irreducible ring \( R_i \) and \( R_i \)

does not have identity, we have \( e_i = 0 \). Hence \( x_i^q = 0 \) a contradiction, since \( x_i \) is not nilpotent.

This contradiction proves \( R_i \subseteq N_i \cup C_i. \)
From 4.3.23 we see that \([x, -x, k, y_i, y_i'] = 0, k \geq 1, r \geq 1, x, y_i \in R\). 4.3.28

Now by a trivial minimality argument, it is easily verified that 4.3.28 implies

\([a, b] = 0\), for all nilpotents \(a, b\) in \(R\),

i.e., \(N\) is Commutative.

Combining 4.3.25 and 4.3.29, we see that \(R\) is commutative.

**Case 2:** \(R\) has an identity.

Since the homomorphic image of a generalized periodic ring is also generalized periodic, it follows that \(R\) is commutative by corollary 4 of [28].

Since each \(R\) in the subdirect sum representation 4.3.25 is commutative therefore the ground ring \(R\) itself is also commutative and the theorem is proved. \(\Box\)

We consider the following two examples, which show that neither centrality of idempotents nor commutativity of nilpotent elements implies commutativity of a generalized periodic ring. We note that, in each case, central elements are zero divisors.

**Example 4.3.13:** Let

\[
R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, 0, 1 \in GF(2) \right\}.
\]
It is easily verified that $R$ is a generalized periodic ring with commuting nilpotents but its idempotents are not in the center.

**Example 4.3.14:** Let

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in GF(3) \right\}.$$

It can be seen that $R$ is a generalized periodic ring with central idempotents but its nilpotents do not commute with each other.