CHAPTER- 2
BOTTLENECK LINEAR PROGRAMMING: SOME ASPECTS

This chapter consists of two sections. The first sections describe a method of ranking of the solutions of the bottleneck linear programming problem. Ranking of the solutions is useful in situations where the optimal solution is not practically applicable. For instance, in a Time Minimizing Transportation problem, which is an important class of bottleneck linear programming problems, the optimal solution which gives the least transportation time may involve a very high cost and thus may not be implementable because of monetary constraints. Such problems may arise in Time Minimizing Assignment problems also. In these problems the feasible solutions can be ranked in increasing order of values of the bottleneck objective function, and that solution implemented which suits the decision maker the best.

Ranking of the feasible solutions in bottleneck linear programming problems also helps in finding efficient solutions to bicriteria and multicriteria linear programming problems in which one or more objectives are of the bottleneck type.

It has been shown here that for ranking of the feasible solutions of a bottleneck linear programming problem only the basic feasible solutions (i.e., extreme point solutions.) are required to be considered. Kirby et al [] and Murty [] have developed methodologies for ranking of the extreme point solutions of a linear programming problem.
Consider an optimization problem of the following type

\[ \text{Min } F(X) \quad \text{(P0)} \]

Subject to \( g_i(X) \leq 0, \quad i = 1, \ldots, p, \)

Where \( X \) is an extreme point of \( S \{X/X \in \mathbb{R}^n, AX = b, X \geq 0\}, \)
And \( F(X) \) is defined as
\[
F(X) = \max_{1 \leq j \leq n} (f_j(X_j)), -f_j(X_j) = f_j, \quad X_j > 0 \quad (f_j \in \mathbb{R}^+)
\]
= 0, \quad X_j = 0,

\( X_j \) being the \( j \)th component of \( X \) and \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \).

An optima solution of \((P_0)\) can only be obtained by ranking the extreme point solutions of the related bottleneck linear programming problem
\[
\min F(X) \quad (P)
\]
\[ x \in S \]

If \( X_1 \) is an optimal basic feasible solution of \((p)\) and satisfies the constraints \( g_i(X) \leq 0, i = 1, \ldots, p \), then it is an optimal solutions to \((P_0)\).

If \( X_1 \) does not satisfy \( g_i(X) \leq 0, i = 1, \ldots, p \), then the basic feasible solutions of \((P)\) are ranked in increasing order of the values of \( F(X) \) until some \( k \)th best basic feasible solution \( X_k \) is found which also satisfies the constraints \( g_i(X) \leq 0, i = 1, \ldots, p \), and this \( X \) is then an optimal solution to \((P)\).

Method for ranking of the feasible solution to \((P)\) is presented in section 2.1, whereas section 2.2 in this chapter introduces and studies three variants of the bottleneck linear programming problem. In these problems the objective function is the sum, ratio and product of two bottleneck functions respectively. The development of the algorithm to solve these problems is quite interesting. Also, these problems have various practical applications, some of which are described in the beginning of this section.
METHOD OF RANKING OF THE SOLUTIONS

As mentioned earlier, (P) is the bottleneck linear programming problem. The following assumptions have been made throughout this chapter:

i) S is bounded (for, in case S is unbounded, it can be made bounded by adding a constraint of the form \( \sum_{j=1}^{n} X_j + S = M \), where M is a suitably large positive number and \( S > 0 \) with \( f_s = 0 \).

ii) \( f_j > 0, \quad j = 1, \ldots, n \).

The following notations have been used:

\( X_j \): the j\(^{th}\) component of X.

\( X_k \): a k\(^{th}\) best solution of (P).

\( F_k = F(X_k) \).

\( ^\wedge X_k \): the best basic feasible solution at the given value \( F_k \) of F (X).

Rel.Int (S): the set of points in the relative interior of S. The fact that F (X) is a concave function has been proved in [ ]. Therefore F (X) attains its minimum value over S at an extreme point of S [12,50].

For the development of the ranking algorithm the following concepts and related ideas are required:

\(^\wedge \) Definition 2.1.1: Linear programming problem LP (F)

\(^\wedge \)

For a given value of F of F (X), the related linear programming problem LP
(F) Is defined as:

\[
\min_{X \in S} Z(X) = \sum_{j=1}^{n} C_j X_j
\]

Where the \( C_j \)s are given by

\[
C_j = \begin{cases} 
0 & \text{if } f_j < F \\
-1 & \text{if } f_j = F \\
\infty & \text{if } f_j > F 
\end{cases}
\]

Remark 2.1.1: Let \( X \) be an optimal basic feasible solution of the linear Programming problem LP (F).

If \( Z(X) = 0 \) then there does not exist any basic feasible solution \( X \in S \) with \( F(X) = F \).

If \( Z(X) < 0 \), then a basic feasible solution \( X \) in \( S \) with \( F(X) = F \) exists, and
X is one such basic feasible solution.

Definition 2.1.2: If \( X = (X_j) \) and \( X = (X_j) \) are two feasible solutions of (P),

Then \( X \) is said to be a better solution of (P) than \( X \) if either

1) \( F(X) < F(X) \), or

2) \( F(X) = F(X) = F \) (say) and,

\[
\sum X_j < \sum X_j
\]

\( j/\sum X_j = F \) \( j/\sum X_j = F \)

Definition 2.1.3: Best basic feasible solution at a given value \( F \) of \( F(X) \).

A basic feasible solution \( X \) of (P) yielding the value \( F \) of \( F(X) \) is said to be the best basic feasible solution at \( F \) if

\[
\sum X_j = \text{Min} \left[ \sum X_j : X = (X_j) \right]
\]

is a basic feasible solution of (P) with \( F(X) = F \).
Remark 2.1.2: if $F_1$ is the optimal value of $F(X)$ in (P), then a best basic feasible solution with respect to $F_1$ is the optimal solution of the linear programming problem

$$\min_{x \in S} \sum_{j=1}^{n} C_j x_j \quad (P_1)$$

where

$$C_j = \begin{cases} 0 & , \quad f_j < F_1 \\ 1 & , \quad f_j = F_1 \\ \infty & , \quad f_j > F_1 \end{cases}$$

Remark 2.1.3: If $F_k (k>1)$ is the $k^{th}$ best value of $F(X)$ then a best basic feasible solution with respect to $F_k$ is any second best basic feasible solution of the linear programming problem:

$$\min_{x \in S} \sum_{j=1}^{n} C_j x_j \quad (P_K)$$

Where

$$C_j = \begin{cases} 0 & , \quad f_j < F_K \\ 1 & , \quad f_j = F_K \\ \infty & , \quad f_j > F_K \end{cases}$$
Here, the second best feasible solution, say \( X^2 \), of (P_\( k \)) is defined as \( Z_k (X^2) = \text{Min} \left( \frac{Z_k (X)}{X_S}, \frac{Z_k (X)}{X} > Z_k (X^1) \right) \), \( X \) is a basic feasible solution and \( X^1 \) is a optimal basic solution of (P_\( k \)).

It is to be noted that this second best basic feasible solution is that extreme point of \( S \) for which the value of the objective function in (P_\( k \)) is the smallest positive real number.

Theorem 2.1.1 : Let \( X \) be a point (other than an extreme point) lying on a face, say \( S_i \), of \( S \) [57]. Then there exists at least one extreme point \( X \) of \( S \) lying on \( S \) such that \( F(X) = F(X) \).

Proff : Since \( X \) is not an extreme point, it is a convex combination of some or all of the extreme points of \( S \) lying on \( S \).  

\[
X = \sum_{i=1}^{r} \mu_i X^i, \quad \sum_{i=1}^{r} \mu_i = 1, \quad 0 < \mu_i < 1, \quad i = 1, \ldots, r
\]

And \( X^1, \ldots, X^r \) are (some of / all) the extreme points of \( S \) lying on \( S \).

Also, let \( F(X) = f_k (X_k) \) \( (X_k > 0) \)

\[
= f_k, \quad \text{for some } k, \quad 1 \leq k \leq n.
\]

Therefore, \( X_j = 0 \) for \( j : f_j > f_k \)

This implies, \( X^i_j = 0 \) for all \( j : f_j > f_k, \quad i = 1, \ldots, r \),

\( X^i_j \) being the \( j^{th} \) component of \( X^i \).

Since \( X_k > 0 \), therefore from (2.1)

\[
X^s_k > 0 \quad \text{for some } S, \quad 1 \leq S \leq r.
\]

Hence \( F(X^s) = f_k = F(X) \).

Thus there exists an extreme point \( X = X^s \) of \( S \) lying on \( S \) such that \( F(X) = F(X) \).
Theorem 2.1.2: For any pair of points $X$ and $X$ in $S$ such that $X$ rel. Int $(S)$

and $X$ & Rel. Int $(S)$,

$F(X) \leq F(X)$.

Proof: since $X \in$ Rel Int $(S)$, therefore

$X_j > 0 \quad \forall j, \ j = 1, \ldots, n.$

Hence, $F(X) = \max \left( f_j \left( X_j \right) \right)$

$= \max \left( f_j \right)$

$\quad \mathrm{j}$

$\quad \mathrm{\wedge}$

Now for $X$, $X_j = 0$ for at least one $j, \ 1 \leq j \leq n$

$\quad \mathrm{\wedge}$

Therefore, $F(X) = \max \left( f_j \left( X_j \right) \right)$

$\quad \mathrm{\wedge}$

$= \max \left( f_j / X_j > 0 \right)$

$\quad \leq \max \left( f_j \right) \quad \mathrm{j}$

$= F(X),

\text{that is,}

$F(X) \leq F(X)$. 
Remark 2.1.4: From the theorems 2.1.1 and 2.1.1 and 2.1.2 it can be concluded that only the extreme points of S need to be examined for ranking of the values of F (X). Since every solution point in the relative interior of S gives the worst case value of F (X), therefore these points need not be considered in the ranking process.

The Ranking Algorithm

Step 1: Solve \((P)\) to get the optimal value \(F\) of \(F (X)\) [12].

   The optimal basic feasible solution of \((P'_{1})\) is the best basic feasible solution at \(F_{1}\).

Step 2: Arrange the \(f_{j} \in S\) \((f_{j} > F_{1})\) in ascending order, that is,

\[ F_{1} < f_{j_{1}} < \ldots < f_{j_{p}} \]

Set \(K = 2\) and \(q = 1\).

Step 3: Solve LP \((f_{j_{q}})\).

   If the optimal value of the objective function in LP \((f_{j_{q}})\) is strictly negative, then \(f_{j_{q}}\) is the \(k^{th}\) best value of \(F (X)\).

   Set \(F_{k} = f_{j_{q}}\) and go the step 4.

   If the optimal value of the objective function in LP \((f_{j_{q}})\) is zero, return to step 3 with the next higher value of \(q\).

Continue until \(q = p\).

Step 4: Solve \((P'_{k})\). to find the best basic feasible solution at the value \(F_{k}\). Return to step 3 with the next higher values of \(k\) and \(q\).

Remark 2.1.5: The following algorithm may be implemented if one is interested in testing two consecutive values of \(F (X)\) simultaneously.
Step 1: Solve (P) to get the optimal value $F_1$ of $F(X)$ [12].

Step 2: Arrange that $f_j \in S \ (f_j > F_i)$ in ascending order, that is, $F_1 < f_{j_1} < \ldots < f_{j_p}$

Set $K = 2$ and $q = 1$.

Step 3: Solve the linear programming problem

$$\min_{X \in S} \sum_{j=1}^{n} C_j \ X_j \ \text{LP} \ (f_j^q, f_{j+1}^q)$$

Where $C_j = 0$ if $f_j = F_1$

$= -1$ if $f_j = f_j^q$

$= -\infty$ if $f_j = f_{j+1}^q$

$= \infty$ if $f_j > f_{j+1}^q$

Let $X^*$ be its optimal solution with the optimal value $Z(X)$.

Step 4: If $Z(X^*) = 0$, neither $f_j^q$ nor $f_{j+1}^q$ are attainable values of $F(X)$. Replace $q$ by $q+2$ and go to step 3. If $Z(X^*) < 0$ and finite, then $f_j^q$ is the $k^{th}$ best value of $F(X)$ while $f_{j+1}^q$ is not an attainable value. Set $F_k = f_j^q$, replace $k$ by $k+1$, $q$ by $q+2$ and go the step 3. If $Z(X^*) = -\infty$, continue ranking the basic feasible solutions of LP ($f_j^q, f_{j+1}^q$) in ascending order of values of the objective function $Z(X)$, until a basic feasible function say $X^*$, yielding a finite objective function value is obtained. If $Z(X^*) = 0$, the $f_{j+1}^q$ is the $k^{th}$ best value of $F(X)$, and clearly, $f_j^q$ is not an attainable value. Set $F_k = f_{j+1}^q$ Replace $k$ by $k+1$ and $q$ by $q+2$, and go to step 3. If $Z(X^*) < 0$ and finite, then $f_j^q$ and $f_{j+1}^q$ are the $k^{th}$ and $(k+1)^{th}$ best values of $F(X)$ respectively.

Set $F_k = f_{jq}$, $F_{k+1} = f_{jq+1}$ and go to step 3 after replacing $k$ by $k+2$ and $q$ by $q+2$. 
Continue until \( q = p - 1 \).

**Remark 2.1.6:** As may be seen when \( Z(X^*) \) is finite, two values of \( F(X) \), namely \( f_{j,q} \) and \( f_{j,q+1} \), are simultaneously tested, but when \( Z(X^*) = -\infty \), testing becomes quite involved. If the aim is to show that \( f_{j,q} \) and \( f_{j,q+1} \) are both not attainable values, then this approach is useful.

**Example:** Considering the bottleneck linear programming problem

**Min** \( F(X) \)

\[ x \]

**Subject to**

\[ x_1 + 2x_2 + 4x_3 + x_4 + 2x_5 = 6 \]
\[ 3x_1 + x_2 + 4x_3 + 2x_4 + x_6 = 5 \]
\[ x_1 + x_2 + 2x_3 + x_4 + x_6 = 3 \]
\[ x_j \geq 0, \ j = 1, \ldots, 6, \]

where the \( f_j \)s are:
\[ f_1 = 8, \ f_2 = 4, \ f_3 = 6, \ f_4 = 15, \ f_5 = 10, \ f_6 = 3. \]
The optimal solution of the problem is

\[ x = (0 \ 1 \ 1 \ 0 \ 0 \ 0) \]

with \( F(X_1) = F_1 = 6 \). It is also the best basic feasible solution with respect to \( F_1 = 6 \), that is, \( X_1 = X_1 \).

The \( f_j \)s greater than \( F_1 \) are \( f_{j1} = 8, \ f_{j2} = 10, \ f_{j3} = 15 \).

The optimal solution of LP \((f_{j1} = 8)\) is \( x_1 = (0 \ 1 \ 1 \ 0 \ 0 \ 0) \)

With \( Z(X) = 0 \).

Therefore, 8 is not the second best value of \( F(X) \).
The optimal solution of LP \( f_{j_2} = 10 \) is \( x_2 = (1 0 0 0 2.5 2) \). With \( Z(X) = -2.5 < 0 \), hence \( F_2 = 10 \) is the second best value of \( F(X) \).

The best basic feasible solution with respect to \( F_2 = 10 \) is \( x_2 = (1 2 0 0 0.5 0) \).

The optimal solution of LP \( f_{j_3} = 15 \) is \( x_3 = (0 0 0 2 2 1) \) with \( Z(X_3) = -2 < 0 \). Hence \( F_3 = 15 \) is the third best value of \( F(X) \). \( x_3 \) is also a best basic feasible solution with respect to \( F \), that is, \( x_3 = x_3 \).

**VARIANTS OF THE BOTTLENECK LINEAR PROGRAMMING PROBLEM**

In this section a new class of bottleneck optimization problems is studied in which the objective function is the sum, ratio or product of two bottleneck functions.

Consider the two bottleneck functions \( F(X) \) and \( T(X) \) where

\[
F(X) = \max_{1 \leq j \leq n} (f_j(x_j)), \quad f_j(x_j) = f_j, \quad x_j > 0 \ (f_j \in \mathbb{R}^+) \\
\text{with} \quad x_j = 0, \quad x_j = 0, \text{ and}
\]

\[
T(X) = \max_{1 \leq j \leq n} (t_j(x_j)), \quad t_j(x_j) = t_j, \quad x_j > 0 \ (t_j \in \mathbb{R}^+) \\
\text{with} \quad x_j = 0, \quad x_j = 0, \text{ and}
\]

Then the problems studied here are:

\[
\min \ Z_1(x) = F(X) + T(X) \quad (P1)
\]
\[
\begin{align*}
x \in \mathcal{S} \\
\text{Min } Z_2(x) &= \frac{F(X)}{T(X)} & (P2) \\
x \in \mathcal{S} \\
\text{Min } Z_3(x) &= F(X) \cdot T(X) & (P3) \\
x \in \mathcal{S}
\end{align*}
\]

where \( \mathcal{S} = \{ x \mid x = (x_j) \in \mathbb{R}^n, Ax = b, x \geq 0 \} \), \( A \in \mathbb{R}^{mxn}, b \in \mathbb{R}^m \) and it is assumed to be regular.

The importance of the problem lies in the fact that these occur in various real-life situations and also contribute to the completeness in the study of bottleneck programming.

Suppose in a production set-up a number of products are being produced and they have to pass through two phases of production in such a way that the second phase may begin only when the first has ended. If \( x \) is a feasible solution with respect to the constraints \( Ax = b, x \geq 0 \), then the components \( x_j, j = 1, \ldots, n \) of \( x \) give the quantity of various goods to be produced, and \( f_j(x_j), t_j(x_j) \) give the time spent on the \( j^{th} \) product in the two phases respectively, irrespective of the quantity \( x_j \) of the product produced. (This can happen, for example, in an agricultural setting, where the time taken to cultivate a crop depends not on the quantity produced, but the kind of crop itself). Then \( F(X) \) and \( T(X) \) are the times spent in each of the phases and the total production time is \( F(X) + T(X) \), which is the objective function in (P1).

A transportation problem in which a complete schedule consists of an onward and a return journey, with the restriction that all the return trips on the various routes can start only when all the onward trips have ended, also lends itself to solution by the method developed here to solve (P1). Again, the objective function becomes the sum of two bottleneck
functions, the first giving the time taken to complete the return journeys. It is assumed that all the carriers start simultaneously for the onward journeys and also for the return journeys. On any route, the time taken for the onward trip is allowed to be different from the time taken for the return trip, to accommodate the possibility of different modes of transportation being used in the two.

If again, in a production set-up, the time taken for the completion of the schedule $x$ is $T(X)$, and the fixed rental or hiring charge on the equipment used for manufacture of the $j^{th}$ product is $f_j$, and the manufacturer is charged only the maximum rental component, then the rental cost per unit time becomes $F(X)$ which is to be minimized over all feasible schedules $x$.

$T(X)$

But if the rental cost is given in terms of per unit time, with all the equipment being returned only when the production is equipment being returned only when the production is complete, then the objective is to minimize the total cost which is now $F(X)T(X)$. In these two situations the problem structure is that of (P2) and (P3) respectively.

The three problems (P1), (P2) and (P3) are treated together in this section as they can be solved by the same approached. For each of these problem and optimal solution exists at an extreme point of $S$. A related bottleneck programming problem in which the objective function is a single bottleneck function is formulated for each problem. The objective function or this related problem provides bounds on the value of the objective function of the problem to be solved. The unifying factor which gives rise to a common solution approach is the way that the bounds on the objective function values of (P1), (P2) and (P3) are tightened by successive ranking on the objective function values of the related problem.
A considerable amount of theoretical development follows. The theorems stated and proved here lead to the algorithm in a systematic and natural manner.

**Theorem 2.2.1**: The optimal solution of (P1) is attainable at an extreme point of S.

**Proof**: The objective function (P1), being the sum of two concave functions, is again a concave function and hence its global minimum attained at an extreme point of S.

**Theorem 2.2.2**: The optimal solution of (P2) is attainable at an extreme point of S.

**Proof**: Let \( X \in \text{Rel. Int.} (S) \), the set of points in the relative interior of S. Then \( X \) can be expressed as a convex combination of some or all of the extreme points of S \([43]\)

\[
X = \sum_{i=1}^{P} \mu_i x^i, \quad \sum_{i=1}^{P} \mu_i = 1, \quad 0 < \mu_i < 1 \quad \text{for} \quad i = 1, \ldots, P
\]

Where \( x^1, \ldots, x^P \) are the extreme points of S which strictly contributed in the convex combination.

Let \( X = (X_j) \) and \( X^1 = (X^1_j) \).

Let \( T(X) = \overline{t}_k (x_k), \quad x_k > 0 \)

Then, \( x_j = 0 \quad j : t_j > t_k \) \( \ldots \) (2.3)

And \( x^S_k > 0 \) for some \( S, 1 \leq S \leq P \) \( \ldots \) (2.4)

Also (2.2) and (2.5) imply

\( X^1_j = 0 \quad \forall \quad j : t_j > t_k \) and \( i = 1, \ldots, P \)

in particular for \( i = S \)

\( X^S_j = 0 \quad \forall \quad j : t_j > t_k \) \( \ldots \) (2.5)

(2.4) and (2.5) imply \( T(X^S) = t_k = T(X) \).
Now since $X$ is a point in the relative interior of $S$ and $X^S$ is an extreme point of $S$, therefore $F(X^S) \leq F(X)$.

Hence, \[
\frac{F(X^S)}{T(X^S)} \leq \frac{F(X)}{T(X)} \quad \text{(since $T(X) > 0$)}
\]

\[
\wedge
\]
Let $X = X^S$.

Thus for any point $X$ in the relative interior of $S$, there exists an extreme point $X$ of $S$ such that $\frac{F(X^S)}{T(X^S)} \leq \frac{F(X)}{T(X)}$.

Similarly, the above statement can also be proved when $X$ is a point on a bounded face or facet of $S$.

The above two statements lead to the conclusion that an optimal solution of $(P2)$ is attainable at an extreme point of $S$.

**Theorem 2.2.3** : The optimal solution of $(P3)$ is attainable at an extreme point of $S$.

**Proof** : Let $x$ be a point in the relative interior or on a bounded face or facet of $S$. Then, as in the previous theorem, there exists an extreme point $X$ of $S$ such that $\frac{F(X^S)}{T(X^S)} = \frac{F(X)}{T(X)}$.

\[
\wedge
\]
Also, $F(X) \leq F(X)$. 

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Therefore, \( F(X) \leq F(X) \leq T(X) \).

Thus an extreme point of \( s \) yields the minimum value of \( Z_3(X) \).

Aliter it can be proved that \( Z(X) \) is concave.

Let \( X_1, X_2 \in S \) and \( X = \lambda X_1 + (1- \lambda) X_2 \) for \( 0 < \lambda < 1 \).

The following four cases arise:

i) \( F(X_1) > F(X_2), \ T(X_1) \geq T(X_2) \)

ii) \( F(X_1) > F(X_2), \ T(X_1) < T(X_2) \)

iii) \( F(X_1) \leq F(X_2), \ T(X_1) \geq T(X_2) \)

iv) \( F(X_1) \leq F(X_2), \ T(X_1) < T(X_2) \).

Consider case (i).

Here \( F(X) = F(X_1) \) and \( T(X) = T(X_1) \).

Therefore \( Z_3(X) = F(X) T(X) \)

\[
= F(X_1) T(X_1)
\]

\[
= \lambda F(X_1) T(X_1) + (1- \lambda) F(X_1) T(X_1)
\]

\[
> \lambda F(X_1) T(X_1) + (1- \lambda) F(X_2) T(X_2)
\]
as $F(X_1) > F(X_2)$ and $T(X_1) \geq T(X_2)$

$$= \lambda Z_3(X_1) + (1 - \lambda) Z_3(X_2).$$

Hence $Z(X)$ is concave.

Consider case (ii)

Here $F(X) = F(X_1)$ and $T(X) = T(X_2)$.

Therefore $Z_3(X) = F(X) T(X)$

$$= F(X_1) T(X_2)$$

$$= \lambda F(X_1) T(X_1) + (1 - \lambda) F(X_1) T(X_2)$$

$$> \lambda F(X_1) T(X_1) + (1 - \lambda) F(X_2) T(X_2)$$

as $T(X_2) > T(X_1)$ and $F(X_1) > F(X_2)$

$$= \lambda Z_3(x_1) + (1 - \lambda) Z_3(X_2).$$

Hence $Z(X)$ is concave.

In the cases (iii) and (iv) also, the concavity of $Z_3(x)$ can be similarly proved. Therefore, the minimum value of $Z_3(X)$ is attained at an extreme at an extreme point of $S$.

A Related Bottleneck Linear programming problems

Consider the bottleneck linear programming problems
Min \( R_1(X) \) \( x \in S \) \( \quad \) (RP1)

Min \( R_2(X) \) \( x \in S \) \( \quad \) (RP2)

and

Min \( R_3(X) \) \( x \in S \) \( \quad \) (RP3)

Where

\[
R_1(X) = \max_j r^1_j (X_j), \quad r^1_j(X_j) = f_j(X_j) + t_j(X_j) \]

\[
= \begin{cases} 
  f_j + t_j, & X_j > 0 \\
  0, & X_j = 0
\end{cases}
\]

\[
R_2(X) = F(X) \quad \text{where } T_M = \max_{1 \leq j \leq n} (t_j)
\]

\[
R_3(X) = \max_j r^3_j (X_j), \quad r^3_j(X_j) = f_j(X_j) + t_j(X_j) \]

\[
= \begin{cases} 
  f_j + t_j, & x_j > 0 \\
  0, & x_j = 0
\end{cases}
\]

Notice that the feasible region in each of the above problems is \( S \), the same as in (P1), (P2) and (P3).
Each of $R_1(X)$, $R_2(X)$ and $R_3(X)$ is a concave bottleneck function and hence, optimal solutions to (RP1), (RP2) and (RP3) are attainable at the extreme points of $S$.

The following theorem shows how (RP1), (RP2) and (RP3) are related to (P1), (P2) and (P3) respectively.

**Theorem 2.2.4**: $R_i(X) \leq Z_i(X) \forall X \in S$, $i = 1, 2, 3$.

**Proof** :

(i) $R_i(X) \leq Z_i(X) \forall X \in S$.

For any $X = (X_j) \in S$,

$$R_i(X) = \overline{f}_i(X) + \overline{t}_i(X) = (\text{Max } \overline{f}_j(X_j)) + (\text{Max } \overline{t}_j(X_j)) = F(X) + T(X)$$

$$= Z_i(X).$$

Therefore, $\text{Max } r_i(X) \leq Z_i(X)$,

which implies $R_i(X) \leq Z_i(X), \forall X \in S$.

(ii) $R_2(X) \leq Z_2(X) \forall X \in S$

as $T_M = \text{Max } (t_j)$, therefore $T(X) \leq T_M$ for all $X \in S$,

which implies $\frac{1}{T_M} \leq \frac{1}{T(X)} \forall X \in S$ (since $T_M > 0$, $T(X) > 0 \forall X \in S$)

And therefore $\frac{F(X)}{T_M} \leq \frac{F(X)}{T(X)} \forall X \in S$ (since $F(X) > 0 \forall X \in S$)

that is, $R_2(X) < Z_2(X) \forall X \in S$. 

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