5. SOME APPLICATIONS OF NONPARAMETRIC INFERENCE IN SINGLE SERVER QUEUES

5.1 INTRODUCTION

Several authors have considered the problem of estimation and tests of hypothesis concerning the queueing parameters such as the arrival rate, service rate or the system parameter like the traffic intensity parameter $\rho$. These techniques mainly depend on the assumption made on the form of arrival and service time distributions. Though in this work some generality has been achieved by allowing the service time distributions to be of the Erlang type etc., there are many situations where the exact form of the arrival and service time distributions are unknown. In this section we discuss certain applications of nonparametric inference such as point estimation, interval estimation, tests of hypothesis for the parameters of the input/output distributions of a single server queueing system.

For stable queues it is necessary that $\rho$ must be less than one. If $\rho \geq 1$, the queues are saturated and become unstable leading to congestion. We will discuss some applications of nonparametric tests for testing $H_0: \rho \geq 1$ against the alternative $H_1: \rho < 1$. As opposed to the earlier work, where parametric tests were used as a vehicle for the control of queues, the proposed nonparametric test will be useful for the operator to control queues by taking suitable actions such as
altering the service rate or checking for the arrival rate etc., so as to maintain steady state conditions. The efficacy of the proposed nonparametric tests are compared with the available parametric tests through the vehicle of asymptotic relative efficiency. Some nonparametric tests for the exponentiality of queueing systems are also suggested.

5.2 NONPARAMETRIC INFERENCE ABOUT THE PARAMETERS OF THE INPUT/OUTPUT DISTRIBUTIONS

Consider a single server queue. The interarrival times are assumed to be independent identically distributed (i.i.d) random variables with distribution function (d.f) $F(x)$ with mean $\mu_x$ and variance $\sigma_x^2$. The service times are also assumed to be i.i.d random variables with d.f $G(y)$ with mean $\mu_y$ and variance $\sigma_y^2$. The variances $\sigma_x^2$ and $\sigma_y^2$ are assumed to be finite. We shall denote by $\rho$ the traffic intensity parameter where $\rho = \frac{\mu_y}{\mu_x}$.

5.2.1 Estimation of $\mu_x$ and $\sigma_x^2$:

Let $X_1, X_2, \ldots, X_n$ be $n$ observations from $F(x)$. The $r$th order population moment and sample moment are denoted by $\mu_r^x$ and $m_r^x$ respectively, where

$$\mu_r^x = E(X^r) \quad \text{and} \quad m_r^x = \frac{\sum X_i^r}{n}.$$ 

The sample moment $m_r^x$ is a consistent and unbiased estimator of $\mu_r^x$. Further $m_r^x \sim N(\mu_r^x, v_r)$ for large $n$, where $v_r = \text{Var}(m_r^x)$.

Thus $\frac{1}{n} \sum X_i$ is a consistent and unbiased estimate
of $\mu_x$.

The moment estimate of $\sigma_x^2$ is given by

$$s_x^2 = \frac{\sum (x_i - \bar{x})^2}{n} \quad \text{.. (5.1)}$$

$s_x^2$ is a consistent estimate of $\sigma_x^2$ but not unbiased.

5.2.2 A Large Sample Test for $\mu_x$ and Interval Estimation:

The sample mean $\bar{x}$ is an unbiased and consistent estimate of $\mu_x$. For any distribution function $F(x)$ with $\sigma_x^2 < \infty$, $\text{Var}(x) = \frac{\sigma_x^2}{n}$.

Hence $\bar{x} \sim N(\mu_x, \sigma_x^2 / n)$ for large $n$. Therefore the statistic

$$Z' = \frac{\bar{x} - \mu_x}{\sigma_x / \sqrt{n}}$$

has a normal distribution with mean 0 and variance 1 for large $n$. Since $\sigma_x$ is not known it may be replaced by its consistent estimate $s_x$. The statistic

$$Z = \frac{\bar{x} - \mu_x}{s_x / \sqrt{n}} \quad \text{.. (5.2)}$$

has also a standard normal distribution for large $n$ (see for example Rao [1973], p.385).

Large sample tests concerning $\mu_x$ can be performed using (5.2).

An 100(1-$\alpha$)% confidence interval for $\mu_x$ is given by

$$\left[ \bar{x} - Z_{\alpha/2} \frac{s_x}{\sqrt{n}}, \ \bar{x} + Z_{\alpha/2} \frac{s_x}{\sqrt{n}} \right]$$

where $Z_{\alpha/2}$ is the standard normal deviate.

5.2.3 Large Sample Test for $\sigma_x$ and Interval Estimation:

The moment estimate of $\sigma_x^2$ is given by $s_x^2$, where
\[ s_x^2 = \frac{\sum (x_i - \bar{x})^2}{n} \]

\[ E(s_x^2) = \frac{n-1}{n} \sigma_x^2 \]

\[ \text{Var}(s_x^2) = \frac{\mu_4 - \mu_2^2}{n} - \frac{2(\mu_4 - 2\mu_2^2)}{n^2} + \frac{\mu_4 - 3\mu_2^2}{n^3} \]

where \( \mu_x = E(X - \mu_x)^2 \) being the \( r \)th central moment of \( X \).

Since moment estimates are asymptotically normal, \( s_x^2 \) has a normal distribution with mean \( E(s_x^2) \) and variance \( \text{Var}(s_x^2) \) for large \( n \).

Therefore, \( Z = \frac{s_x^2 - \frac{n-1}{n} \sigma_x^2}{\sqrt{\text{Var}(s_x^2)}} \sim N(0,1) \) for large \( n \).

where \( V(s_x) = \sqrt{\text{Var}(s_x^2)} \)

If we denote by \( m_r(x) \) the moment estimate of \( \mu_x \) then

\[ v^2(s_x) = \frac{m_4(x) - m_2(x)}{n} - \frac{2(m_4(x) - 2m_2(x))}{n^2} + \frac{m_4(x) - 3m_2(x)}{n^3} \]

is a consistent estimate of \( \text{Var}(s_x^2) \).

Therefore,

\[ Z = \frac{s_x^2 - \frac{n-1}{n} \sigma_x^2}{V(s_x)} \quad \ldots \quad (5.3) \]

has a normal distribution for large \( n \). Hence large sample tests concerning \( \sigma_x^2 \) can be performed using (5.3).

For large \( n \), an \( 100(1-\alpha)\% \) confidence interval for \( \sigma_x^2 \) is given by

\[ \left\{ \frac{s_x^2 - Z_{\alpha/2} V(s_x)}{n-1} \leq \sigma_x^2 \leq \frac{s_x^2 + Z_{\alpha/2} V(s_x)}{n-1} \right\} \]
5.2.4 A Large Sample Test for Population Median:

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics for $X_1, X_2, \ldots, X_n$.

The empirical distribution function of $X$ is given by

$$
F^*_n(x) = \begin{cases} 
0 & \text{if } x < X_{(1)} \\
\frac{k}{n} & \text{if } X_{(k)} \leq x \leq X_{(k+1)}, \ (k=1,2,\ldots,n-1) \\
1 & \text{if } x \geq X_{(n)}
\end{cases} \quad \ldots \quad (5.4)
$$

Thus $nF^*_n(x)$ gives the number of $X_k$'s $(1 \leq k \leq n)$ that are $\leq x$. Then we have (see for example Rohatgi [1976], p.299),

$$E\{F^*_n(x)\} = F(x) \quad \ldots \quad (5.5)$$

$$\text{Var}\{F^*_n(x)\} = \frac{F(x)[1-F(x)]}{n} \quad \ldots \quad (5.6)$$

$$F^*_n(x) \overset{P}{\to} F(x) \text{ as } n \to \infty, \text{ i.e. } F^*_n(s) \text{ is a consistent estimate of } F(s). \quad \ldots \quad (5.7)$$

$$Z = \frac{\sqrt{n}[F^*_n(s) - F(s)]}{\sqrt{F(s)[1-F(s)]}} \sim N(0,1) \text{ for large } n \quad \ldots \quad (5.8)$$

If we denote by $M_0$ the population median, then from (5.8)

$$Z = \sqrt{4n}[F^*_n(M_0) - \frac{1}{2}] \sim N(0,1) \text{ for large } n \quad \ldots \quad (5.9)$$

Therefore large sample tests concerning population median $M_0$ can be performed using (5.9). Tests concerning any population percentile can also be performed using (5.8). For symmetric distributions (5.9) can be used for testing the population mean.

5.2.5 Identification of the distribution $F(x)$:
Assume that \( n \) observations \( X_1, X_2, \ldots, X_n \) are available on the input distribution \( F(x) \). The problem is to test whether the observations come from a specified distribution say \( F_0(x) \) or from some alternative distribution. Such a hypothesis can be tested, for example, by making use of the Chi-square goodness of fit test or the Kolmogorov-Smirnov test. For distributions like uniform, exponential or normal simple graphical test procedures are available. We will describe the Kolmogorov-Smirnov test below.

**Kolmogorov-Smirnov Goodness of fit Test:**

Let \( X_1, X_2, \ldots, X_n \) be a sample of \( n \) observations from distribution \( F(x) \). Let \( F_n^*(x) \) be the corresponding empirical distribution function as given in (5.4). The Kolmogorov-Smirnov statistic for testing \( H_0: X_1 \sim F_0(x) \) against the alternative \( H_1: X_1 \not\sim F_0(x) \) \((F_0(x)\) is completely specified) is given by

\[
D_n = \text{Sup} \left[ F_n^*(x) - F_0(x) \right]
\]

Large values of \( D_n \) or small values of \( D_n \) tends to the rejection of \( H_0 \). The critical values of \( D_n \) can be found in Kolmogorov [1941] and Miller [1956].

**Graphical Test For The Exponential Distribution:**

Let \( f(x) = \beta e^{-\beta x} \quad x > 0 \)

then \( F(x) = 1 - e^{-\beta x} \)

Let \( H(x) = 1 - F(x) = e^{-\beta x} \)

then \( \ln H(x) = -\beta x \)

It may be seen that the complementary d.f of the exponential
distribution is linear in $X$. Therefore, the plot of $H(x)$ on a semi-log paper would be a straight line. The complementary empirical d.f $F^*_n(x)$ can be used as an estimate of $H(x)$.

A number of techniques to test whether the distribution is exponential or not has been discussed in detail by Epstein [1960].

5.2.6 Test for Exponentiality of Queue:

The exponential distribution and the Poisson process play very important role in queueing theory. The simplest and most commonly used test for testing exponentiality/Poisson characteristic of data is the Chi-square goodness of fit test. A few other tests which are powerful than the Chi-square test and can be performed even for small samples are the F-test, Kolmogorov-Smirnov test and the Anderson-Darling test. These tests can be applied to data on interarrival and service times seperately for exponentiality character. (see Gibbons [1971])

It may be noted that the output process of $M|M|1$ and $M|M|c$ are also Poisson. In fact the output process of $M|G|c$ and $GI|M|c$ are Poisson iff $M=G$ and $GI=M$ respectively. Therefore exponentiality of the queue can be tested using the output data. We suggest the following test based on output data.

Consider a single or multiserver queueing system. The system is observed for the departures in $(0,T]$, $T$ denote the time fixed for experiment. Let there be $n$ departures. Let $t_i$ $(i=1,2,...,n)$ denote the time at which the $i^{th}$
departure takes place. Clearly \( t_1 < t_2 < \ldots < t_n \).

Then the random variables \( t_1, t_2, \ldots, t_n \) have the same distribution as the \( n \)-order statistics corresponding to the independent random variables say \( U_1, U_2, \ldots, U_n \) which are uniformly distributed in the interval \((0, T]\). (see Gross and Harris [1974], p.28)

Let \( S_n = \sum_{i=1}^{n} t_i \)
then \( E(S_n) = \frac{nT}{2} \) and \( \text{Var}(S_n) = \frac{nT^2}{12} \) since \( t_i \sim U(0, T] \).

Therefore, under the hypothesis that the underlying process is Markovian, i.e. the input and service time distributions both are exponential, the statistic

\[
Z = \frac{\sum t_i - \frac{nT}{2}}{\sqrt{\frac{nT^2}{12}}}
\]

follows a normal distribution with mean 0 and variance 1 for large \( n \). This statistic can be used to test for the exponential or Markovian character of queue.

### 5.3 DISTRIBUTION-FREE TESTS FOR \( p \)

#### 5.3.1 A Large Sample Test For \( p \):

Consider a single server queue. The interarrival time distribution and the service time distributions are denoted by \( f(x) \) and \( g(y) \) respectively. The corresponding d.f's are denoted by \( F(x) \) and \( G(y) \). \( \mu_x \) and \( \mu_y \) denote the mean inter-arrival time and mean service times.

Let \( X_1, X_2, \ldots, X_m \) be \( m \) observations from the input
distribution $F(x)$ and let $Y_1, Y_2, \ldots, Y_n$ be $n$ observations from the service time distribution $G(y)$.

We shall state our hypothesis as $H_0: \rho \geq 1$ which is to be tested against $H_1: \rho < 1$.

Let $\Delta(F,G) = P(X > Y) = \int_0^\infty \int_0^x g(y)f(x)dydx = \int_0^\infty G(x)f(x)dx$

When the two distributions are identical, i.e. when $F(x) = G(x)$ we have $\mu_x = \mu_y \implies \rho = 1$, so that $P(X > Y) = \frac{1}{2}$.

When $F(x) \geq G(x)$ (strict inequality for some $x$) $\implies \mu_y \geq \mu_x$ $\implies \rho \geq 1$, we have $P(X > Y) \leq \frac{1}{2}$.

Define $Z_{ij} = \begin{cases} 1 & \text{if } X_i > Y_j \\ 0 & \text{if } X_i \leq Y_j \end{cases}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Let $U = \sum_{i=1}^m \sum_{j=1}^n Z_{ij}$.

An unbiased estimate of $\Delta(F,G)$ is given by $\hat{\Delta}(F,G) = \frac{U}{mn}$.

In particular if $F = G$, $E[\hat{\Delta}(F,G)] = \Delta(F,G) = P(X > Y) = \frac{1}{2}$ and $\text{Var}(\hat{\Delta}(F,G)) = \frac{m+n+1}{12mn}$.

Therefore, for large $m$ and $n$, the statistic

$$T = \frac{U}{\sqrt{mn}} - \frac{1}{2}$$

has a standard normal distribution. If $U$ is large, the values of $Y$ tend to be larger than the values of $X$, and this supports the hypothesis, $F(x) \geq G(x)$ ($\implies \rho \geq 1$). Similarly, if $U$ is small the $Y$ values tend to be smaller than $X$ values, and
this supports the hypothesis \( F(x) < G(x) \) (\( \Rightarrow P < 1 \)).

Thus to test \( H_0 : \rho \geq 1 \) against \( H_1 : \rho < 1 \) at level \( \alpha \), the procedure is to reject \( H_0 \) if \( T < T_\alpha \) and accept \( H_0 \) if \( T > T_\alpha \) where \( \alpha = \Phi(T_\alpha) \), \( \Phi(.) \) being the d.f of the standard normal variate.

5.3.2 Applications of Classical Nonparametric Tests:

In the literature of nonparametric inference for the general two sample problem the hypothesis is stated as

\[ H_0 : G(x) = F(x) \quad \text{for all } x \quad \ldots \quad (5.10) \]

That is, the two samples come from identical populations.

The general two sided alternative is stated as

\[ H_A : G(x) \neq F(x) \quad \text{for some } x \quad \ldots \quad (5.11) \]

and a one sided alternative as

\[ H_B : G(x) \geq F(x) \quad \text{for all } x, \quad \text{and} \]
\[ G(x) > F(x) \quad \text{for some } x \quad \ldots \quad (5.12) \]

The condition \( G(x) \geq F(x) \) implies the random variable \( X \) is stochastically larger than the random variable \( Y \). An arbitrary representation of such a situation is shown in Fig.1 below.

![Fig. 1](image-url)
Now consider
\[ G(x) > F(x) \]
\[ \implies \int [1 - G(x)] \, dx < \int [1 - F(x)] \, dx \]
\[ \implies \mu_y < \mu_x \]

Therefore the alternative that \( X \) is stochastically larger than \( Y \) is general alternative which includes the more specific alternative \( \mu_y < \mu_x \).

Now the hypothesis \( H_0 : \rho \geq 1 \) to be tested against \( H_1 : \rho < 1 \) is equivalent to testing \( H_0' : \mu_y \geq \mu_x \) against \( H_1' : \mu_y < \mu_x \), since \( \rho = \mu_y / \mu_x \).

Since the one sided alternative given in (5.13) includes the alternative \( \mu_y < \mu_x \), the hypothesis \( H_0' : \mu_y \geq \mu_x \) vs \( H_1' : \mu_y < \mu_x \) can be restated as, testing

\[ H_0^* : G(x) \leq F(x) \text{ for all } x, \text{ and} \]
\[ G(x) < F(x) \text{ for some } x \]

against

\[ H_1^* : G(x) \geq F(x) \text{ for all } x, \text{ and} \]
\[ G(x) > F(x) \text{ for some } x \]

Therefore, for testing \( H_0^* \) against \( H_1^* \) (in terms of \( H_0^* \) and \( H_1^* \)) the classical tests such Kolmogorov-Smirnov two sample test or the Mann-Whitney test can be made use of.

The hypothesis that two populations are identical against that they are different, can also be stated as

\[ H_0 : G(x) = F(x) \text{ for all } x, \text{ vs} \]
\[ H_0^* : G(x) = F(x - \Theta) \text{ for all } x, \Theta \neq 0 \]

\[ \cdots (5.14) \]
as the alternative. This is called the location alternative. At $\Theta = 0$ we have the null case, $\Theta < 0 \Rightarrow \mu_y < \mu_x$ and $\Theta > 0 \Rightarrow \mu_y > \mu_x$. Therefore $H_0$ can be tested against $H_1$ through the location alternative (5.14). Thus, through the location alternative a number of nonparametric tests such as the Median test, Wilcoxon test, Terry-Hoefding test, Van Der Waerden test etc., (for details see Gibbons [1971]) can be made use of for testing $H_0$ against $H_1$.

5.4 ILLUSTRATIONS

In this section we illustrate the Kolmogorov-Smirnov, Mann-Whitney, Median and Wilcoxon test procedures for testing the hypothesis $H_0: \rho \geq 1$ against $H_1: \rho < 1$ in terms of the hypotheses as stated in (5.13) and (5.14)

Assume that a single server queueing system is observed starting from time $0$ at which the first customer arrives to the time the $n^{th}$ departure takes place. Denote by $Y_1, Y_2, \ldots, Y_n$ the service times of these $n$ customers. Let there be $m$ arrivals before the $n^{th}$ departure takes place. The interarrival times of these $m$ customers are denoted by $X_1, X_2, \ldots, X_m$. These $mX$ and $nY$ observations constitute the random samples for the purpose of testing the hypothesis.

5.4.1 Kolmogorov-Smirnov Two-Sample Test:

Let $H_0: \rho \geq 1$ and $H_1: \rho < 1$. This hypothesis is equivalent to testing $H_0^*$ against $H_1^*$ as given in (5.13).

For the observations $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$
define the order statistics by \( X(1), X(2), \ldots, X(m) \) and \( Y(1), Y(2), \ldots, Y(n) \) respectively. Their empirical distributions are denoted by

\[
S_m(x) = \begin{cases} 
0 & \text{if } x < X(1) \\
\frac{k}{m} & \text{if } X(k) \leq x \leq X(k+1) \text{ for } k=1,2,\ldots,m-1 \\
1 & \text{if } x \geq X(m) \\
& \text{if } x < Y(1) \\
& \text{if } x \geq Y(n)
\end{cases}
\]

\[
T_n(x) = \frac{k}{n} \text{ if } Y(k) \leq x \leq Y(k+1) \text{ for } k=1,2,\ldots,n-1
\]

The one sided statistic for testing \( H_0 \) against \( H_1 \) is given by

\[
D_{m,n} = \sup_x \left[ T_n(x) - S_m(x) \right]
\]

The rejection region for this test is given by \( D_{m,n} \geq D_{m,n,\alpha} \)

For small samples the critical values \( D_{m,n,\alpha} \) are given by Messey [1952] and for larger samples Smirnov [1939] gives the rule for determining the critical region.

5.4.2 Mann-Whitney U Test:

Let \( H_0 : \rho \geq 1 \) to be tested against \( H_1 : \rho < 1 \).

Based on the observations \( X_1, X_2, \ldots, X_m \) and \( Y_1, Y_2, \ldots, Y_n \) on the interarrival and service times, define an indicator variable as

\[
\delta_{ij} = \begin{cases} 
1 & \text{if } Y_j < X_i \text{ for } i=1,2,\ldots,m \text{ and } j=1,2,\ldots,n \\
0 & \text{if } Y_j > X_i
\end{cases}
\]

For simplicity, we shall assume that there are no ties.

The Mann-Whitney U Statistic for the one sided
alternative is given by

\[ U = \sum_{i=1}^{m} \sum_{j=1}^{n} (1 - D_{ij}) \]

The rejection region for testing \( H_0 \) against \( H_1 \) is given by \( U \leq U_\alpha \). The critical values for small samples are available in Auble [1953] or Mann and Whitney [1947].

For large \( m \) and \( n \) it has been shown that \( E(U) = \frac{mn}{2} \) and \( \text{Var}(U) = \frac{mn(m+n+1)}{12} \). Hence for large samples, normal approximation can be used for testing \( H_0 \) against \( H_1 \). In fact, this test procedure is equivalent to the large sample test discussed in Section 5.3.1.

5.4.3 Median Test:

\( H_0^* \) and \( H_1^* \) given by (5.13) are a general set of hypotheses which includes the hypothesis \( H_0: \mu_Y \geq \mu_X \) and \( H_1: \mu_Y < \mu_X \). Since we are interested in testing only \( H_0 \) against \( H_1 \), the general one sided location alternative

\[ H_C: G(x) = F(x - \theta) \text{ for all } x \text{ and } \theta < 0 \]

can be used.

In Median test, the procedure is to arrange the combined samples in increasing order of magnitude and the sample median is to be determined. Denote by \( u \) the number of \( X \) observations less than the combined sample median.

Whenever \( u \leq c_\alpha \) we accept \( H_C \) with \( \theta < 0 \). The critical value \( c_\alpha \) can be found from the distribution of \( U \) given by,
\[ P(U=u) = \binom{m}{u} \binom{n}{t-u} \binom{m+n}{t} \quad u=0,1,2,...,t, \ t = \lceil \frac{n}{2} \rceil \]

for large \( m \) and \( n \), approximate normal test can be used.

**5.4.4 Wilcoxon Test**:

Wilcoxon test is applicable for the general location alternative as stated in \( H_c \).

Define indicator random variables \( Z_1, Z_2, ..., Z_{m+n} \)
where \( Z_i = 1 \) if the \( i^{th} \) random variable in the combined ordered sample is an \( X \), and \( Z_i = 0 \) if it is a \( Y \), \( i = 1,2, ..., m+n \). The Wilcoxon test statistic is given by,

\[ W = \sum_{i=1}^{m+n} Z_i \]

Large values of \( W \) indicates \( \Theta < 0 \) which implies \( \mu_Y \geq \mu_X \).

The probability distribution of \( W \) when \( \Theta = 0 \) can be derived by enumeration (see for example Gibbons [1971]). However, for \( m+n < 20 \) the critical values have been tabulated by Wilcoxon [1945].

Since a linear relationship exists between the Wilcoxon \( W \) statistic and the Mann-Whitney \( U \) statistic, the two tests are similar. Therefore the efficiency of the Wilcoxon test is same as that of the Mann-Whitney test.

**5.5 ASYMPTOTIC RELATIVE EFFICIENCY**

A nonparametric test can be compared with a parametric or nonparametric test through the asymptotic relative efficiency (ARE). The ARE of the test \( T \) relative to
another test $T^*$ is given by (see Gibbons [1971]),

$$\text{ARE}(T, T^*) = \lim_{n \to \infty} \frac{e(T_n)}{e(T^*_n)}$$

where $e(T_n)$ is the efficacy of the test statistic $T_n$ when used to test the hypothesis $\Theta = \Theta_0$ and

$$e(T_n) = \left\{ \frac{dE(T_n)}{d\Theta} \right\}_{\Theta = \Theta_0} \sigma^2(T_n)$$

5.5.1 ARE of Mann-Whitney test Relative to t-test

For the general location problem in the case of two independent random samples of sizes $m$ and $n$, the assumed distribution model is $G(x) = F(x - \Theta)$, and the null hypothesis of identical distribution is $H_0: \Theta = 0$. The corresponding classical test statistic for testing $H_0: \Theta = 0$ against $H_1: \Theta \neq 0$ is the two-sample Student’s t-test statistic given by

$$T^*_{m,n} = \sqrt{\frac{mn}{m+n}} \frac{\bar{Y}_n - \bar{X}_m}{S_{m+n}}$$

The efficacy of $T^*_{m,n}$ for any population is $e(T^*_{m,n}) = \frac{mn}{\sigma^2(m+n)}$ where $\sigma^2 = \text{Var}(X)$. The efficacy of the Mann-Whitney test is

$$e(U_{m,n}) = 12 mn \left\{ \int f^2(x) dx \right\}^{2} \frac{1}{m+n+1}$$

where $f(x)$ is the p.d.f of $X$. The ARE of $U_{m,n}$ relative to
for few selected distributions are summarized in Table 1.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>ARE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>3.0</td>
</tr>
<tr>
<td>k-Erlangian</td>
<td></td>
</tr>
<tr>
<td>k=1</td>
<td>3.0</td>
</tr>
<tr>
<td>k=2</td>
<td>1.5</td>
</tr>
<tr>
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<td>1.2656</td>
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<tr>
<td>k=4</td>
<td>1.1718</td>
</tr>
<tr>
<td>k=5</td>
<td>1.1215</td>
</tr>
<tr>
<td>Uniform</td>
<td>1.0</td>
</tr>
<tr>
<td>Normal</td>
<td>0.9545</td>
</tr>
</tbody>
</table>

From Table 1, it may be noted that for exponential and Erlangian distributions (for small k) the Mann-Whitney test is powerful than the t-test. This is not a surprise, since the performance of t-test is poor for extremely asymmetric distributions.

5.5.2 ARE of the Mann-Whitney Test Relative to the F-Test:

When the interarrival and service times have exponential or Erlangian distributions, Cox [1965], has suggested an F-test for testing hypothesis concerning the traffic intensity $\rho$.

Consider an $M|M|1$ queue where the arrival rate is $\lambda$ and the service rate is $\mu$. Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ denote the random samples from the arrival and service time distributions respectively. Then

$$F = \frac{\frac{2\mu \sum X_i}{2m}}{\frac{2\mu \sum Y_j}{2n}}$$

(5.15)
has an \( F \)-distribution with \( 2m \) and \( 2n \) degrees of freedom. This \( F \)-statistic can be used to test the hypothesis \( H_0: \rho \geq 1 \) against \( H_1: \rho < 1 \), where \( \rho = \lambda / \mu \).

Since the Mann-Whitney and such other nonparametric tests were used to test the similar hypothesis, we shall compare the ARE of Mann-Whitney test relative to the \( F \)-test.

Let \( \mu_x = 1/\lambda \), \( \mu_y = 1/\mu \), \( \rho = \mu_y / \mu_x \), \( \mu_y = \mu_x + \Theta \)

Then testing \( H_0: \Theta \geq 0 \) vs \( H_1: \Theta < 0 \) is equivalent to testing \( H_0: \rho \geq 1 \) vs \( H_1: \rho < 1 \). From (5.15), the \( F \)-statistic reduces to

\[
F = \left( \frac{\mu_x + \Theta}{\mu_x} \right) \frac{n \sum X_i}{m \sum Y_j},
\]

Consider

\[
\frac{n \sum X_i}{m \sum Y_j} = \left( 1 + \frac{\Theta}{\mu_x} \right)^{-1} F
\]

Efficacy of \( Z \) is same as the efficacy of \( F \), since when \( \Theta = 0 \), \( F = Z \).

Now,

\[
e(Z) = \left( \frac{dE(Z)}{d\Theta} \right)^2 \left| \frac{\text{Var}(Z)}{\Theta = 0} \right. = \frac{mn}{\mu_x^2(m+n)} = \frac{mn}{\sigma^2(m+n)}
\]

It may be noted that the efficacy of the \( F \)-test is same as the efficacy of the Student's t-test. Hence the ARE of Mann-Whitney test relative to the \( F \)-test is same as the ARE of Mann-Whitney test relative to the t-test.