CHAPTER II
ELECTRON SCATTERING

A brief survey on electron scattering has been given in this chapter because of the fact that the primary motivation for the experiments described in the later chapters of this thesis came from the results of the electron scattering experiments highlighted in this chapter. A comparison is often made between the results of our experiments and electron scattering experiments.

Electron scattering is quite an old field. Yet, due to development of high energy and high resolution electron machines and high resolution detection systems, work with electron scattering is now pursued but with much more precision. Extensive work already performed on the scattering of electrons by the nuclear charge distribution, the results, and their interpretation have been summarized in a number of review papers ([Ho 58, De 66, Ub 71, Ba 74, Do 75, Ba 77, Ci 80]). At the same time considerable work has also been carried out with magnetic electron scattering probing the structure of nuclei, viz., configuration mixing, size of radial wave functions, etc. Results from magnetic electron scattering have been discussed in detail by Platchkov [Pl 82] and also have been reviewed recently by Donnelly and Sick [Do 84].

Electron scattering is known to be an excellent tool for the study of nuclear structure. First of all the interaction is well known and is described by an exact theory, viz., quantum electrodynamics. Therefore, the nuclear properties can be extracted from the data in a quantitative way. Calculations performed in the impulse and plane-wave approximations are already quite close to reality since the interaction measured by $\alpha=1/137$ is weak and any modifications or
refinements needed can be done with a manageable complexity. Secondly one can map out the Fourier transforms of the static and transition densities by varying the three momentum transfer \( \vec{q} \) for a fixed energy loss. By inverting these Fourier transforms one can get the density distributions of the nuclei. The larger the value of \( q \), the finer are the details of the charge density that can be probed. Finally, by varying \( \omega \), the energy transferred to the target, one can obtain an excitation profile of the nucleus. At low momentum transfer, this is similar to the excitation functions seen in photo-nuclear reactions dominated by electric dipole transitions. Furthermore, because the interaction of the electron with the nuclear spin-magnetization current is enhanced at high \( q \) and large scattering angle, it is possible to examine states of magnetic character, including high-spin magnetic excitations.

Fig. 2.1 shows electron scattering in the one-photon-exchange approximation. An incident electron with four momentum \( k_\mu = (\vec{k}, \epsilon) \) is scattered through an angle \( \theta \) to four momentum \( k'_\mu = (\vec{k}', \epsilon') \). During the process a virtual photon with four momentum \( q_\mu = (\vec{q}, \omega) \) is exchanged with the nucleus. Conservation of four momentum implies that \( q_\mu = k_\mu - k'_\mu \); also there energy transfer \( \omega = \epsilon - \epsilon' \).

In the extreme relativistic limit \( k = |\vec{k}| = \epsilon, k' = |\vec{k}'| = \epsilon' \) and therefore

\[
q = |\vec{q}| = [\omega^2 + 4\epsilon\epsilon'\sin^2\frac{1}{2}\theta]^\frac{1}{2} \tag{2.1}
\]

and

\[
q^2_\mu = q^2 - \omega^2 = 4\epsilon\epsilon'\sin^2\frac{1}{2}\theta \tag{2.2}
\]
Fig. 2.1. Electron scattering in the one-photon exchange or first Born approximation.
In the plane-wave Born approximation, the cross section for unpolarized electron scattering, involving a transition from state \(|i\rangle\) to state \(|f\rangle\), is given by [Pl 82]

\[
\left[ \frac{d\sigma}{d\omega} \right]_{i \rightarrow f} = \left( \frac{d\sigma}{d\omega} \right)_{Mott} \eta^{-1} F^2(q, \theta). \tag{2.3}
\]

In this expression

\[
\left( \frac{d\sigma}{d\omega} \right)_{Mott} = \left[ \frac{\alpha \cos \frac{1}{2} \theta}{2 \varepsilon \sin^2 \frac{1}{2} \theta} \right]^2 \tag{2.4}
\]

where \(\alpha\) is the fine structure constant, \(\theta\) the scattering angle, and \(\eta\) the recoil factor \(1 + (2\varepsilon/M_T) \sin^2 \frac{1}{2} \theta\) with \(M_T\) being the target mass.

The expression for the cross section in Born aproximation is considerably simplified in the case of elastic scattering. Neglecting the electron mass and nuclear recoil, the cross section is proportional to the lowest-order Mott cross-section and is given by

\[
\frac{d\sigma}{d\omega} = \left( \frac{d\sigma}{d\omega} \right)_{Mott} [F_L^2(q) + (\frac{1}{2} + \tan^2 \frac{1}{2} \theta) F_T^2(q)] \tag{2.5}
\]

where \(F_L(q)\) and \(F_T(q)\) are the longitudinal and transverse form factors, respectively.
2.1 CHARGE ELECTRON SCATTERING

The nucleus creates a distribution of the electrostatic potential $V(r)$ determined by the distribution of charge density $\rho(r)$. Scattering an electron from this potential allows one to deduce its shape and to determine $\rho(r)$. Ignoring the transverse form factor which is exclusively of magnetic character Eq. (2.5) can be written in the simple form

$$\frac{d\sigma}{d\omega} = \left(\frac{d\sigma}{d\omega}\right)_{Mott} F^2(q). \quad (2.6)$$

Here the cross section simply factorizes into a kinematical term, $\sigma_{Mott}$, and a form factor that contains all the information on $\rho(r)$. This form factor depends on the momentum transfer $q$ only, and represents the Fourier transform of the charge density that is to be extracted from the experimental data. It is given by:

$$F(q) = \int_0^\infty \frac{\sin(qr)}{qr} \rho(r) 4\pi r^2 \, dr. \quad (2.7)$$

PWBA is sufficiently precise for the very lightest nuclei ($Z \leq 2$) only. For most of the heavier nuclei the exact solution of the Dirac equation that describes the electron in the electrostatic potential of the nucleus has to be used. Therefore, in order to determine $\rho(r)$ from the experimental data one has to postulate a nuclear charge density and solve the Dirac equation for this $\rho(r)$ numerically, then compute the elastic cross sections and compare them with the experimental data. Here one varies the parameters until agreement with experiment is satisfactory. This method, although exact, is indirect.

One can avoid this by expanding the density in terms of a complete orthogonal set of functions. The success of an analysis in which $\rho(r)$ is expanded in a
complete set of functions depends critically on the choice of the functions. One criterion for the choice of a set of functions is that the functions whose coefficients are determined by \( F(q) \) should be orthogonal to those whose coefficients depend on \( F(q) \) for \( q < q_{\text{max}} \). A simple expansion which satisfies this criterion is a Fourier sine series

\[
\rho(r) = \frac{1}{r} \sum_{n=1}^{\infty} C_n \sin(n\pi r/R), \quad r \leq R. \tag{2.8}
\]

R is chosen so that the density \( \rho(r) \) is negligible for \( r > R \). The coefficients are given by

\[
C_n = q_n F(q_n)/2\pi R, \tag{2.9}
\]

where

\[
q_n = n\pi/R. \tag{2.10}
\]

Such an expansion was used by Meyer-Berkhout et al. [Me 59] where they terminated the series for a maximum value \( n_{\text{max}} = 5 \) or 6. Later this approach was modified by Friar and Negele [Fr 73] who took the tail of \( \rho(r) \) (which is poorly determined by experiment) be given by a Hartree-Fock calculation. They assumed that the density is of the form \( \rho_0 + \delta\rho \), where \( \rho_0 \) is the Hartree Fock density and \( r\delta\rho \) is assumed to be zero outside a maximum radius \( R \), and that the density is expanded for \( r < R \) as a Fourier sine series. Now solving the Dirac equation exactly for the density \( \rho_0 \), one can obtain cross sections and muonic atom x-ray energies. Then for a change \( \delta\rho \) in the density, first-order perturbation gives integrals for the changes in these quantities which are linear in \( \delta\rho \) so that the coefficients of the series may be obtained from the best fit to the data.
The error $\Delta \rho$ in the charge density is given by the errors on the coefficients, and hence the uncertainty in integral quantities of the form

$$< G >= \int_0^\infty g(r)\rho(r)r^2 \, dr$$

(2.11)

can be easily obtained. Besides Fourier sine series, a number of other complete sets of functions viz. Hermite polynomials (Me 59), Laguerre polynomials (Fr 73) and cosine, spline, and unspline functions have also been used. In this method, since one has to truncate at a finite maximum momentum transfer $q_{\text{max}}$ in an experiment, only a few coefficients of the expansion can be determined; hence this method is not so useful.

In order to derive a true model-independent density, one somehow has to account for the properties which were not measured ($q > q_{\text{max}}$). For this one has to have some theoretical input (Hartree-Fock calculations) on the maximum amount of structure to be expected in $\rho(r)$. Now a constraint on the finest structure to be allowed in $\rho(r)$ can be applied directly in radial space by a new approach introduced by Lenz [Le 69], Friedrich and Lenz [Fr 72] and modified by Dreher et al. [Dr 74] and Sick [Si 73a,b]. The main feature of this approach is the complete freedom to vary one region of the density without affecting any other part. In this method the charge density is given by an expansion of a set of spherical shells

$$\rho(r) = \sum_{i=1}^N P_i \delta(r - R_i)/r\pi R_i^2,$$

(2.12)

where $R_i$ is the radius of each shell and $P_i$ is the fraction of the total charge which it contains; $R_i$ and $P_i$ are treated as parameters of the charge distribution.
Later Sick [Si 74] replaced the sum of spherical shells by a sum of gaussians (SOG) centered about radii $R_i$ given by the expression

$$\rho(r) = \sum_i \frac{P_i}{2\pi^2 \gamma (\gamma^2 + 2R_i^2)} \left[ e^{-\gamma(r-R_i)^2} + e^{-\gamma(r+R_i)^2} \right],$$

(2.13)

where $P_i$ is now the fraction of charge in the $i^{th}$ Gaussian and the second term is added to eliminate a cusp at $r=0$. The width $\gamma$ is chosen so that the mean-square radius of the Gaussian is equal to that of the proton or, the full width $\Gamma = 2\gamma \ln 2$ of the Gaussian is equal to the minimum full width of oscillation obtained using Hartree-Fock wave functions. The form factor corresponding to the SOG charge density is given by

$$F(q) = e^{-q^2 \gamma^2 / 4} \sum_i \frac{P_i}{\gamma^2 + 2R_i^2} \left[ \gamma^2 \cos(qR_i) + 2R_i^2 \frac{\sin(qR_i)}{qR_i} \right]$$

with $\gamma \to 0 \sum_i P_i \frac{\sin(qR_i)}{qR_i}$

(2.14)

and in the limit $\gamma \to 0$ this gives the form factor for the $\delta$-function shell density.

Sick (Si 74) linearizes the calculation to reduce the amount of computation by replacing the experimental cross-sections by model density $\rho_{mod}$ which gives cross-sections $\sigma_{mod}$ close to the experimental values. Here, the direct Fourier transform of the experimental PWBA form factor is defined by [Si 82] as

$$F_{exp}(q) = \left[ \frac{\sigma_{exp}(E_0, \theta)}{\sigma_{mod}(E_0, \theta)} \right]^{1/2} F_{mod}(q),$$

(2.15)

where the index “mod” refers to quantities computed using a model density $\rho_{mod}$ that fits the experimental data. This model density can be used to connect $\sigma_{mod}$ to $F_{mod}$ in the PWBA. Since $\sigma_{exp}$ and $\sigma_{mod}$ do not differ much, one can quantitatively determine $F_{exp}$ using the above equation.
Once one has translated the experimental data into the PWBA form factors, the determination of $\rho(r)$ can be done by the Fourier transform of equation

$$\rho(r) = \frac{1}{2\pi^2} \int_0^\infty \frac{\sin(qr)}{qr} F_{\text{exp}}(q)q^2 dq. \quad (2.16)$$

If the data are of sufficiently good quality this integration over the experimental form factor yields $\rho(r)$. However, this straightforward determination of $\rho(r)$ is of course not possible because $F(q)$ is known only over a limited range of $q = q_{\text{max}}$. Nevertheless, one might ask whether the unknown parts of $F(q)$ are sufficiently unimportant so that the charge distribution can be approximately inferred directly from the data. In order to estimate the error (which one makes by stopping at $q = q_{\text{max}}$), one can plot the integral of Eq. 1.5 as a function of the upper integration limit which changes sign at every diffraction zero and $\rho(0, q_{\text{max}})$ has a minimum or maximum and the difference between the last measured minimum and maximum can be taken as an estimate of the error $\delta\rho_{\text{exp}}(r)$.

Cavedon et al. [Ca 82] measured the elastic electron scattering cross sections on $^{206}\text{Pb}$ and $^{205}\text{Tl}$ in order to find the nature of the $3s_{1/2}$ proton hole in $^{205}\text{Tl}$. The cross section ratio $\sigma_{205}/\sigma_{206}$ shown in Fig. 2.2 has a spike in the region of momentum transfer $q=2 \text{ fm}^{-1}$. This spike was due to the effect of the $3s_{1/2}$ proton hole in $^{205}\text{Tl}$. Furthermore, the charge densities [Fr 83] shown in Fig. 2.3 were determined from fits to the elastic electron scattering cross sections, combined with five moments deduced from the measurement of muonic x-ray transitions and a model-independent analysis made with an expansion of the charge density in a sum of Gaussians [Si 74]. The resulting charge density difference $\Delta\rho(r) = \rho_{206} - \rho_{205}$ is shown in Fig 2.4. This density difference
between the two isotones is expected to give the nature of the $3s_{1/2}$ proton hole in $^{205}$Tl. Indeed, the shape of the $3s_{1/2}$ orbital determined by this experiment was strikingly close to the one expected for a $3s_{1/2}$ wave function.

The experimental results were compared with the Hartree Fock calculations [Ca 72 and De 80] using a $3s$ spectroscopic factors of 1.0 and 0.7 and the resulting curves are shown in Fig. 2.2. The agreement between the HF calculation using 0.7 and the experiment is excellent, and the difference in charge density $\Delta \rho(r) = \rho_{206} - \rho_{205}$ is attributed to a difference in the proton occupation numbers between the two nuclei of $z = n(206) - n(205) = 0.7 \pm 0.1$

Unfortunately, the size of the $3s_{1/2}$ orbital, such as the root-mean-square (RMS) radius of the orbital, could not be extracted with precision from this charge density difference because of the need for correction for core polarization effects, i.e., the increase in overall size of the $^{205}$Tl core due to addition of one nucleon.
Fig. 2.2. Experimental cross section ratio between $^{205}$Tl and $^{206}$Pb. The solid curve represents the HF calculation performed with a 3s hole strength of 0.7, and the dashed line with a hole strength of 1.0.
Fig. 2.3. Experimental charge densities of $^{205}\text{Pb}$ and $^{206}\text{Pb}$. 
Elastic scattering of electrons on a nucleus gives us a handle on the charge distribution (protons), but magnetic electron scattering opens a possibility to look also at neutrons in the nuclei. The individual proton and neutron spins and angular momenta pair off in a nucleus, hence the total nuclear magnetization typically comes from the last valence nucleon. Thus, the magnetic moments of the nuclear ground states are generally determined by the valence nucleons, and hence one can directly examine the properties of these valence nucleons. Since neutrons possess a small intrinsic magnetic moment, they will also contribute to elastic magnetic scattering. By measuring the scattered electrons’ diffraction pattern to high values of momentum transfer, one has the capability of examining the spatial distribution of the last valence particle.

Elastic magnetic scattering from the highest order magnetic multipole in the high momentum transfer region can be interpreted experimentally in terms of the radial wave function of the valence nucleon with negligible uncertainty due to configuration mixing. Such a measurement, because of its sensitivity to the inner region of the valence wave function, yields the RMS radius of the valence nucleon orbit to 1% accuracy for both protons and neutrons.

Single-nucleon transfer reactions using hadronic probes are also often used to measure the wave function of the transferred nucleon. The main disadvantage with hadronic probes is that they are strongly absorbed in nuclei, so these methods are primarily sensitive to the tail region of the wave function. Therefore, any sizes of valence orbits determined from transfer reactions suffer from larger uncertainties.
On the other hand, magnetic electron-scattering experiments are quite sensitive to the shape of the valence proton and neutron valence wave function throughout the nuclear volume. As such, the size of the valence nucleon orbits can be determined precisely from magnetic electron scattering and these values can then be used in the study of transfer reactions in order to deduce the single-particle strengths to a greater accuracy.

The theory of magnetic electron scattering has been reviewed in Ref. [Do 75, Do 84] and has been discussed in detail by Platchkov et al. [Pl 82]. In this section the notations used by Platchkov et al. are followed for a brief summary of the theory.

The cross section for elastic scattering of electrons by a target is given by Eq. 2.5. In this equation the separation of transverse and longitudinal form factors can be performed with an appropriate combination of data at far backward and forward scattering angles at the same momentum transfer. In elastic scattering the transverse form factor $F_T(q)$ is exclusively of magnetic character and is generated by the current and magnetization density distribution of the nuclear ground state.

The squared form factor for scattering from a target with spin $J_0$ is written [Pl 82] as an incoherent sum over odd multipoles $J$ as:

$$F_T^2(q) = \sum_{J=1}^{2J_0} F_M^2 J(q)$$

$$= \frac{4\pi}{2J_0 + 1} \sum_{J=1}^{2J_0} | \langle \psi_A | \hat{T}_J^m(q) | \psi_A \rangle |^2$$

(2.17)

where $\hat{T}_J^m(q)$ is the many-body matrix element of the magnetic current operator.
and can be expanded in a complete basis of orbital states as

$$< \psi_A | \hat{T}^m_J (q) | \psi_A > = \sum_{\alpha, \beta} \rho^J_{\alpha, \beta} < \alpha | \hat{T}^m_J (q) | \beta > .$$  \hspace{1cm} (2.18)

In this expression $\rho^J_{\alpha, \beta}$ is the one-body density matrix with notation $\alpha = (n_{\alpha})$ used to indicate the quantum numbers of the state $|\alpha >$. In the independent particle shell model without the residual interactions, the current density is determined exclusively by the odd-nucleon wave function $|\alpha >$ i.e. only $\rho^J_{\alpha_0, \alpha_0} \neq 0$.

The dominant portion of $F_T$ comes largely from the highest magnetic multipole $F_{M\Lambda}$. Thus $F_{M\Lambda}$ can be measured with only minor interference from the lower multipole components. Since scattering of multipolarity $\Lambda = 2J_0$ corresponds to a complete spin-flip of the nucleus, shells with $J$ values $J < J_0$ cannot contribute. The highest multipolarity receives no contribution from the convection current. Ignoring admixtures with $J > J_0$, one can write

$$F_{M\Lambda}(q) = \left[ \frac{4\pi}{2J_0 + 1} \right]^{\frac{1}{2}} \rho^{\Lambda}_{\alpha_0, \alpha_0} < \alpha_0 | \hat{T}^m_{\Lambda} (q) | \alpha_0 > .$$  \hspace{1cm} (2.19)

The $q$ dependence of the form factor $F_{M\Lambda}$ lies in $|\alpha_0 >$. Further, this form factor is related to the radial wave function $R_{\alpha_0}(r)$ by

$$F_{M\Lambda}(q) = C_{\Lambda} \alpha_\Lambda \mu \frac{q}{M} \int_0^\infty R_{\alpha_0}(r) j_{\Lambda-1} (qr) r^2 \, dr.$$  \hspace{1cm} (2.20)

For the $1g_{9/2}$ neutron of $^{87}\text{Sr}$ the M9 form factor is

$$F_{M9}(q) = \left[ \frac{3 \times 7}{(11 \times 13 \times 17/5)} \right]^{\frac{1}{2}} \frac{q}{M} \int_0^\infty R_{\alpha_0}^2 (r) j_8 (qr) r^2 \, dr,$$  \hspace{1cm} (2.21)

where $\alpha_\phi$ is the ratio of the many-body to single particle matrix element of the M9 operator, and $M_N$ and $\mu$ are the nuclear mass and magnetic moment.
Table 2.1. Results of three-parameters fit to the experimental form factors along with the RMS radii of valence orbitals.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Configuration</th>
<th>$-V_0$ (MeV)</th>
<th>$r_0$ (fm)</th>
<th>$&lt;r^2&gt;^{1/2}$ (fm) (No MEC)</th>
<th>$&lt;r^2&gt;^{1/2}$ (fm) (with MEC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{49}$Ti</td>
<td>$1f_{7/2}^n$</td>
<td>52.85(53)</td>
<td>1.170(7)</td>
<td>3.986(52)</td>
<td>4.042(56)</td>
</tr>
<tr>
<td>$^{81}$V</td>
<td>$1f_{7/2}^p$</td>
<td>58.97(51)</td>
<td>1.232(6)</td>
<td>4.007(48)</td>
<td>4.063(53)</td>
</tr>
<tr>
<td>$^{87}$Sr</td>
<td>$1g_{9/2}^n$</td>
<td>50.00(99)</td>
<td>1.201(18)</td>
<td>4.756(71)</td>
<td>4.823(76)</td>
</tr>
<tr>
<td>$^{93}$Nb</td>
<td>$1g_{9/2}^p$</td>
<td>56.82(36)</td>
<td>1.265(8)</td>
<td>4.897(59)</td>
<td>4.946(64)</td>
</tr>
</tbody>
</table>

The radial dependence of the magnetization density $R^2(r)$ for a typical $1g_{9/2}$ neutron wave function of $^{87}$Sr, together with the integrand of Eq. (2.21) for two typical values of $q$, is shown in Fig. 2.5. The angular dependence which is dictated by the corresponding spherical harmonic [Si 80] is shown in Fig. 2.6 for a $2^9$-pole.

The data shown in Fig. 2.7 have been fitted using phenomenological radial wave functions computed from a Woods-Saxon potential with $a=0.65$ fm, $V_{so}=7.0$ MeV and varying the radius of the well $r_0$ with a fixed separation energy for the ground state. The value of the surface thickness $a=0.65$ fm is determined by fitting charge form factors by densities computed from Woods-Saxon wells and $V_{so}=7.0$ MeV gives the correct prediction for the $g_{7/2}$ and $g_{9/2}$ spin-orbit splitting. The resulting parameters taken from Tables X and XV of Ref. [Pl 82] are listed in Table 2.1.

The RMS radius of the $1g_{9/2}$ neutron orbital in $^{87}$Sr thus determined is 4.756 fm. This radius is subsequently corrected for meson exchange currents (MEC), which increases the RMS radius by 1.4% and leads to a value of 4.823
fm. These RMS radii, thus determined with an accuracy of 1% can be used in the analyses of transfer reaction; this aspect is also reviewed in Ref. [Pl 82].
Fig. 2.5. Squared $1g_0/2$ radial wave function of $^{87}\text{Sr}$ (solid curve); dotted curves represent integrand of Eq. 2.21 for two momentum transfers.
Fig. 2.6. Angular shape of the $2^9$ pole (Pentakosisadodekapole), magnetization density.
Fig. 2.7. The $^{87}$Sr data of Platchkov et al. [Pl 82]. Solid curve represents the fit obtained using Woods-Saxon radial wave function; the dashed curves represent the M7 and M9 contributions.