Chapter 3
Parameter Estimation of Unstable System without Augmentation

3.1 Introduction

The output error method (OEM) is the most widely used technique for parameter estimation of dynamical systems [6]. Literature abounds in its application to parameter estimation of stable systems [6-7]. However, due to the problem of numerical divergence associated with the integration of state equations during parameter estimation using the OEM, identification of unstable systems is very difficult if not impossible [18]. Also, in highly unstable systems, the output data could grow so rapidly that the identification procedure may not be able to generate useful results. To overcome this problem, (i) limited data records could be used or (ii) the unstable system could be stabilised by using a suitable control system and then the open loop characteristics could be identified from the closed loop data. The problem with the first approach, due to the use of limited time records, is that, the identification result will be unbiased only when the system is noise free. The second approach is investigated in Chapter 6. There are other approaches like equation error method [32] for parameter estimation of unstable systems. Some of these are investigated in Chapter 5.

In practice, parameter estimation of unstable systems is necessary in various situations. In applications of adaptive control, it might so happen that the system becomes unstable due to sensor failures (feedback signals) and large dynamic changes in the system. This situation could occur in case of satellite launch vehicles or unstable aircraft operating in closed loop. Under these conditions,
analysis of the data would give clues to the cause of the failure. This knowledge can be utilised for reconfiguration of control laws or systems.

In this Chapter, two approaches for parameter estimation of unstable system without augmentation are presented. One is based on UD factorisation Kalman filtering approach (applicable to linear as well as nonlinear systems) [25] and the other is a new approach based on eigen transformation applicable to linear continuous time systems.

In the first approach, the UD factorisation filtering algorithm is used since it can handle process as well as measurement noise further. UD factorisation has certain advantages: triangular structure of matrices, which avoids time-consuming square rooting operations, numerical reliability, stability and accuracy [26]. The validation of this algorithm for unstable systems is presented using simulated data of a second order dynamical system with varying degrees of instability.

In the second approach, a technique of transformation of input/output data of a continuous time unstable system is presented, so that the conventional parameter estimation methods, like an OEM could be utilised. The technique is applicable to linear continuous time systems. In [33], a similar method has been applied to discrete systems, but it was demonstrated only for transfer function identification. However in many practical cases, we require to estimate the parameters of the state space models of continuous time systems.

An objective criterion for choosing the linear transformation parameter, based on the real part of the largest unstable eigenvalue of the system is proposed. The numerical divergence problem associated with the identification of unstable
A general nonlinear system can be described as follows:

\[
\dot{x}(t) = f(x,u,\Theta) + Gw(t); x(0) = x_0. 
\tag{3.1}
\]

\[
y(t) = h(x,u,\Theta) 
\tag{3.2}
\]

\[
y_m(k) = y(k) + v(k), k = 1,2,\ldots N 
\tag{3.3}
\]

where \( \Theta \) is the unknown parameter vector to be estimated, \( w \) and \( v \) represent the process and measurement noise vectors, and \( y_m \) represents the discrete measurements. Extended Kalman filter is a sub-optimal solution to a nonlinear filtering problem. The nonlinear functions are linearised about each new estimated/filtered state trajectory as soon as it becomes available. Simultaneous estimation of states and parameters is achieved by augmenting the state vector with the unknown parameters and applying the filtering algorithm to the augmented nonlinear model.

The new augmented state vector is

\[
x_a = [x^T \ \Theta^T]^T 
\tag{3.4}
\]

\[
\dot{x}_a = \begin{bmatrix} f(x_a,u,t) \\ 0 \end{bmatrix} - \begin{bmatrix} G \\ 0 \end{bmatrix} w(t) - f_u(x_a,u,t) \cdot G_u w(t) 
\tag{3.5}
\]

\[
y(t) = h_u(x_a,u,t) 
\tag{3.6}
\]

\[
y_m(k) = y(k) + u(k) \quad k = 1, \ldots , N 
\tag{3.7}
\]
where \( f_a(t) = [f^T \ 0^T]^T; \ G_a = [G^T \ 0^T]^T \) 

The estimation algorithm is obtained by linearising eqs. (3.5) and (3.6) around the prior/current best estimate of the state at each time and then applying the filtering algorithm to the linearised model. The linearised system matrices are defined as:

\[
A(k) = \frac{\partial f_a}{\partial x_a} \quad \text{at} \quad x_a = x_a(k), u - u(k)
\]

\[
H(k) = \frac{\partial h_a}{\partial x_a} \quad \text{at} \quad x_a = x_a(k), u - u(k)
\]

and the state transition matrix is defined as

\[
\Phi(k) = \exp(-A(k)\Delta t) \quad \text{where} \quad \Delta t = t_{k+1} - t_k
\]

3.2.1 UD Factorisation Filtering

In the UD filter, the covariance update formulae and the estimation recursion are reformulated, so that the covariance matrix does not appear explicitly. Specifically, we use recursions for \( U \) and \( D \) factors of covariance matrix \( P = UDU^T \) where \( U \) is unit upper triangular matrix and \( D \) is a diagonal matrix. Computing and updating with triangular matrices involve fewer arithmetic operations and thus greatly reduce the problem of round off errors which might cause ill-conditioning and subsequent divergence of the algorithm. The filter algorithm is given in two parts:

**Time Update:**

We have for the covariance update

\[
\hat{P}(k+1 \mid k) = \Phi \hat{P}(k) \Phi^T + G_a Q G_a^T
\]
Given $\hat{P} = \hat{U} \hat{D} \hat{U}^T$ and $Q$ as the process noise covariance matrix, the time update factors $\hat{U}$ and $\hat{D}$ are obtained through modified Gram Schmidt orthogonalisation process.

We may define $W = [\Phi \hat{U} | G_a]$ and $D = \text{diag}(\hat{D}, Q)$ with $W^T = [w_1, w_2, ..., w_n]$. $P$ is then reformulated as $\tilde{P} = \tilde{W} \tilde{D} \tilde{W}^T$. The $U$ and $D$ factors of $\tilde{W} \tilde{D} \tilde{W}^T$ may be computed as described below.

For $j=n, ..., 1$ the following equations are recursively evaluated.

$$\tilde{D} = \langle w_j, w_j \rangle_D$$

$$\tilde{U}_{ij} = (1 / \tilde{D}_j) \langle w_i, w_i \rangle_D, \quad i = 1, 2, ..., j-1$$

$$w_i = w_i - \tilde{U}_{ij} w_j$$

where $\langle w_i, w_j \rangle_D = w_i^T D w_j$ is the weighted inner product between $w_i$ and $w_j$.

**Measurement Update**

The measurement update in Kalman filtering combines a priori estimate $\tilde{x}$ and error covariance $\tilde{P}$ with a scalar observation $z = a^T \tilde{x} + v$ to construct an updated estimate and covariance given as:

$$K = \frac{\hat{P} a}{a}$$

$$\hat{x} = \tilde{x} + K(z - a^T \tilde{x})$$

$$\alpha = a^T \hat{P} a + r$$

$$\hat{P} = \hat{P} - K a \tilde{P}$$

where $\tilde{P} = \tilde{U} \tilde{D} \tilde{U}^T$, $a$ = measurement matrix, $r$ is the measurement noise covariance and $z$ = noisy measurements.
Kalman gain $K$ and updated covariance factors $\hat{U}$ and $\hat{D}$ can be obtained from the following equations:

$$f = \bar{U}^T a; \quad f^T = (f_1, ..., f_n)$$
$$v = \bar{D} f;$$
$$\hat{d}_j = d_j / \alpha_j, \quad \alpha_j = r + v_j f_j$$

(3.16)

For $j = 2, ..., n$ recursively the following equations are evaluated:

$$\alpha_j = \alpha_{j-1} + v_j f_j$$
$$d_j = d_j \alpha_{j-1} / \alpha_j$$
$$u_j = u_j + \lambda_j k_j, \quad \lambda_j = -f_j / \alpha_{j-1}$$
$$K_{j+1} = K_j + v_j f_j$$

where $U = [u_1, ..., u_n]$ and Kalman gain is given by $K = K_{n+1} / \alpha_n$ (3.17)

and $\hat{d}_j$ is the predicted diagonal element, and $\hat{d}_j$ is the updated diagonal element of the $D$ matrix.

As already mentioned the calculation of the matrices $A(k)$ and $H(k)$ for nonlinear system is accomplished by finite difference method. In this method there is no need to make any programming changes when alternative nonlinear models are to be used.

$$A_y = \frac{\partial f_i}{\partial x_j} = \frac{f_i(x_j + \Delta x_j) - f_i(x_j)}{\Delta x_j}$$

(3.18)

for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n; \Delta x_j = $perturbation step size $= \varepsilon x_j$. 
For a small perturbation $\Delta x$ in each of the unknown variables, the perturbed values $f(x_j + \Delta x)$ of each of the unperturbed values $f = x_j$ are computed. The corresponding elements of the matrix $A_{ij}$ are given by finite difference in functions to changes in that parameter. A step size of $\varepsilon = 10^{-5}$ is considered to be adequate.

### 3.2.2 Numerical Validation

In order to study the applicability of the UD method to handle unstable data, a second order system with varying degrees of instability is simulated.

**Example 3.2.1:** The simulated second order plant has the following structure

$$
\begin{align*}
x_1 &= \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\
x_2 &= \begin{bmatrix} a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{align*}
$$

(3.19)

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

(3.20)

By varying the element $a_{22}$ four data sets are generated. Table 3.1 gives the eigenvalues of the second order plant for the various cases studied. The data is generated by using a doublet signal at the input to the dynamical system (with sampling time=0.1 sec.). For comparison purposes, the same set of data was analysed using OEM. In Fig. 3.1, two of the estimated parameters using UD filter are compared with the true values and those estimated using OEM for the four cases listed in Table 3.1. As the instability increases, despite very close start up values, OEM does not converge to a stable solution and the iterations repeat indefinitely. Also it was noted that as the instability increases, the standard deviations of the estimated parameters increase. The parameters are estimated with
incorrect signs/magnitudes and correlation between the parameter estimates increases rendering inaccurate parameter estimates when OEM is used for analysis. With the UD filter, the parameter convergence is good although with increasing instability closer initial parameter estimates are required to ensure convergence and the results are closer to the true values. The UD filter thus gives improved results even for unstable systems because of the inherent stabilisation present in the filter. As is clear from eq. (3.15), a feedback proportional to the fit error updates the state variables. This feedback numerically stabilises the filter algorithm and improves the convergence of the estimation algorithm. In subsequent Chapters of the thesis, Kalman UD filter is used as a basic algorithm for future studies: Recursive Mixed Estimation and Two Step Bootstrap Method.

### 3.3 Eigenvalue Transformation Method

In this section, theory of the transformation applicable for continuous time linear system is dealt with. The aim is to transform the linear unstable system to a stable one. Essentially the math model is transformed. The general continuous time linear system description is given by:

\[
\dot{x} = Ax + Bu \quad \text{with } x(0) = x_0 \\
y = Hx
\]  

(3.21) \hspace{1cm} (3.22)

Assume that a suitable parameter \( p \) is defined. Then, the states, input and output are transformed as:

\[
x(t) = \tilde{x}(t)e^{\alpha t} \quad ; \quad x(0) = \tilde{x}(0); \\
y(t) = \tilde{y}(t)e^{\alpha t} \quad ; \quad u(t) = \tilde{u}(t)e^{\alpha t}
\]

(3.23) \hspace{1cm} (3.24)

where \( ' \sim ' \) denotes the transformed variables.

From eq. (3.23), we have
\[ \dot{x}(t) = x(t)e^{\alpha t} + \rho e^{\alpha t} \tilde{x}(t) \]  
(3.25)

Substituting eqs. (3.23)-(3.25) in eq. (3.21), (3.22) we have

\[ \dot{x}(t)e^{\alpha t} + \rho e^{\alpha t} \tilde{x}(t) = A \tilde{x}(t)e^{\alpha t} + Bu(t)e^{\alpha t} \]  
(3.26)

Since \( e^{\alpha t} \) is a scalar, eliminating it from eq. (3.26), we get

\[ \dot{x}(t) + \rho \tilde{x}(t) = A \tilde{x}(t) + Bu(t) \]  
(3.27)

which on simplification yields

\[ \dot{x}(t) = (A - I \rho) \tilde{x}(t) + Bu(t) \]  
(3.28)

with \( \tilde{y} = H \tilde{x} \)  
(3.29)

It can be observed from eqs. (3.28) and (3.29) that the new system is in terms of the transformed data. Also, the eigenvalues of the new system are altered. If the scalar parameter is chosen suitably, then the transformed matrix \((A - I \rho)\) can have stable eigenvalues. In the next section, a procedure for parameter estimation using the transformed data and OEM is presented.

### 3.3.1 Parameter Estimation Procedure

As shown in the previous section, first a set of transformed data is obtained using eqs. (3.23) and (3.24) as follows:

\[ \tilde{x}(t) = x(t)e^{-\alpha t} \]  
(3.30)

\[ \tilde{y}(t) = y(t)e^{-\alpha t} \]  
(3.31)

\[ \tilde{u}(t) = u(t)e^{-\alpha t} \]  
(3.32)

The measurements can be expressed as

\[ \tilde{z}(k) = \tilde{y}(k) + \tilde{v}(k), \quad k=1,2...,N \]  
(3.33)
\( \nu \) is the measurement noise, with zero mean and covariance matrix \( R \). The parameter vector to be estimated is given by \( \Theta = \{ A, B, H \} \). To estimate the parameters \( \Theta \), the cost function to be minimised is defined as:

\[
E(\Theta) = \frac{1}{2} \sum_{k=1}^{N} [\hat{z}(k) - \hat{y}(k)]^T R^{-1} [\hat{z}(k) - \hat{y}(k)] + \frac{N}{2} \ln |R| \tag{3.34}
\]

Minimisation of the above cost function w.r.t. \( \Theta \) yields the maximum likelihood estimates of \( \Theta^* \):

\[
\hat{\Theta}_{l+1} = \hat{\Theta}_l + \mu \Delta \Theta_l
\tag{3.35}
\]

\[
\Delta \Theta_l = \left\{ \sum_k \left( \frac{\partial y(k)}{\partial \Theta} \right)^T R^{-1} \left( \frac{\partial y(k)}{\partial \Theta} \right) \right\}^{-1} \left\{ \sum_k \left( \frac{\partial y(k)}{\partial \Theta} \right)^T R^{-1} [z_m(k) - y(k)] \right\}
\tag{3.36}
\]

The first bracketed term in eq. (3.36) is the Gauss Newton approximation to the second gradient of the cost function \( E(\Theta) \) and is called the information matrix. Eq. (3.35) in terms of the first and second gradients can be written as

\[
\hat{\Theta}_{l+1} = \hat{\Theta}_l + \nabla^2 E(\Theta) \nabla E(\Theta)
\tag{3.37}
\]

Here, \( l \) stands for iteration number. The constant \( \mu \) is called the damping factor which can be used to improve the convergence of the algorithm. Thus, to compute the first and second gradients, we need to compute the term \( \frac{\partial y(k)}{\partial \Theta} \). This term is called the sensitivity matrix and is obtained by finite difference method. For aircraft parameter estimation the OEM is the most widely used estimator since it has many desirable statistical properties [1,2]. We see that the estimator of eq. (3.35), on convergence will give the parameters of the transformed systems of eqs. (3.28) and (3.29). Then the parameter estimates of the original system can be retrieved as:

\[
A = \tilde{A} + l \rho
\tag{3.38}
\]
since matrices $B$ and $H$ remain unaffected. The transformation scalar $p$ may be taken as the real part of the largest unstable eigenvalue of the system. Generally this information is available from the design considerations of the control system or some apriori information.

3.3.2 Numerical Validation

The approach presented in the previous section is now illustrated using the numerical simulation data.

Example 3.5.1: (Same as Example 3.2.1). The input/output data generated using eqs. (3.19) and (3.20), as in example 3.2.1, are then transformed by using eqs. (3.31) and (3.32). Table 3.1 lists the eigenvalues of the transformed system, which is now a neutrally stable system.

This data is then used to estimate six parameters of the eqs. (3.19) and (3.20) using OEM. Tables 3.2 and 3.3 give the results using the original and transformed data (for SNR= $\infty$) respectively. From Table 3.2 it is clear that OEM can handle unstable data when the instability is less pronounced (as in cases 1-3). For the case 4, the parameters, especially the control related ones, $b_1$ and $b_2$ are estimated with wrong sign and/or large variances. The algorithm's iterations repeat many times before convergence is reached. The results in Table 3.3 indicate that the transformation effectively avoids the problem of numerical divergence. Fig. 3.2 shows the time history match for (a) the original data and (b) the transformed data. The figure also shows the original and transformed doublet input. The time history match which is a necessary condition for convergence is very good. It is clear from the values of PEEN that when the instability is large, the estimation error is very
large with the unstable data but with the transformed data, the error reduces to a minimum.

To study the effectiveness of the transformation to retrieve the correct estimates in the presence of measurement noise, parameter estimation is performed with data with \( \text{SNR}=10 \). Tables 3.4 and 3.5 give the results for these data with and without transformation. and Fig. 3.3 gives the time history match. With the unstable data, the results of parameter estimation show deviations even in case 3 and the results for case 4 are worse as is evident from a comparison of PEENs in Tables 3.4 and 3.5. However, even in the presence of noise, it is clear that the transformation method can be used to estimate the parameters of the unstable system. The parameter estimation error norm shows a remarkable reduction for the case with maximum instability.

**Example 3.5.2:** Next, fourth order unstable system is simulated. Table 3.6 gives the eigenvalues of the unstable and transformed systems. Table 3.7 gives the parameter estimates and Fig. 3.4 shows the time history match. Although the time history match for the unstable data is very good, the parameter estimates deviate from the true values and the standard deviation of the parameters are very high. The results using transformed data is very good indicating the effectiveness of the eigenvalue transformation method to estimate parameters of unstable systems directly using an OEM.

For the present, the value of \( \rho \) is taken as the numerical value of the real part of the largest unstable eigenvalue. However, in practice while handling real data, the value of \( \rho \) can be obtained based on apriori information of the system. Alternatively, it is possible to obtain an approximate value of \( \rho \) by determining the
slope from successive values of the peaks of the oscillatory data. This will suffice, since it gives the positive trend of the data which is numerically growing as the time elapses. The transformation then effectively tries to detrend the data which become suitable to be used in OEM.

In this Chapter, two approaches to parameter estimation of unstable/unaugmented system are presented: (i) UD filter based approach and (ii) a practical and simple approach using transformed data and an output error method for a continuous time dynamical system. The methods are validated using second order and fourth order unstable/unaugmented systems. The methods are found to give satisfactory results.

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Eigenvalues (unstable system)</th>
<th>Eigenvalues (transformed system)</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0253 ± j 1.4140</td>
<td>0.00 ± j 1.4140</td>
</tr>
<tr>
<td>2</td>
<td>0.2250 ± j 1.4043</td>
<td>0.00 ± j 1.4043</td>
</tr>
<tr>
<td>3</td>
<td>0.6250 ± j 1.2920</td>
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</tr>
<tr>
<td>4</td>
<td>1.0250 ± j 1.2956</td>
<td>0.00± j 1.0244</td>
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Table 3.2 Parameter estimates (unstable data, Example 3.5.1, 2nd order system, SNR=∞)

<table>
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<tr>
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<tbody>
<tr>
<td>a11</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
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<td></td>
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<td>-1.00</td>
<td>-1.057</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>b1</td>
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<td>-0.4</td>
<td>-0.400</td>
<td>-0.4</td>
<td>-0.399</td>
<td>-0.4</td>
<td>-8.770</td>
<td></td>
<td>-20.99</td>
<td>(22.7)</td>
</tr>
<tr>
<td>b2</td>
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<td>1.0</td>
<td>1.0004</td>
<td>1.0</td>
<td>1.0004</td>
<td>1.0</td>
<td>-0.175</td>
<td></td>
<td>-0.575</td>
<td>(19.53)</td>
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<td>PEEN %</td>
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<td>0.0236</td>
<td>-</td>
<td>0.3982</td>
<td>-</td>
<td>0.1817</td>
<td>-</td>
<td>738.00</td>
<td></td>
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</table>

(\%) % standard deviations of the estimated parameters

Table 3.3 Parameter estimates (transformed data, Example 3.5.1, 2nd order system, SNR=∞)

<table>
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<td></td>
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<td>-0.999</td>
</tr>
<tr>
<td>a12</td>
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<td>-0.999</td>
<td>-1.0</td>
<td>-0.999</td>
<td>-1.00</td>
<td>-0.999</td>
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<td>-0.999</td>
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<td>2.0003</td>
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<td>1.9999</td>
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<td>0.175</td>
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<td>0.975</td>
<td>0.975</td>
<td></td>
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</tr>
<tr>
<td>b1</td>
<td>-0.4</td>
<td>-0.400</td>
<td>-0.4</td>
<td>-0.400</td>
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<td>-0.400</td>
<td>-0.4</td>
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<td></td>
</tr>
<tr>
<td>b2</td>
<td>1.0</td>
<td>1.0004</td>
<td>1.0</td>
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<tr>
<td>PEEN %</td>
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<td>-</td>
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<td>-</td>
<td>0.0190</td>
<td></td>
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Table 3.4 Parameter estimates (unstable data, Example 3.5.1, 2nd order system, SNR=10)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Case 1 True</th>
<th>Case 1 Estm.</th>
<th>Case 2 True</th>
<th>Case 2 Estm.</th>
<th>Case 3 True</th>
<th>Case 3 Estm.</th>
<th>Case 4 True</th>
<th>Case 4 Estm.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a11</td>
<td>0.05</td>
<td>0.067 (69.5)</td>
<td>0.05</td>
<td>0.0956 (50.4)</td>
<td>0.05</td>
<td>0.108 (63.4)</td>
<td>0.05</td>
<td>-0.193 (119.3)</td>
</tr>
<tr>
<td>a12</td>
<td>-1.0</td>
<td>-0.987 (3.33)</td>
<td>-1.0</td>
<td>-0.985 (3.24)</td>
<td>-1.00</td>
<td>-1.042 (3.54)</td>
<td>-1.00</td>
<td>-1.111 (9.75)</td>
</tr>
<tr>
<td>a21</td>
<td>2.0</td>
<td>2.029 (3.27)</td>
<td>2.0</td>
<td>2.003 (3.29)</td>
<td>2.0</td>
<td>1.845 (4.92)</td>
<td>2.0</td>
<td>1.907 (19.3)</td>
</tr>
<tr>
<td>a22</td>
<td>0.0005</td>
<td>0.030 (157.3)</td>
<td>0.4</td>
<td>0.383 (12.4)</td>
<td>1.2</td>
<td>1.158 (5.33)</td>
<td>2.0</td>
<td>2.047 (9.94)</td>
</tr>
<tr>
<td>b1</td>
<td>-0.4</td>
<td>-0.375 (12.6)</td>
<td>-0.4</td>
<td>-0.517 (27.72)</td>
<td>-0.4</td>
<td>3.453 (76.64)</td>
<td>-0.4</td>
<td>-5.138 (2225)</td>
</tr>
<tr>
<td>b2</td>
<td>1.0</td>
<td>1.0051 (6.45)</td>
<td>1.0</td>
<td>0.9403 (20.19)</td>
<td>1.0</td>
<td>-2.469 (201.3)</td>
<td>1.0</td>
<td>196.9 (107.2)</td>
</tr>
<tr>
<td>PEEN%</td>
<td>-</td>
<td>2.1589 -</td>
<td>5.6120 -</td>
<td></td>
<td>188.14 -</td>
<td></td>
<td>6140.0 -</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.5 Parameter estimates (transformed data, Example 3.5.1, 2nd order system, SNR=10)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Case 1 True</th>
<th>Case 1 Estm.</th>
<th>Case 2 True</th>
<th>Case 2 Estm.</th>
<th>Case 3 True</th>
<th>Case 3 Estm.</th>
<th>Case 4 True</th>
<th>Case 4 Estm.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a11</td>
<td>0.0247</td>
<td>-0.009 (470.2)</td>
<td>-0.175</td>
<td>-0.069 (65.91)</td>
<td>-0.575</td>
<td>-0.558 (8.94)</td>
<td>-0.975</td>
<td>-0.967 (6.55)</td>
</tr>
<tr>
<td>a12</td>
<td>-1.0</td>
<td>-1.05 (3.18)</td>
<td>-1.0</td>
<td>-0.982 (3.23)</td>
<td>-1.00</td>
<td>-1.019 (3.57)</td>
<td>-1.00</td>
<td>-1.029 (4.27)</td>
</tr>
<tr>
<td>a21</td>
<td>2.0</td>
<td>1.921 (3.12)</td>
<td>2.0</td>
<td>2.013 (3.20)</td>
<td>2.0</td>
<td>1.916 (3.54)</td>
<td>2.0</td>
<td>1.927 (4.44)</td>
</tr>
<tr>
<td>a22</td>
<td>-0.025</td>
<td>-0.002 (1946)</td>
<td>0.175</td>
<td>0.0744 (61.2)</td>
<td>0.575</td>
<td>0.575 (8.82)</td>
<td>0.975</td>
<td>0.968 (6.55)</td>
</tr>
<tr>
<td>b1</td>
<td>-0.4</td>
<td>-0.376 (11.26)</td>
<td>-0.4</td>
<td>-0.410 (10.02)</td>
<td>-0.4</td>
<td>-0.436 (8.77)</td>
<td>-0.4</td>
<td>-0.42 (8.99)</td>
</tr>
<tr>
<td>b2</td>
<td>1.0</td>
<td>1.009 (5.77)</td>
<td>1.0</td>
<td>0.985 (5.85)</td>
<td>1.0</td>
<td>0.967 (5.99)</td>
<td>1.0</td>
<td>0.975 (6.27)</td>
</tr>
<tr>
<td>PEEN%</td>
<td>-</td>
<td>4.2364 -</td>
<td>5.9824 -</td>
<td></td>
<td>3.8591 -</td>
<td></td>
<td>3.0234 -</td>
<td></td>
</tr>
</tbody>
</table>
Table 3.6 Eigenvalues of the unstable 4th order system (Example 3.5.2)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Eigenvalues (unstable system)</th>
<th>Eigenvalues (transformed system)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>-1.6112</td>
<td>-2.3489</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>0.7337</td>
<td>0.000</td>
</tr>
<tr>
<td>$\lambda_{3,4}$</td>
<td>-0.0309± j(0.1433)</td>
<td>-0.767± j(0.1433)</td>
</tr>
</tbody>
</table>

Table 3.7 Parameter estimates, (Example 3.5.2, 4th order system, SNR=∞)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unstable system data</th>
<th>Transformed data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>True</td>
<td>Estimated</td>
</tr>
<tr>
<td>$a_{11}$</td>
<td>-0.4193</td>
<td>-0.2002 (687.9)</td>
</tr>
<tr>
<td>$a_{14}$</td>
<td>-1.1955</td>
<td>0.5017 (856.5)</td>
</tr>
<tr>
<td>$a_{21}$</td>
<td>1.3316</td>
<td>1.4376 (1078.9)</td>
</tr>
<tr>
<td>$a_{22}$</td>
<td>-0.4717</td>
<td>-0.5025 (239.0)</td>
</tr>
<tr>
<td>$a_{24}$</td>
<td>0.0226</td>
<td>0.2537 (374.0)</td>
</tr>
<tr>
<td>$a_{41}$</td>
<td>-0.0965</td>
<td>0.1933 (2756.0)</td>
</tr>
<tr>
<td>$a_{43}$</td>
<td>-0.0443</td>
<td>-0.9244 (84.0)</td>
</tr>
<tr>
<td>$a_{44}$</td>
<td>-0.1018</td>
<td>-0.4840 (2001.0)</td>
</tr>
<tr>
<td>$b_{11}$</td>
<td>-0.1500</td>
<td>-0.2615 (1068.0)</td>
</tr>
<tr>
<td>$b_{21}$</td>
<td>-4.8200</td>
<td>-4.9296 (1340.0)</td>
</tr>
<tr>
<td>$b_{41}$</td>
<td>-0.0429</td>
<td>0.0358 (2833.0)</td>
</tr>
<tr>
<td>PEEN%</td>
<td>-</td>
<td>25.3270</td>
</tr>
</tbody>
</table>
Fig. 3.1 Parameter estimates, (Example 3.2.1, 2nd order system)

Fig 3.2 Time history match (Example 3.5.1, 2nd order system, SNR=∞)
Fig 3.2 Time history match (Example 3.5.1, 2\textsuperscript{nd} order system, SNR=\infty)

(a) Unstable data

(b) Transformed data

Fig 3.3 Time history match (Example 3.5.1, 2\textsuperscript{nd} order system, SNR=10)
Fig 3.4 Time history match (Example 3.5.2, 4\textsuperscript{th} order system, SNR=\infty)