Chapter 4

Plane waves in micropolar porous elastic solid

4.1 Introduction

Eringen and his co-worker (1964a, 1964b) developed a non-linear theory of microelastic solids. Later, Eringen (1966/1968) developed a linear theory of micropolar elasticity, which is a subclass of the theory of microelastic solids. A homogeneous isotropic micropolar elastic material is a material characterized by a continuum in which the rigid grains are of dumb-bell shaped (infinitesimal in size) distributed uniformly throughout the continuum. The basic difference between the Eringen’s theory of micropolar elasticity and that of the classical elasticity is the introduction of an independent microrotation vector. In classical elasticity, the motion is described by a displacement vector only, while in micropolar elasticity, it is described not only by a displacement vector but also by a microrotation vector. Thus, in micropolar elasticity, the motion is completely described by six degrees of freedom (three of translation and three of rotation). The force at a point of a surface element of a micropolar material is completely known by a force stress vector and by a couple stress vector at that point. Parfitt and Eringen (1969) have shown that there can exist four plane waves in an infinite micropolar elastic material, two of which disappear below a critical frequency.

In this chapter, possibility of plane wave propagation in an infinite micropolar porous elastic medium is explored. Reflection of plane longitudinal displacement and coupled transverse shear and microrotational waves (called a set of “Coupled waves”)

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impinging against a plane stress-free boundary of a micropolar porous elastic half-space have been studied. It is found that for the case of limitly low frequency incident wave, the presence of voids have significant effect on amplitudes of reflected waves. For the case of limitly high frequency incident longitudinal wave, the reflection coefficients match with those obtained in micropolar elasticity without voids. The energy partitioning of the incident energy at the boundary surface has been presented and the balance of energy is also verified.

4.2 Field equations and constitutive relations

Following Eringen (1966) and Iesan (1985), the field equations in micropolar elastic material with voids, in the absence of body forces and body couple densities are given by

\[ (\lambda + \mu)\nabla (\nabla \cdot u) + (\mu + K^*)\nabla^2 u + K^* \nabla \times \psi + \beta \nabla \phi = \frac{\partial^2 u}{\partial t^2}, \quad (4.1) \]

\[ (\alpha^* + \beta^*)\nabla (\nabla \cdot \psi) + \gamma^* \nabla^2 \psi + K^* \nabla \times u - 2K^* \psi = \rho J^* \frac{\partial^2 \psi}{\partial t^2}, \quad (4.2) \]

\[ \alpha \nabla^2 \phi - \xi \phi - \omega \frac{\partial \phi}{\partial t} - \beta \nabla \cdot u = \rho K \frac{\partial^2 \phi}{\partial t^2}, \quad (4.3) \]

where \( K^*, \alpha^*, \beta^* \) and \( \gamma^* \) are the micropolar parameters; \( \alpha, \beta, \xi, \omega \) and \( K \) are the void parameters discussed already in Chapter 1; \( u(x, t) \) and \( \psi(x, t) \) are the displacement and microrotation vectors, respectively; \( \phi \) is the change in void volume fraction; \( J^* \) is the micro-inertia and \( \rho \) is the density of the medium.

The constitutive relations for the micropolar porous medium are given by

\[ \tau_{ij} = \lambda u_{k,k} \delta_{ij} + \mu (u_{ij} + u_{sj}) + K^* (u_{j,i} - \epsilon_{ijk} \psi_k) + \delta_{ij} \beta \phi, \quad (4.4) \]

\[ m_{ij} = \alpha^* \psi_{k,k} \delta_{ij} + \beta^* \psi_{i,j} + \gamma^* \psi_{j,i}, \quad (4.5) \]

\[ h_i = \alpha \phi_{,i}, \quad (4.6) \]
where $r_{ij}$, $m_{ij}$ and $h_i$ are the force stress tensor, couple stress tensor and equilibrated force vector, respectively.

For time harmonic plane wave propagation (i.e., $\alpha \exp\{-\beta \Omega t\}$), the equations of motion (4.1)-(4.3) reduce to

$$(c_1^2 + c_3^2) \nabla (\nabla \cdot \mathbf{u}) - (c_2^2 + c_4^2) \nabla \times (\nabla \times \mathbf{u}) + c_3^2 \nabla \times \Psi + \frac{\beta}{\rho} \nabla \phi + \Omega^2 \mathbf{u} = 0, \quad (4.7)$$

$$(c_1^2 + c_3^2) \nabla (\nabla \cdot \Psi) - c_4^2 \nabla \times (\nabla \times \Psi) + \omega_0^2 \nabla \times \mathbf{u} - 2\omega_0^2 \Psi + \Omega \Psi = 0, \quad (4.8)$$

$$(\alpha \nabla^2 - \xi + \omega \Omega + \rho \Omega^2) \phi - \beta \nabla \cdot \mathbf{u} = 0, \quad (4.9)$$

where

$$c_2^2 = \frac{K^*}{\rho}, \quad c_4^2 = \frac{\gamma^*}{\rho J^*}, \quad c_3^2 = \frac{\alpha^* + \beta^*}{\rho J^*}, \quad \omega_0^2 = \frac{K^*}{\rho J^*}. \quad (4.10)$$

By using Helmholtz’s decomposition of vectors, we can write

$$\mathbf{u} = \nabla p + \nabla \times \mathbf{q}, \quad \Psi = \nabla \Upsilon + \nabla \times \mathbf{\Pi}, \quad \nabla \cdot \mathbf{q} = \nabla \cdot \mathbf{\Pi} = 0, \quad (4.11)$$

where $p$ and $\Upsilon$ are scalar potentials and $\mathbf{q}$ and $\mathbf{\Pi}$ are vector potentials. Using (4.11) into equations (4.7)-(4.9), we obtain

$$[(c_1^2 + c_3^2) \nabla^2 + \Omega^2] p + \frac{\beta}{\rho} \phi = 0, \quad (4.12)$$

$$[(c_2^2 + c_4^2) \nabla^2 - 2\omega_0^2] \Upsilon + \Omega^2 \Upsilon = 0, \quad (4.13)$$

$$[(c_4^2 - 2\omega_0^2 + \Omega^2] \mathbf{\Pi} + \omega_0^2 \nabla \times \mathbf{q} = 0, \quad (4.14)$$

$$[(c_2^2 + c_3^2) \nabla^2 + \Omega^2] \mathbf{q} + c_3^2 \nabla \times \mathbf{\Pi} = 0, \quad (4.15)$$

$$(\alpha \nabla^2 - \xi + \omega \Omega + \rho \Omega^2) \phi - \beta \nabla^2 p = 0. \quad (4.16)$$
We note that equations (4.12) and (4.16) are coupled in scalar potentials \( p \) and \( \phi \), equations (4.14) and (4.15) are coupled in vector potentials \( q \) and \( \Pi \), while equation (4.13) is uncoupled in scalar potential \( \Upsilon \).

### 4.3 Plane wave propagation

Plane waves propagating in the positive direction of a unit vector \( n \) may be expressed as

\[
\{p, \phi, q, \Pi\} = \{a, b, c, d\} \exp\{ik(n \cdot r - vt)\},
\]

where \( a \) and \( b \) are scalar constants, while \( c \) and \( d \) are vector constants, \( v \) is the phase speed, \( k \) is the wavenumber and \( r \) is the position vector.

Eliminating \( \phi \) from (4.12) and (4.16) and using (4.17), we obtain

\[
\left( k^2 - \frac{\Omega^2}{c_p^2} \right) \left( k^2 - \frac{\Omega^2}{c_\alpha^2} + \frac{1}{l_2^2} + \frac{i\omega \Omega}{\alpha} \right) - \frac{H^*}{l_1^2} k^2 = 0,
\]

where

\[
c_{p}^* = \frac{\lambda + 2\mu + K^*}{\rho}, \quad H^* = \frac{\beta}{\lambda + 2\mu + K^*},
\]

\( c_p^* \) is the speed of longitudinal displacement wave discussed in detail earlier by Parfitt and Eringen (1969) and \( H^* \) is a dimensionless coupling number similar to that introduced earlier by Puri and Cowin (1985) and reduces to it in the absence of micropolarity. The symbols \( l_1^2 \) and \( l_2^2 \) are defined earlier in Chapter 2. Also, equation (4.18) can be written as

\[
\left( k^2 - \frac{\Omega^2}{c_p^2} \right) \left( k^2 - \frac{\Omega^2}{c_\alpha^2} + \frac{1}{l_2^2} + \frac{i\omega \Omega}{\alpha} \right) - \frac{N^*}{l_1^2} k^2 = 0,
\]

where \( N^* = \frac{l_2^2}{l_1^2} H^* \). It can be seen that the dispersion relation (4.19) does match with relation (15) of Ciarletta and Sumbatyan (2003) in the absence of micropolarity. On substituting the void parameter \( \beta = 0 \), the dispersion relation (4.19) yields \( k^2 = \frac{\Omega^2}{c_p^2} \).

This yields the speed of propagation \( c_p^* \) of the longitudinal displacement wave (see Parfitt and Eringen, 1969).

The general solution of the dispersion relation given by (4.19) is complex valued, but
it admits real valued solutions for limitly high and limitly low frequencies. Rewriting the dispersion relation (4.19) as

$$k^4 - \left( \frac{\Omega^2}{c_3^2} + \frac{\Omega^2}{c_p^2} - \frac{1}{l_2^2} + \frac{N^*}{l_2^2} + \frac{i\omega\Omega}{\alpha} \right) k^2 + \frac{\Omega^2}{c_m^2} \left( \frac{\Omega^2}{c_3^2} - \frac{1}{l_2^2} + \frac{i\omega\Omega}{\alpha} \right) = 0. \quad (4.20)$$

For high frequency ($\Omega l_2 >> 1$), adopting the same procedure as discussed in Chapter 1, we obtain following two roots of equation (4.20) given as

$$k_1 = \frac{\Omega}{c_p}, \quad k_2 = \frac{\Omega}{c_3} + \frac{i\omega c_3}{2\alpha}. \quad (4.21)$$

Similarly for low frequency case ($\Omega l_2 << 1$), we obtain the following two roots of equation (4.20) given as

$$k_1 = \frac{\Omega}{c_p\sqrt{1 - N^*}}, \quad k_2 = \frac{\Omega}{c_4} + \frac{i\sqrt{1 - N^*}}{l_2}. \quad (4.22)$$

From (4.21) and (4.22), we conclude that for high frequency, the wave corresponding to $k_1$ propagates with speed that exists in micropolar elastic medium encountered earlier by Parfitt and Eringen (1969) and designated as longitudinal displacement wave and the wave corresponding to $k_2$ propagates with speed that exists in elastic material with voids encountered earlier by Puri and Cowin (1985) and designated as dilational volume fractional wave. Further, we can also infer from (4.21) and (4.22) that (i) as long as the medium is porous (that is, when $0 < N^* < 1$), the speed of longitudinal displacement wave at low frequency will be less than that of at high frequency, however, in non-porous medium (that is, when $N^* = 0$,) the longitudinal displacement wave travels with the same speed as at low and high frequencies, (ii) the attenuation coefficients of volume fractional wave for limitly high and low frequencies are independent of frequency.

Parfitt and Eringen (1969) have shown that equations (4.14) and (4.15) represent two sets of coupled transverse shear and microrotational waves propagating with speeds $V_{3,4}$ given by

$$V_{3,4}^2 = \frac{1}{2(1 - x)} \left\{ c_3^2 + c_4^2 - (c_2^2 + c_3^2/2)x \right\} \pm \left\{ \left( c_2^2 - c_3^2 - (c_2^2 + c_3^2/2)x \right)^2 + 2\epsilon c_3^2 \epsilon x \right\}^{1/2}, \quad (4.23)$$
where \( x = 2\omega_0^2/\Omega^2 \). Equation (4.13) represents longitudinal microrotational wave propagating with phase speed \( V_5 \) given by

\[
V_5^2 = c_t^2 + c_s^2 + \frac{2\omega_0^2}{k^2}.
\]  

(4.24)

Each set of coupled transverse shear and microrotational waves consists of the transverse displacement wave coupled with transverse microrotational wave. The set of coupled transverse waves with speed \( V_3 \) and longitudinal microrotational wave with speed \( V_5 \) exist only when \( \Omega > \sqrt{2}\omega_0 \), below which they degenerate into distance decaying vibrations.

### 4.4 Problem and its solution

Let us consider a homogeneous and isotropic micropolar porous elastic half-space with \( x \)-axis along the free boundary and \( z \)-axis pointing vertically downwards into the half-space. We shall discuss a two-dimensional problem in \( x-z \) plane by taking the displacement vector \( u \), microrotation vector \( \Pi \) and change in void volume fraction \( \psi \) as follows

\[
u = (u_1, 0, u_3), \quad \Pi = (0, \psi_2, 0), \quad \phi = \phi(x, z).
\]  

(4.25)

#### 4.4.1 Boundary conditions

The appropriate boundary conditions at the free surface are the vanishing of force stresses, couple stresses and equilibrated force. These conditions can be written mathematically as

\[
\tau_{xz} = 0, \quad \tau_{xz} = 0, \quad \frac{\partial \psi_2}{\partial z} = 0, \quad \frac{\partial \phi}{\partial z} = 0.
\]  

(4.26)

Using relations given through (4.4)-(4.6) and (4.11), the above boundary conditions at the free surface \( z = 0 \) can be written as

\[
\frac{\partial^2 p}{\partial x^2} \cdot \frac{\partial^2 q}{\partial z \partial x} + \zeta^2 \psi = 0,
\]  

(4.27)
\[(2\mu + K^*) \frac{\partial^2 p}{\partial x \partial z} - (\mu + K^*) \frac{\partial^2 q}{\partial z^2} + \mu \frac{\partial^2 q}{\partial x^2} - K \psi_2 = 0 \quad (4.28)\]

\[\frac{\partial \psi_2}{\partial z} = 0, \quad (4.29)\]

\[\frac{\partial}{\partial z} (\nabla^2 + \zeta_1^2) p = 0, \quad (4.30)\]

where

\[\zeta_1^2 = \frac{\Omega^2}{c_1^2 + 2c_2^2}, \quad \text{and} \quad \zeta_2^2 = \frac{\Omega^2}{c_1^2 + c_3^2}.\]

### 4.4.2 Incidence of longitudinal displacement wave

Let a unit amplitude plane longitudinal displacement wave propagating with wavenumber \(k_1\) and making an angle \(\theta_1\) with the normal be incident at the free surface \(z = 0\).

This incident longitudinal displacement wave will give rise to:

(i) a reflected longitudinal displacement wave of amplitude \(A_1\) making an angle \(\theta_1\) with the normal.

(ii) a reflected longitudinal volume fractional wave with amplitude \(A_2\) making an angle \(\theta_2\) with the normal.

(iii) a reflected set of coupled transverse shear and microrotational waves of amplitude \(A_3\) propagating with speed \(V_3\) and making an angle \(\theta_3\) with the normal.

(iv) a similar set of reflected coupled transverse shear and microrotational waves of amplitude \(A_4\) propagating with speed \(V_4\) and making an angle \(\theta_4\) with the normal.

The full wave structure of the reflected waves can be written as

\[p = \exp\{i k_1 (x \sin \theta_1 - z \cos \theta_1)\} + \sum_{i=1,2} A_i \exp\{i k_i (x \sin \theta_i + z \cos \theta_i)\}, \quad (4.31)\]

\[q = \sum_{i=3,4} A_i \exp\{i k_i (x \sin \theta_i + z \cos \theta_i)\}, \quad (4.32)\]

\[\psi_2 = \sum_{i=3,4} \eta_i A_i \exp\{i k_i (x \sin \theta_i + z \cos \theta_i)\}, \quad (4.33)\]
where $\eta_{3,4}$ are coupling parameters given by Parfitt and Eringen (1969) and rewritten as follows

$$\eta_{3,4} = \omega_0^2 \left[ V_{A,4}^2 - 2 \frac{\omega_0^2}{k_{A,4}^2} - c_i^2 \right]^{-1}.$$  

The amplitudes $A_1$, $A_2$, $A_3$ and $A_4$ can be determined using the above boundary conditions at the free surface $z = 0$. Substituting the values of potentials $p$, $q$, $q$ from (4.31)-(4.33) into equations (4.27)-(4.30), after employing the Snell’s law given by

$$k_1 \sin \theta_1 = k_2 \sin \theta_2 = k_3 \sin \theta_3 = k_4 \sin \theta_4,$$  

we obtain the following system of four simultaneous non-homogeneous equations as follows

$$\begin{align*}
\sum_{i=1,2} (\zeta_i^2 - k_i^2 \sin^2 \theta_i) A_i + \sum_{i=3,4} k_i^2 \sin \theta_i \cos \theta_i A_i &= - (\zeta_i^2 - k_i^2 \sin^2 \theta_i), \\
\sum_{i=1,2} (2 \mu + K^*) k_i^2 \sin \theta_i \cos \theta_i A_i - \sum_{i=3,4} \left\{ \mu \cos 2 \theta_i + K^* \left( \cos^2 \theta_i - \frac{\eta_i}{k_i^2} \right) \right\} k_i^2 A_i &= (2 \mu + K^*) k_1^2 \sin \theta_1 \cos \theta_1, \\
\sum_{i=1,2} (\zeta_i^2 - k_i^2) k_i \cos \theta_i A_i &= (\zeta_i^2 - k_i^2) k_i \cos \theta_i, \\
\sum_{i=3,4} \eta_i k_i \cos \theta_i A_i &= 0.
\end{align*}$$

These equations enable us to provide the formulae for the reflection coefficients of various reflected waves at the plane free surface $z = 0$.

### 4.4.3 Incidence of a set of coupled transverse waves

Let a set of plane coupled transverse shear and microrotational waves of unit amplitude propagating with speed $V_3$ and making an angle $\theta_3$ with the normal be incident at the free surface $z = 0$. This will give rise to same reflected waves as described in case...
of incident longitudinal displacement wave. The full wave structure of incident and reflected waves can be written as

\[ q = \exp\{i k_3 (x \sin \theta_3 - z \cos \theta_3)\} + \sum_{i=3,4} A_i \exp\{i k_i (x \sin \theta_i + z \cos \theta_i)\}, \quad (4.39) \]

\[ \psi_2 = \eta_3 \exp\{i k_3 (x \sin \theta_3 - z \cos \theta_3)\} + \sum_{i=3,4} \eta_i A_i \exp\{i k_i (x \sin \theta_i + z \cos \theta_i)\}, \quad (4.40) \]

\[ p = \sum_{i=3,4} A_i \exp\{i k_i (x \sin \theta_i + z \cos \theta_i)\}. \quad (4.41) \]

The amplitudes \( A_1, A_2, A_3 \) and \( A_4 \) can be determined using above boundary conditions at the free surface \( z = 0 \) given by (4.26).

Substituting the values of potentials \( p, q \) and \( \psi_2 \) from (4.39)-(4.41) into equations (4.27)-(4.30), we obtain the following system of four simultaneous non-homogenous equations given as follows

\[ \sum_{i=1,2} (\zeta_i^2 - k_i^2 \sin^2 \theta_i) A_i + \sum_{i=3,4} k_i^2 \sin \theta_i \cos \theta_i A_i = k_3^2 \sin \theta_3 \cos \theta_3, \quad (4.42) \]

\[ \sum_{i=1,2} (2\mu + K^*) k_i^2 \sin \theta_i \cos \theta_i A_i - \sum_{i=3,4} \left\{ \mu \cos 2\theta_i + K^* \left( \cos^2 \theta_i - \frac{\eta_i}{k_i^2} \right) \right\} k_i^2 A_i \]

\[ = \left\{ \mu \cos 2\theta_3 + K^* \left( \cos^2 \theta_3 - \frac{\eta_3}{k_3^2} \right) \right\} k_3^2, \quad (4.43) \]

\[ \sum_{i=1,2} (\zeta_i^2 - k_i^2) k_i \cos \theta_i A_i = 0, \quad (4.44) \]

\[ \sum_{i=3,4} \eta_i k_i \cos \theta_i A_i = \eta_3 k_3 \cos \theta_3. \quad (4.45) \]

In case, when a set of coupled transverse and microrotational waves propagating with speed \( V_4 \) and making an angle \( \theta_4 \) with the normal becomes incident at the free surface, then adopting the same procedure as above, one can obtain the following system of
four equations

\[
\sum_{i=1,2} (\xi_i^2 - k_i^2 \sin^2 \theta_i) A_i + \sum_{i=3,4} k_i^2 \sin \theta_i \cos \theta_i A_i = k_4^2 \sin \theta_4 \cos \theta_4, \tag{4.46}
\]

\[
\sum_{i=1,2} (2\mu + K^*) k_i^2 \sin \theta_i \cos \theta_i A_i - \sum_{i=3,4} \left\{ \mu \cos 2\theta_i + K^* \left( \cos^2 \theta_i - \frac{\eta_i}{k_i^2} \right) \right\} k_i^2 A_i \]

\[
= \left\{ \mu \cos 2\theta_4 + K^* \left( \cos^2 \theta_4 - \frac{\eta_4}{k_4^2} \right) \right\} k_4^2, \tag{4.47}
\]

\[
\sum_{i=1,2} (\xi_i^2 - k_i^2) k_i \cos \theta_i A_i = 0, \tag{4.48}
\]

\[
\sum_{i=3,4} \eta_i k_i \cos \theta_i A_i = \eta_4 k_4 \cos \theta_4. \tag{4.49}
\]

These equations enable us to provide the formulae for the reflection coefficients of various reflected waves at the plane free surface.

### 4.5 Energy partition

We shall now consider the energy partitioning between various reflected waves at the free surface. The rate of energy transmission per unit area is given by

\[
P^\prime = \tau_{xx} \dot{u}_3 + \tau_{xx} \dot{u}_1 + m_y \dot{\psi}_2. \tag{4.50}
\]

**(i) When a longitudinal displacement wave is incident**

The energy \( E_{\text{inc}}^i \) of incident longitudinal displacement wave is given by

\[
E_{\text{inc}}^i = \rho \Omega^3 k_i \cos \theta_1 \left[ \exp 2\iota \left\{ k_i (z \sin \theta_1 - z \cos \theta_1) - \Omega t \right\} \right].
\]

The energy ratios \( E_i(i = 1, 2, 3, 4) \) corresponding to a reflected longitudinal displacement wave, a reflected longitudinal volume fractional wave, a reflected set of coupled transverse shear and microrotational wave with speed \( V_3 \) and another reflected set of
coupled transverse shear and microrotational wave with speed $V_4$ are given by

$$E_1 = |A_1|^2, \quad E_2 = \frac{k_2 \cos \theta_2}{k_1 \cos \theta_1} |A_2|^2,$$

$$E_3 = \frac{k_3^2 \cos \theta_3}{\rho \Omega^2 k_1 \cos \theta_1} \left[ \mu + K^* - \frac{n_3}{k_3^2} (\gamma' \eta_3 + K^*) \right] |A_3|^2,$$

$$E_4 = \frac{k_4^2 \cos \theta_4}{\rho \Omega^2 k_1 \cos \theta_1} \left[ \mu + K^* - \frac{n_4}{k_4^2} (\gamma' \eta_4 + K^*) \right] |A_4|^2.$$

(ii) When a set of coupled transverse waves is incident

The energy $E_{\text{inc}}^i$ of an incident set of coupled transverse shear and microrotational waves propagating with speed $V_3$ is given by

$$E_{\text{inc}}^i = \left[ \mu + K^* - \frac{n_3}{k_3^2} (\gamma' \eta_3 + K^*) \right] \Omega k_3^2 \cos \theta_3 \exp 2i \left\{ k_3 (z \sin \theta_3 - z \cos \theta_3) - \Omega t \right\}.$$

The energy ratios $E_i (i = 1, 2, 3, 4)$ corresponding to a reflected longitudinal displacement wave, a reflected longitudinal volume fractional wave, a set of reflected coupled transverse shear and microrotational waves with speed $V_3$ and another set of reflected coupled transverse shear and microrotational waves with speed $V_4$ are given by

$$E_1 = \frac{\rho \Omega^2 k_1 \cos \theta_1 |A_1|^2}{\left[ \mu + K^* - \frac{n_3}{k_3^2} (\gamma' \eta_3 + K^*) \right] k_3^2 \cos \theta_3}, \quad E_2 = \frac{\rho \Omega^2 k_2 \cos \theta_2 |A_2|^2}{\left[ \mu + K^* - \frac{n_4}{k_4^2} (\gamma' \eta_4 + K^*) \right] k_4^2 \cos \theta_4},$$

$$E_3 = |A_3|^2, \quad E_4 = \frac{\left[ \mu + K^* - \frac{n_3}{k_3^2} (\gamma' \eta_3 + K^*) \right] k_3^2 \cos \theta_3}{\left[ \mu + K^* - \frac{n_4}{k_4^2} (\gamma' \eta_4 + K^*) \right] k_4^2 \cos \theta_4} |A_4|^2.$$

### 4.6 Special cases

(i) For limitly high frequency longitudinal displacement wave striking at the boundary surface, we substitute $k_1 = \zeta_2$ in the boundary conditions given by (4.37), we obtain $A_2 = 0$. This means that the wave corresponding to volume fraction disappear. The remaining boundary conditions given by equations (4.35), (4.36) and (4.38) match with those given by Parfitt and Eringen (1969) for the relevant problem. Thus for high frequency incident wave, these problems are no different from whatever have been earlier discussed by Parfitt and Eringen (1969).
(ii) If the presence of voids in the medium is neglected then the problem reduces to 
the problem earlier discussed by Parfitt and Eringen (1969). In this case, we put \( \beta = 0 \)
into equation (4.37), we obtain \( A_2 = 0 \), which means that the void volume fraction
wave disappears. Therefore, substituting \( \beta = A_2 = 0 \) into remaining equations (4.35),
(4.36) and (4.38), we obtain

\[
\begin{align*}
(\zeta_i^2 - k_i^2 \sin^2 \theta_i) A_i + \sum_{i=3,4} k_i^2 \sin \theta_i \cos \theta_i A_i &= - (\zeta_i^2 - k_i^2 \sin^2 \theta_i), \\
(2\mu + K^*) k_i^2 \sin \theta_i \cos \theta_i A_i - &\sum_{i=3,4} \left\{ \mu \cos 2\theta_i + K^* \left( \cos^2 \theta_i - \frac{\eta_i}{k_i^2} \right) \right\} k_i^2 A_i \\
&= (2\mu + K^*) k_i^2 \sin \theta_i \cos \theta_i, \\
&\sum_{i=3,4} \eta_i k_i \cos \theta_i A_i = 0.
\end{align*}
\]

These equations match with equations (75) and (76) of Parfitt and Eringen (1969).

(ii) If the micropolarity of the medium is removed, that is, if the parameter \( K^* = 0 \), we
see that \( \eta_4 = 0 \). Therefore, from equation (4.38), we obtain \( A_4 = 0 \). Now substituting
\( K^* = A_4 = 0 \) into equations (4.35)-(4.37), we obtain the following system of equations

\[
\begin{align*}
&\cos 2\theta_3 A_1 + \cos 2\theta_3 A_2 + \sin 2\theta_3 A_3 = - \cos 2\theta_3, \\
&k_1^2 \sin 2\theta_1 A_1^2 + k_2^2 \sin 2\theta_2 A_2^2 - k_3^2 \cos 2\theta_3 = k_1^2 \sin 2\theta_1, \\
&\left( \frac{\Omega^2}{\zeta_1^2} - k_1^2 \right) k_1 \cos \theta_1 A_1 + \left( \frac{\Omega^2}{\zeta_1^2} - k_2^2 \right) k_2 \cos \theta_2 A_2 = \left( \frac{\Omega^2}{\zeta_1^2} - k_1^2 \right) k_1 \cos \theta_1.
\end{align*}
\]

These equations do match with equations (38a)-(38c) of Ciarletta and Sumbatyan (2003). Note that the right hand side of equation (38c) is reported incorrectly in Ciarletta and Sumbatyan (2003), its correct form is given above in equation (4.56).

(iii) If the micropolarity and voids are absent from the half-space, then the problem
reduces to the corresponding problem of classical elasticity. Therefore, in this case,
substituting \( \beta = K^* = 0 \) into equations (4.37) and (4.38), we obtain \( A_2 = A_4 = 0. \)
Using $A_2 = A_4 = 0$ into equations (4.35) and (4.36), we obtain

$$\cos 2\theta_3 A_1 + \sin 2\theta_3 A_3 = -\cos 2\theta_3,$$  \hspace{1cm} (4.57)

$$k_1^2 \sin 2\theta_1 A_1 - k_2^2 \cos 2\theta_3 A_3 = k_1^2 \sin 2\theta_1.$$  \hspace{1cm} (4.58)

These equations do match with those obtained for reflection problem from the free boundary of a uniform elastic half-space due to incident longitudinal wave (see Kolsky 1963).

4.7 Numerical results and discussion

Since for limitly high frequency incident wave, the analysis is same as in case of micropolar elastic material, therefore we shall seek the effect of presence of voids on reflection coefficients numerically only for limitly low frequency incident wave, which is important from seismological point of view. We shall study the variations of these reflection coefficients and their associated energy with the angle of incidence. For this purpose, the following numerical values of relevant micropolar elastic parameters are taken from Chiroiu and Munteanu (2002), while numerical values of other relevant parameters corresponding to voids are taken arbitrarily.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>value $\times 10^{10}$</th>
<th>unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>7.55</td>
<td>N/m²</td>
</tr>
<tr>
<td>$\mu$</td>
<td>6.19</td>
<td>N/m²</td>
</tr>
<tr>
<td>$K^*$</td>
<td>14.50 $\times 10^7$</td>
<td>N/m²</td>
</tr>
<tr>
<td>$\gamma^*$</td>
<td>2.86 $\times 10^6$</td>
<td>N</td>
</tr>
<tr>
<td>$J^*$</td>
<td>2.12 $\times 10^{-6}$</td>
<td>m²</td>
</tr>
<tr>
<td>$\rho$</td>
<td>1.16 $\times 10^3$</td>
<td>kg/m³</td>
</tr>
<tr>
<td>$\xi$</td>
<td>12 $\times 10^{10}$</td>
<td>N/m²</td>
</tr>
<tr>
<td>$\beta$</td>
<td>10 $\times 10^{10}$</td>
<td>N/m²</td>
</tr>
<tr>
<td>$\omega$</td>
<td>10 $\times 10^{7}$</td>
<td>N sec/m²</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.001 $\times 10^6$</td>
<td>N</td>
</tr>
<tr>
<td>$K$</td>
<td>10 $\times 10^5$</td>
<td>m²</td>
</tr>
</tbody>
</table>

We have solved equations (4.35) - (4.38) for amplitudes of various reflected waves when longitudinal displacement wave becomes incident obliquely, and equations (4.42)
- (4.45) for amplitudes of various reflected waves when coupled transverse shear and transverse microrotational wave becomes incident obliquely. Gauss elimination method is used through a FORTRAN program. We have chosen the value of \( \alpha \) in such a way that \( l_2 \Omega \ll 1 \). In all numerical computations, we have used \( \Omega = 25 \text{ rad/s} \), wherever it is not mentioned.

Figure 4.1: Variation of modulus of amplitude ratios with angle of incidence of longitudinal displacement wave. (Curve - I: \( |A_1| \), Curve - II: \( |A_2| \times 10^6 \), Curve - III: \( |A_3| \times 10^5 \), Curve - IV: \( |A_4| \)).

Figure 4.2: Variation of modulus of energy ratios with angle of incidence of longitudinal displacement wave. (Curve - I: \( E_1 \), Curve - II: \( E_2 \times 10^{11} \), Curve - III: \( E_3 \times 10^5 \), Curve - IV: \( E_4 \)).

Figure 4.1 depicts the variation of modulus of reflection coefficients (amplitude ratios) of various reflected waves with respect to the angle of incidence, when a longitudinal displacement wave strikes at the free surface with frequency \( \Omega = 25 \text{ rad/s} \). We have plotted the graphs of reflection coefficients \( |A_2| \) and \( |A_3| \) by magnifying with factors \( 10^6 \) and \( 10^5 \) respectively. It is clear from this figure that the amplitude \( |A_2| \) of reflected volume fractional wave is quite small as compared to the amplitudes \( |A_1| \) and \( |A_4| \) corresponding to reflected longitudinal displacement wave and coupled transverse shear and microrotational waves at speed \( V_4 \). Also, the amplitude of coupled transverse shear and microrotational waves with speed \( V_3 \) is small as compared to the amplitudes \( |A_1| \) and \( |A_4| \), but it increases abruptly between angles 15\(^\circ\) to 25\(^\circ\) and then decreases smoothly with the increase of the angle of incidence. The amplitude of reflected longitudinal displacement wave has maximum values at 0\(^\circ\), 75\(^\circ\) and 90\(^\circ\) angles of incidence and minimum values at 36\(^\circ\) and 89\(^\circ\) angles of incidence. On the other hand, the ampli-
tude of reflected coupled transverse shear and microrotational waves with velocity $V_4$ has maximum values at $34^\circ$ and $87^\circ$ angles of incidence and minimum values at $0^\circ$, $75^\circ$ and $90^\circ$ angles of incidence. Thus, the amplitudes $|A_1|$ and $|A_4|$ are possessing almost reverse behaviors.

Figure 4.2 depicts the variation of energy ratios of various reflected waves with the angle of incidence. It is found that $\sum_{i=1}^{4} E_i = 1$, which means that there is no dissipation of energy at the free surface. It is also noted from this figure that the reflected longitudinal displacement wave and coupled transverse shear and microrotational waves with speed $V_4$ carry maximum amount of incident energy as compared to other reflected waves.

Figure 4.3 depicts the variation of modulus of reflection coefficients (amplitude ratios) of various reflected waves with respect to the angle of incidence, when a coupled transverse shear and microrotational wave with speed $V_4$ strikes obliquely at the free surface. We have plotted the graphs of reflection coefficients $|A_1|$, $|A_2|$ and $|A_4|$ by magnifying with factors 10, $10^6$ and 10 respectively. We noticed from this figure that the amplitude $|A_2|$ corresponding to reflected volume fractional wave is quite small in comparison to other reflected waves. The value of amplitude ratio $|A_1|$ corresponding
to reflected longitudinal displacement wave increases with angle of incidence achieving its maximum value at 45° angle of incidence and then decreases with further increase of the angle of incidence. The value of the amplitude \(|A_3|\) remains constant and is maximum among all amplitude ratios. The variation of the amplitude \(|A_4|\) is similar to the behavior of \(|A_1|\) except that it attains maximum value at 60° angle of incidence. The behavior of the amplitude ratio \(|A_2|\) corresponding to volume fractional wave is found to be oscillatory throughout the entire range 0° < \(\theta_3\) < 90°, though amplitude of oscillation is very small.

Figure 4.5: Variation of wavenumbers with frequency (Curve - I: \(k_1 \times 10^2\), Curve - II: \(k_2\)).

Figure 4.6: Variation of modulus of amplitude ratios with frequency of longitudinal displacement wave striking at \(\theta_1 = 45^\circ\). (Curve - I: \(|A_1|\), Curve - II: \(|A_2| \times 10^7\), Curve - III: \(|A_3| \times 10^8\), Curve - IV: \(|A_4|\)).

Figure 4.4 depicts the variation of energy ratios of various reflected waves with respect to the angle of incidence of a set of coupled transverse waves with speed \(V_3\). It is clear from this figure that the maximum amount of incident energy is carried by reflected coupled transverse waves with amplitude \(A_3\) in comparison to the amount of energies carried by the other reflected waves. Here, we have also found that there is no dissipation of energy at the surface i.e. \(\sum_{i=1}^{4} E_i = 1\).

From Figures 4.1-4.4, we note that at grazing incidence (i.e. at 90° angle of incidence), the incident wave propagates along the boundary and no reflection phenomena take place. Hence, at grazing incidence, there is no effect of presence of pores in the medium. Also at normal incidence (i.e. at 0° angle of incidence) there is an effect of
presence of voids, through it is very small.

Figure 4.5 depicts the variation of wavenumbers $k_1$ and $k_2$ with frequency ($\Omega$). We note that $k_1$ increases linearly with the increase of frequency, while $k_2$ increases exponentially for the range $0 < \Omega < 3$, thereafter it increases linearly with the increase of frequency. Hence, we conclude that the speed of wave corresponding to volume fractional wave is less than that of the speed of longitudinal displacement wave.

Figures 4.6 and 4.7 depict the variations of modulus of amplitude ratios with the frequency of incident wave striking at an angle 45° angle of incidence. We note from Figure 4.7 that the amplitude ratios $|A_1|$ and $|A_4|$ are independent of frequency ($\Omega$), while the amplitude ratios $|A_2|$ and $|A_3|$ depend on frequency. However, we notice from Figure 4.8 that all amplitude ratios are influenced with the frequency except the amplitude ratio $|A_3|$ corresponding to reflected coupled transverse waves with speed $V_3$. Also, it is found that at all frequencies $\sum_{i=1}^{4} E_i = 1$.

Figures 4.8 and 4.9 show a comparison of modulus of amplitudes of reflected waves against angle of incidence of longitudinal displacement and coupled transverse shear and microrotational waves with speed $V_3$ respectively. In these figures, the curve with centered symbol represents the variation of amplitude of reflected volume fractional wave. The solid curves represent the amplitudes of reflected waves in micropolar elastic half-space and the dotted curves represent the corresponding reflected waves in...
micropolar porous elastic half-space. It is clear from these figures that there is significant effect of presence of voids on the reflected waves and there exists a reflected volume fractional wave in case of reflection from the boundary of a micropolar porous half-space. This reflected wave is new and not encountered during reflection from the boundary of a micropolar elastic half space without voids. From Figure 4.8, a comparison of amplitudes of reflected waves in micropolar elastic half-space with voids and micropolar elastic half-space without voids indicates that the effect of presence of voids on $|A_1|$ is 92.78% at 71° angle of incidence, on $|A_3|$ it is 93.41% at 17° angle of incidence and on $|A_4|$ it is 98.03% at 75° angle of incidence, for the data used. A similar conclusion can be drawn from Figure 4.9. In these figures, we have found that the sum of energies at the boundary surface is equal to unity showing that there is no dissipation of energy during reflection.

4.8 Conclusions

In this chapter, we have studied the plane wave propagation in an infinite micropolar porous elastic half-space. It is found that there can exist five plane waves: one is longitudinal displacement wave, second is longitudinal microrotational wave, third is
longitudinal volume fractional wave and the remaining two waves are sets of transverse displacement wave coupled with transverse microrotational wave propagating with different speeds. Reflection of longitudinal displacement wave and of coupled transverse shear and microrotational waves from the free boundary of a micropolar porous elastic half-space have been discussed. It is found that the reflection coefficients are functions of the angle on incidence, frequency and elastic properties of the half-space. It can be concluded that

(i) The speed of wave carrying a change in void volume fraction is found to be less than the speed of the longitudinal displacement wave.

(ii) Coupled transverse shear and microrotational waves and longitudinal micro-rotational waves are independent of void effects.

(iii) For limitly high frequency case, it is found that the longitudinal displacement wave propagates with the same speed as it propagates in case of micropolar elastic medium without voids.

(iv) For limitly low frequency case, the longitudinal displacement wave travels with a speed less than the speed that of in high frequency case.

(v) When coupled transverse shear and microrotational wave is incident at the free plane boundary, the amplitude of reflected volume fractional wave oscillates with the angle of incidence, however the amplitude of oscillation is very small. The volume fractional wave has maximum amplitude when the incident wave strikes normal to the boundary.

(vi) Amplitude of that reflected wave which is alike to the incident wave is found to be more than the amplitudes of other reflected waves and hence the same is true for amount of energy carried along with each reflected wave.

(vii) In high and low frequency cases, the wave corresponding to the change in volume fraction is not influenced by the micropolarity of the medium. It travels with the same speed as in case of elastic material with voids.

(viii) The sum of the energy ratios at each angle of incidence is found to be unity, showing that there is no dissipation.