Chapter 3

Reflection and transmission of transverse waves at a plane interface between two different porous elastic solid half-spaces

3.1 Introduction

Ciaretta and Sumbatyan (2003) investigated the reflection of obliquely incident plane transverse wave from the stress-free boundary of a porous elastic half space. They obtained the reflection coefficients asymptotically in closed form for high and low frequency incident transverse wave. In this chapter, we have extended the problem of Ciaretta and Sumbatyan (2003) to the reflection and transmission of transverse wave at a plane interface between two different porous elastic half-spaces. The expressions for reflection and transmission coefficients are derived and presented graphically for a particular model. The energy partitioning at the interface has also been derived. The effect of presence of voids on the coefficients corresponding to various reflected and transmitted waves has also been studied. The results of Ciaretta and Sumbatyan (2003) have been reduced as a special case of the present problem by neglecting one of the half-space.

3.2 Problem and its solution

We consider two distinct homogeneous and isotropic porous elastic half spaces $M$ and $M'$. Let us take the $x-$ axis along the interface between $M[0 < Y < \infty]$ and $M'[-\infty < Y < 0]$ in such a way that the $y-$ axis is pointing downwards vertically into the half-space $M$. Let us denote the quantities relevant to the medium $M'$ with primes and the quantities relevant to the medium $M$ without primes. We know that when a shear horizontally polarized plane wave ($SH-$ wave) becomes incident obliquely at the interface between two elastic solid half-spaces, then it give rise to a reflected $SH-$ wave and a transmitted $SH-$ wave with different amplitudes (see Ewing et al., 1957). Since the horizontally polarized transverse wave is independent of void effects (see Chapter 2), therefore its reflection and transmission phenomena from the plane interface will not differ from that what exists in case of classical elasticity. However, in case of incidence of vertically polarized transverse wave, the reflected and transmitted longitudinal waves would be influenced by the presence of voids as the waves corresponding to the potential $p$ and corresponding to the void volume fraction $\phi$ are coupled (see Chapter 2). Therefore, it is interesting to discuss the reflection and transmission of vertically polarized transverse wave at the plane interface between the half-spaces $M$ and $M'$.

Let a unit amplitude plane transverse wave ($SV-$ wave) propagating through the lower half-space be incident at the interface $y = 0$ and making an angle $\theta$ with the normal. We postulate that this incident transverse plane wave will give rise to:

**Reflected waves:**
(i) A transverse plane wave of amplitude $R_q$ traveling in medium $M$ and making an angle $\theta$ with the normal.
(ii) a longitudinal plane wave of amplitude $T_1$ propagating in medium $M$ and making an angle $\gamma_1$ with the normal.
(iii) a longitudinal void volume fractional wave of amplitude $T_2$ propagating in medium $M$ and making an angle $\gamma_2$ with the normal.

**Transmitted waves:**
(i) A transverse plane wave of amplitude $T_q'$ traveling in medium $M'$ and making an angle $\theta'$ with the normal.
(ii) a longitudinal plane wave of amplitude $T_1'$ propagating in medium $M'$ and making an angle $\gamma_1'$ with the normal.
(iii) a longitudinal void volume fractional wave of amplitude $T_2'$ propagating in medium $M'$ and making an angle $\gamma_2'$ with the normal. The complete geometry of the problem is shown in Figure 3.1.

The full wave structure in the two half-spaces can be written as

In medium $M$:

$$\begin{aligned}
q &= \exp\{ik_s(x \sin \theta - y \cos \theta)\} + R_q \exp\{ik_s(x \sin \theta + y \cos \theta)\}, \quad (3.1) \\
p &= T_1 \exp\{i\psi_1(x \sin \gamma_1 + y \cos \gamma_1)\} + T_2 \exp\{i\psi_2(x \sin \gamma_2 + y \cos \gamma_2)\}. \quad (3.2)
\end{aligned}$$

In medium $M'$:

$$\begin{aligned}
q' &= T_q' \exp\{iK_s'(x \sin \theta' - y \cos \theta')\}, \quad (3.3) \\
p' &= T_1' \exp\{i\psi_1'(x \sin \gamma_1' - y \cos \gamma_1')\} + T_2' \exp\{i\psi_2'(x \sin \gamma_2' - y \cos \gamma_2')\}. \quad (3.4)
\end{aligned}$$
where the symbols $k_s, k'_s, \psi_1, \psi'_1, \psi_2$ and $\psi'_2$ denote the wavenumbers of respective waves and have been defined earlier in Chapter 2. As in Chapter 2, the various amplitudes, namely $R_q, T_1, T_2, R'_q, T'_1$ and $T'_2$ corresponding to various reflected and refracted waves will be determined in the following section, using appropriate boundary conditions at the interface $y = 0$ between two media.

### 3.2.1 Boundary conditions

The boundary conditions at the plane interface $y = 0$ between two porous elastic half-spaces are the same as given by (2.23) - (2.28) in Chapter 2. Making use of relations (1.136), (2.7) and the equation (2.8) into the boundary conditions given by (2.23) - (2.28) and then inserting the potentials $p, q, p'$ and $q'$ from (3.1)-(3.4), with the help of Snell’s law given by

$$
\psi_1 \sin \gamma_1 = \psi_2 \sin \gamma_2 = k_s \sin \theta = \psi'_1 \sin \gamma'_1 = \psi'_2 \sin \gamma'_2 = k'_s \sin \theta',
$$

and at $z = 0, \Omega = \Omega'$, we obtain the following system of six simultaneous equations given by

$$
\begin{align*}
\mu k_s^2 \cos 2\theta R_q - \mu' k'_s^2 \cos 2\theta' T'_q - \mu \psi_1^2 \sin 2\gamma_1 T_1 &= 0, \\
-\mu \psi_2^2 \sin 2\gamma_2 T_2 - \mu' \psi'_1^2 \sin 2\gamma'_1 T'_1 - \mu' \psi'_2^2 \sin 2\gamma'_2 T'_2 &= -\mu k_s^2 \cos 2\theta, \\
\mu k_s^2 \sin 2\theta R_q + \mu' k'_s^2 \sin 2\theta' T'_q + \mu \psi_1^2 \cos 2\theta T_1 &= 0, \\
+\mu k_s^2 \cos 2\theta T_2 - \mu' k'_s^2 \cos 2\theta' T'_1 - \mu' \psi'_2^2 \cos 2\theta' T'_2 &= \mu k_s^2 \sin 2\theta, \\
k_s \cos \theta R_q + k'_s \cos \theta' T'_q - k_s \sin \theta T_1 &= 0, \\
-k_s \sin \theta T_2 + k'_s \sin \theta' T'_1 + k'_s \sin \theta' T'_2 &= k_s \cos \theta, \\
k_s \sin \theta R_q - k'_s \sin \theta' T'_q + \psi_1 \cos \gamma_1 T_1 &= 0, \\
+\psi_2 \cos \gamma_2 T_2 + \psi'_1 \cos \gamma'_1 T'_1 + \psi'_2 \cos \gamma'_2 T'_2 &= -k_s \sin \theta, \\
(k_p^2 - \psi_1^2) T_1 + (k_p^2 - \psi_2^2) T_2 - \frac{H}{\rho_l}[(k'_p^2 - \psi'_1)^2 T'_1 + (k'_p^2 - \psi'_2)^2 T'_2] &= 0,
\end{align*}
$$

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These equations enable us to provide the formulae for the reflection and transmission coefficients of various reflected and refracted waves at the interface.

### 3.3 Energy partition

As in Chapter 2, the energy of incident plane transverse wave is given by

\[ E = \frac{1}{2} \rho k_2^2 \Omega^2 \{ \exp [ik_x(x \sin \theta - y \cos \theta) - \Omega t] \}^2. \]  

The energy ratios of the reflected and transmitted transverse waves to that of the incident transverse wave are given by

\[ E_1 = |R_1|^2, \quad E_2 = \frac{\rho \ k_x^2}{\rho \ k_x^2} |T_2|^2. \]

The energy ratios of the reflected and transmitted longitudinal waves to that of the incident transverse wave are given by

\[ E_3 = \frac{\psi_1^2}{k_s^2} |T_1|^2, \quad E_4 = \frac{\rho \ \psi_1^2}{\rho \ k_s^2} |T_4|^2, \quad E_5 = \frac{\psi_2^2}{k_s^2} |T_2|^2, \quad E_6 = \frac{\rho \ \psi_2^2}{\rho \ k_s^2} |T_6|^2. \]

Since both the media are non-dissipative, therefore at the interface between two media considered, the energy of incident wave must be equal to the sum of energies of reflected and transmitted waves. Therefore, we obtain the following relation for the balance of energy at the interface \( y = 0 \)

\[ |R_1|^2 + \frac{\psi_1^2}{k_s^2} |T_1|^2 + \frac{\psi_2^2}{k_s^2} |T_2|^2 + \frac{\rho \ k_x^2}{\rho \ k_x^2} |T_4|^2 + \frac{\rho \ \psi_1^2}{\rho \ k_s^2} |T_1|^2 + \frac{\rho \ \psi_2^2}{\rho \ k_s^2} |T_2|^2 = 1. \]  

### 3.4 Special cases

(i) When the plane transverse wave is incident normally at the interface, then \( \theta = 0 \).

In this case, using Snell’s law given by (3.5), the boundary conditions (3.6) to (3.11)
reduce to

\[ \mu k^2_{s_R} R_q - \mu' k^2_{s_T} T_q' = -\mu k^2, \quad (3.14) \]

\[ k_s R_q + k'_s T_q' = k_s, \quad (3.15) \]

\[ \mu k^2_{s_s} (T_1 + T_2) - \mu' k^2_{s_s} (T_1' + T_2') = 0, \quad (3.16) \]

\[ \psi_1 T_1 + \psi_2 T_2 + \psi_1' T_1' + \psi_2' T_2' = 0, \quad (3.17) \]

\[ (k^2_p - \psi^2_1) T_1 + (k^2_p - \psi^2_2) T_2 = \frac{H}{H'} [(k'^2_p - \psi'^2_1) T_1' + (k'^2_p - \psi'^2_2) T_2'] = 0, \quad (3.18) \]

\[ (k^2_p - \psi^2_1) \psi_1 T_1 + (k^2_p - \psi^2_2) \psi_2 T_2 + \frac{H}{H'} [(k'^2_p - \psi'^2_1) \psi_1' T_1' + (k'^2_p - \psi'^2_2) \psi_2' T_2'] = 0. \quad (3.19) \]

We see that equations (3.16)-(3.19) constitute a set of four homogeneous equations whose determinant of the coefficients is nonzero giving \( T_1 = T_2 = T_1' = T_2' = 0. \)
The remaining equations (3.14) and (3.15) yield the following values of reflection and transmission coefficients

\[ R_q = \frac{\rho' c'_2 c_2 - \mu}{\rho' c'_2 c_2 + \mu}, \quad T_q' = \frac{2\mu c'_2}{c_2 (\mu + \rho' c'_2 c_2)}. \quad (3.20) \]

Thus, we conclude that in the case of normal incidence of plane transverse wave, no longitudinal wave gets reflected or transmitted in the problem.

(ii) When one of the media \( M' \) is absent, then we shall be left with the problem of reflection of plane transverse wave from the free surface of a elastic porous half-space already discussed by Ciarletta and Sumbatyam (2003). In this case, putting \( c'_2 = c'_1 = 0 \) into the boundary conditions (3.6) to (3.11), we see that the three equations given by (3.8) to (3.10) gives \( T_1' = T_2' = 0 \) and rest of the equations reduce to

\[ k_s^2 \cos 2\theta R_q - \psi^2_1 \sin 2\gamma_1 T_1 - \psi^2_2 \sin 2\gamma_2 T_2 = -k_s^2 \cos 2\theta, \quad (3.21) \]
\[
\sin 2\theta R_q + \cos 2\theta T_1 + \cos 2\theta T_2 = \sin 2\theta, \quad (3.22)
\]
\[
(k_p^2 - \psi_1^2)\psi_1 \cos \gamma_1 T_1 + (k_p^2 - \psi_2^2)\psi_2 \cos \gamma_2 T_2 = 0. \quad (3.23)
\]

These equations are exactly the same equations given by Ciarletta and Sumbatyam (2003) for the relevant problem. Also, it is easy to see from equation (3.20) that in the present case \( R_q = -1 \) and \( T'_q = 0 \).

(iii) If the presence of voids from both the media are neglected then the problem reduces to the reflection and transmission of SV - wave at the interface of two uniform elastic half-spaces in welded contact. In this case, putting \( H = H' = 0 \) and \( T_1 = T'_2 = 0 \) into the boundary conditions given by (3.6) to (3.11), we see that equations (3.10) and (3.11) are satisfied identically, while the remaining equations (3.6) to (3.9) reduce to

\[
\mu k_s^2 \cos 2\theta R_q - \mu' k'_s^2 \cos 2\theta' T'_q - \mu \psi_1^2 \sin 2\gamma_1 T_1 - \mu' \psi'_1^2 \sin 2\gamma'_1 T'_1 = -\mu k_s^2 \cos 2\theta, \quad (3.24)
\]
\[
\mu k_s^2 \sin 2\theta R_q + \mu' k'_s^2 \sin 2\theta' T'_q + \mu k_s^2 \cos 2\theta T_1 - \mu' k'_s^2 \cos 2\theta' T'_1 = \mu k_s^2 \sin 2\theta, \quad (3.25)
\]
\[
k_s \cos \theta R_q + k'_s \cos \theta' T'_q - k_s \sin \theta T_1 + k'_s \sin \theta T'_1 = k_s \cos \theta, \quad (3.26)
\]
\[
k_s \sin \theta R_q - k'_s \sin \theta' T'_q + \psi_1 \cos \gamma_1 T_1 + \psi'_1 \cos \gamma'_1 T'_1 = -k_s \sin \theta. \quad (3.27)
\]

One can verify that the reflection and transmission coefficients obtained from these equations are the well known coefficients in the classical elasticity.

### 3.5 Numerical results and discussion

In order to study the problem in greater detail and to find the nature of dependence of reflection and transmission coefficients on the angle of incidence and frequency of the incident wave, we shall solve equations (3.6) to (3.11) numerically. For this purpose, we take the values of relevant physical parameters occurring in the problem from the table given in Chapter 2. Equations (3.6)- (3.11) are solved through FORTRAN programming by using Cramer’s rule and the computations are carried out on a PC.
Figure 3.2: Limitly high frequency: Variations of reflection and transmission coefficients with angle of incidence (Curve 1 - $|R_1|$, Curve 2 - $|T_2^0|$, Curve 3 - $|T_1|$, Curve 4 - $|T_4^0|$, $|T_2^0| = |T_4^0| = 0$).

Figure 3.3: Limitly high frequency: Variations of reflection and transmission energy versus angle of incidence (Curve 1 - $E_1$, Curve 2 - $E_2$, Curve 3 - $E_3$, Curve 4 - $E_4$).

Figure 3.4: Limitly high frequency: Variations of reflection and transmission coefficients with frequency when incidence angle is 25$^\circ$ (Curve 1 - $|R_4|$, Curve 2 - $|T_4^0|$, Curve 3 - $|T_1|$, Curve 4 - $|T_4^0|$, $|T_2^0| = |T_4^0| = 0$).

Figure 3.5: Limitly low frequency: Variations of reflection and transmission coefficients with angle of incidence (Curve 1 - $|R_4|$, Curve 2 - $|T_4^0|$, Curve 3 - $|T_1|$, Curve 4 - $|T_4^0|$).

Figure 3.2 shows the variation of the modulus of reflection and transmission coefficients with respect to the angle of incidence. We note from this figure that for the physical data used, a critical angle occurs at 30$^\circ$ angle of incidence, beyond which no
reflection and transmission appears. We find that for limitly high frequency case, the reflection and refraction coefficients \(|T_2|\) and \(|T'_2|\) do not exist. In the range \(0^\circ < \theta \leq 30^\circ\), the reflection coefficient \(|R_\theta|\) increases gently with the increase of the angle of incidence, while the transmission coefficient \(|T'_\theta|\) exhibits reverse behavior. Since the amplitudes \(|T_2|\) and \(|T'_2|\) corresponding to reflected and transmitted void volume fractional waves respectively are found to be zero, therefore we can say that for a limitly high frequency case, the porous half-spaces behave like a classical elastic medium.

Figure 3.3 shows the variation of energy ratios \((E_i)\) of various reflected and refracted waves with the angle of incidence. It is found that at the interface, the sum of energy ratios is approximately equal to unity as was expected beforehand for non-dissipative media. It can be noticed from this figure that the energy associated with the waves of amplitudes \(|T_1|\) and \(|T'_1|\) is very small in comparison to the energy associated with the waves of amplitudes \(|R_\theta|\) and \(|T'_\theta|\).

![Figure 3.6: Limitly low frequency: Variations of reflection and transmission coefficients with angle of incidence (Curve 1- \([T_2]\), Curve 2 - \([T'_2]\)).](image)

![Figure 3.7: Limitly low frequency: Variations of energy ratios of transmission longitudinal wave \(E_2\) with angle of incidence.](image)

Figure 3.4 shows the variation of modulus of reflection and transmission coefficients with the frequency of the transverse wave striking at \(25^\circ\) angle of incidence. We notice from this figure that the coefficients \(|R_\theta|\), \(|T_1|\) and \(|T'_1|\) do not alter with the increase in the frequency of the striking wave, while the coefficient \(|T'_\theta|\) is very much influenced by the frequency and decreases with increase of the frequency of the incident wave.
Figure 3.5 shows the variation of the modulus of reflection and transmission coefficients with the angle of incidence for the limitly low frequency case. We see that the behavior of the coefficients $|R_q|$, $|T_q'|$, $|T_1|$ and $|T_2'|$ do not differ much as was found in case of limitly high frequency case, except that the value of critical angle is here $33^\circ$. However, it is important to note that in this case, i.e., in limitly low frequency case, there exists a dilational wave (the wave corresponding to the change in void volume fraction), which does not exist in limitly high frequency case. The behavior of these reflected and transmitted dilational waves with the angle of incidence is depicted in Figure 3.6. It is noted that the nature of dependence of the coefficients $|T_2|$ and $|T_2'|$ on the angle $\theta$ is almost same, except that the magnitude of the former is larger than the later in the range $0^\circ < \theta \leq 33^\circ$.

Figures 3.7 and 3.8 depict the variation of the energy ratios $E_i$ ($i = 1, 2, 3, 4$) with the angle of incidence $\theta$, while Figure 3.9 shows the same for $E_5$ and $E_6$. It is found that the sum of the energy carried with reflected and transmitted waves is approximately equal to the energy of the incident wave. It is clear from Figure 3.9 that the energy carried along the waves corresponding to amplitudes $|T_2|$ and $|T_2'|$ is very small in comparison to the energy carried by the other reflected and transmitted waves.

Figure 3.10 shows the variation of coefficients $|R_q|$, $|T_q'|$, $|T_1|$ and $|T_2'|$ with the fre-
frequency of transverse wave striking at 25° angle of incidence. This figure shows that these coefficients are influenced by the frequency $\Omega$ in the range $10^2 < \Omega < 10^4$ only, while for larger values of $\Omega$, they are frequency independent. Figure 3.11 shows the variation of the coefficients $|T_2|$ and $|T_3|$ with frequency $\Omega$ at $\theta = 25^\circ$ angle of incidence. The behavior of these coefficients with $\Omega$ is similar to those of the other coefficients.

### 3.6 Conclusions

A mathematical study is made to discuss the problem of reflection and transmission of plane transverse shear wave striking obliquely at a plane interface between two different elastic solid half-spaces with voids. The theoretical and numerical results reveal that voids have significant effect on the reflection and transmission coefficients. It is concluded that

(i) For limitly high frequency case, the dilational wave corresponding to change in void volume disappears completely. However, for limitly low frequency case, this dilational wave exists. The amplitude of the reflection and transmission coefficients corresponding to this dilational wave is found to be small in magnitude.

(ii) At normal incidence, only reflected and transmitted wave of alike nature exist.
Moreover, there is no effect of presence of voids on reflection and transmission coefficients in this case.

(iii) It is found that the sum of energy ratios at the interface is approximately equal to unity for both limitly low and limitly high frequency cases. Also, the energies carried along the predominantly waves carrying a change in the void volume friction are very small, but significant.

(iv) The values of the coefficients $|R_q|$, $|T_1|$ and $|T'_1|$ are found to be less in magnitude than the coefficients $|T_4'|$ for all values of the angle of incidence well before the critical angle. Near the critical angle, the values of the former coefficients increase, while the values of the later decrease.

(v) At $\theta = 25^0$ angle of incidence, the coefficients $|R_q|$, $|T_1|$ and $|T'_1|$ remain constant, but the coefficient $|T'_4|$ decreases with increase in the value of the frequency of the incident wave.

(vi) For limitly low frequency case, the coefficients $|T_2|$ and $|T'_2|$ behave alike with the angle of incidence except that the magnitude of former is greater than the later.