Appendix C

A Handbook on Developing Problem Solving Ability
While teaching mathematics it is the teacher who, generally, goes on solving the problems most of the time. He hardly provides any chance to pupils to solve problems for themselves. The result is that pupils become passive which, in turn, affects their learning. To maximise learning the pupils should be allowed to participate actively in the process of learning and solve problems co-operatively with the teacher. After understanding the importance of pupil participation in the class-room activities one may look into the ways and means of increasing it.

Pupil participation can be maximised by developing in them the problem solving ability. In doing so the teacher should equip himself with the skill of developing problem solving ability in his pupils. This skill in short can be named as 'skill of developing problem solving ability'. In order to understand what this skill is about, study the following two episodes in classroom situations.

Episode 1.

Problem - Find the sum of series
1 + 2 + 4 + 8 + 16 + 32 + ......... 15 terms

Teacher: Well, students! you see that starting with 1 every term of the series is double of its preceding term and this process continues for 15 terms. We are to find the sum of this series. Could you?
May I try, sir?

No, you will not be in a position to solve this problem. Try to sit calmly and see how I solve it.

Sir, it is not very difficult. Give us chance to solve it.

Do not waste the time and listen to me very carefully. The one way of solving the problem is to find all the terms of the series and then add them up. But this process is very lengthy. Now, I tell you the other method.

Study a portion of the problem. The sum of the series for 3 terms, 4 terms and 5 terms is:

For 3 terms, $1 + 2 + 4 = 7$

For 4 terms, $1 + 2 + 4 + 8 = 15$

For 5 terms, $1 + 2 + 4 + 8 + 16 = 31$

From this it is very clear that the sum of the series up to any term is one less than the next term. We can calculate that 16th term is 32,768. The sum of the series up to 15 term, therefore, is 32,767. Hence, the problem is solved.

Episode 2.

Problem: Find the sum of the series

$1 + 2 + 4 + 8 + 16 + 32 + \ldots \ldots \text{15 terms}$
Teacher: The above problem can be stated in an interesting way as: A man earns Re. 1 on the first day of a job. The second day he is paid Rs. 2, the third day Rs. 4, the fourth day Rs. 8, and so on. Each day he gets twice that of the preceding day. He plans to stay on the job for 15 days and wants to know his total earnings.

Amit: May I try, sir?

Teacher: Yes, what is given in the problem?

Amit: Sir, starting with Rs. 1, everyday he gets double to its preceding day and this continues for 15 days.

Teacher: What is to be calculated?

Vibhu: Sir, sum total of wages for fifteen days.

Teacher: What could be the different ways for finding the sum?

Anjali: One of the methods is to add up the wage of each day for 15 days.

Vandana: We could study a small portion of the problem and draw conclusions from the pattern of that small portion.

Teacher: Very good, try it.

Vandana: Let us consider what he gets if he works for 3 days, 4 days, and 5 days only.
For 3 days, \(1 + 2 + 4 + 7\)
For 4 days \(1 + 2 + 4 + 8 = 15\)
For 5 days \(1 + 2 + 4 + 8 + 16 = 31\)

Now we can generalize these calculations.

Raman: The total wages for 3 days (Rs. 7) is one less than the wage for fourth day (Rs. 8).

Ashok: The total wages for 4 days (Rs. 15) is one less than the wage for fifth day (Rs. 16).

Anil: Sir, I guess that the total wages for 5 days, on the above pattern, will be one less than the wages for sixth day.

Annu: This is so because I find that the wage for the sixth day is Rs. 32.

Atul: The problem is solved, sir, the wages for 15 days is one less than the wage for sixteenth day.

Armitabh: The wage for sixteenth day is calculated to be 32,768. His total wages for 15 days, therefore, is 32,767.

Teacher: This is O.K, could you generalize it.
Archana: Yes, sir, in a series like
\[ 1 + 2 + 4 + 8 + 16 + \ldots \text{ n terms} \]
the sum total = \(( n + 1 ) \text{ th term} - 1\).

Teacher: That is very fine.

In the two episodes given above which of the two teachers has imparted instructions effectively? We are right if our answer is: the teacher in episode 2.

What makes the behaviour of teacher in episode 2 more effective than the teacher in episode 1 in imparting classroom instructions? The teacher in episode 1 made no attempt to present the problem in an interesting way. He discouraged the pupils to solve problem themselves. He did not analyse the problem. He neither suggested probable solution nor ways of checking the reasonableness of answer. Whereas, the teacher in episode 2 created a right situation for problem solving, analysed the problem properly, helped in knowing the probable solution, facilitated the process of problem solving, etc.

To summarise, what are the desirable teacher behaviours which you have noticed as effective means of developing problem solving ability. Your response should
include the teacher behaviours namely

(1) creating right situation
(2) analysing the problem
(3) suggesting probable solution
(4) facilitating the problem solving
(5) checking reasonableness of answer

Thus, the skill of developing problem solving ability involves increasing the above mentioned behaviours (components of the skill) by a teacher. Now let us know in detail about these components and analyse them into subcomponents through the following examples:

(a) Creating Right Situation

study the following three examples to know the subcomponents of the component 'creating right situation'.

Example 1

Episode I

Teacher: If A and B are two sets, having a total of 20 elements. The number of elements in the set A is 12, the number of elements common to both A and B is 4. Find the number of elements in the set B.

Episode II

Teacher: If A and B are two sets, having a total of 20 elements. The number of elements in the set A
is 12, the number of elements common to both A and B is 4. Find the number of elements in the set B. Well, students! This problem can be restated as: There are 20 teachers, in a school, who teach mathematics or physics. of these, 12 teach mathematics and 4 teach both physics and mathematics. How many of them teach physics?

How do you feel about the teachers in the two episodes, which of them is an effective teacher in developing problem solving ability? If we say the teacher in episode II, we are right. Why? Because he created a right situation for problem solving by stating the problem in an interesting way.

Example 2

Episode I

Problem: A lady has only 10 paise and 25 paise coins in her purse. If in all, she has 60 coins totalling Rs. 8.25, how many of each does she have?

Teacher: Let us try to solve this problem. Who will help me?

Sudhir: Sir, I may try it?

Teacher: Oh! Yes, you may try.
Sudhir: Here the type of coins, their total number and the total amount are known. However, the number of coins of each kind is unknown.

Teacher: Therefore, you may suppose $x$ as the number of 10 paise coins and $y$, the number of 25 paise coins.

Vibhu: Sir, since the total amount is Rs. 8.25, the problem reduces to a system of linear equations as follows:

\[
x + y = 60 \\
10x + 25y = 825
\]

Teacher: How?

Seema: Sir, because there are $x$ coins of ten paise each and $y$ coins of 25 paise each, therefore, the sum total of coins is sixty and their value is equal to 825 paise.

Teacher: You are very much right.

Atul: Sir, The above equations can be simplified to

\[
x + y = 60 \quad - (1) \\
2x + 5y = 165 \quad - (2)
\]

Adarsh: Sir, we can solve this system by the method of elimination by substitution.

Teacher: A good idea, try it.
Seema: From (1) we obtain

\[ Y = 60 - x \]  \hspace{1cm} (3)

Substituting this value of \( y \) in (2), we get

\[ 2x + 5 (60 - x) = 165 \]

\[ x = 45 \]

Hence, from (3) \( y = 15 \)

Thus, the lady has 45 coins of 10 paisa each and 15 coins of 25 paisa each.

Episode II
Problem: Same as in Episode I
Teacher: Let us try to solve this problem
Raman: May I try it, sir?
Teacher: You cannot solve it, therefore, I will solve it.
Vandana: Sir, it is not a difficult problem, give us opportunity to try it.
Teacher: Don't waste time. I have yet to solve many more problems. Try to follow what I tell you.
The teacher solves the problem and pupils listen to him.

How do you react to the behaviours of the teachers in the two episodes? The teacher in episode-I tried to develop the problem solving ability by creating a right situation for problem solving through the development of congenial atmosphere for problem solving.

Example 3
Episode-I
Teacher: Multiply the number 12,345,679 by 9.
Amitabh: sir, it is very lengthy.

Teacher: Right you are, it is very lengthy but at the same time the number 12,345,679 is a magic number.
Vibhush: What is magic about this number, Sir?
Teacher: O.K. If you want to know how it is a magic number, divide yourselves in five rows ( students divide themselves in five rows ). The students of first row will multiply it by 9; of second by 18; of third by 27; of fourth by 36; and of fifth by 45.

Annu (of first row): sir, 9 x 12,345,679 = 111,111,111
Amit (of second row): Sir, 18 x 12,345,679= 222,222,222
Archana (of third row): Sir, 27 x 12,345,679= 333,333,333
Vandana (of fourth row): Sir, 36 x 12,345,679= 444,444,444
Anjali (of fifth row): sir, 45 x 12,345,679= 555,555,555
Vibhu: Beautiful, really it is a magic number.

Episode II

Teacher: Multiply the number 12,345,679 by 9.
Vibhu: sir, it is very lengthy.
Teacher: I say, you will have to solve it.
Atul: sir, what is the need to make such lengthy calculations?
Teacher: If you don't want to study properly, you may
leave the class.
Atul leaves the class and the others start trying it.
Out of the above two episodes you may feel that
teacher in episode-I imparted instructions in a better way
than the teacher in episode-II. Why? Because he created a
right situation for developing the problem solving
ability by motivating the pupils.

From the above examples, it is clear that the
component 'Creating Right Situation' has the following
subcomponents:
(a) stating the problem in interesting ways.
(b) congenial atmosphere
(c) motivating the students.

To find the subcomponents of 'Analysing the
problem' let us study the following examples:

Example 4

Problem: Solve the system

\[ x + y = 5 \quad (1) \]
\[ -2x + y = 2 \quad (2) \]

Episode I:

Teacher: Well, students! In the given system of equations
we note that coefficient of \( y \) in both the equations
is the same. We can, therefore, eliminate \( y \) by
subtracting, say (2) from (1) and obtain
3x = 3
or x = 1

We now substitute this value of x in either
(1) or (2) and obtain
y = 5 - x = 4
Thus, the solution is x = 1, y = 4.

Episode II:
Problem: Same as in episode I
Teacher: From (1) and (2) we get
3x = 3
x = 1

\[ y = 4 \]
Hence, the solution.

How do you feel about the two episodes quoted above. You
may feel that teacher in episode I has tried to develop the
problem solving ability among his pupils because he rightly
analysed the problem by analysing the goals to be achieved.

Example 5
Problem: Find the greatest common divisor to 12,18 and 27.
Teacher: To solve this problem the basic issue is to
find the set of divisors of all the three numbers
and then to find their intersection.
Thus, the set of divisors of 12 = \{1,2,3,4,6,12\}
The set of divisors of 18 = \{1,2,3,\#6,9,18\}
the set of divisors of 27 = \{1,3,9,27\}
Therefore, the set of intersection of the three
The set of intersection of all the three sets consists of 1,3 only of which 3 is the greatest. Hence the greatest common divisor is 3.

Episode II
Teacher: The divisors of 12 are 1,2,3,4,6,12
The divisors of 18 are 1,2,3,6,9,18
The divisors of 27 are 1,3,9,27
Hence the greatest common divisor is 3.

In the above two episodes we see that teacher in episode I tried to analyse the basic issues underlying the problem thus helped the pupils in rightly analysing the problem. Whereas the teacher in episode II did not make such efforts thus he did not try to develop problem solving ability among his pupils.

Example 6
Problem (theorem): The sum of the three angles of a triangle is 180°

Episode I
Teacher: Well, students! We are given a triangle ABC.
We wish to prove that

\[ \angle 1 + \angle 2 + \angle 3 = 180^\circ \]

How should we proceed?

There can be two methods to prove this theorem. The one can be to draw a line \( \ell \) from \( A \) parallel to the base line \( BC \) and establish that

\[ \angle 1 + \angle 2 + \angle 3 = \text{Straight Angle at } A. \]

The other method can be to produce the side \( BC \) and draw a line \( CE \parallel AB \)

and show that

\[ \angle 1 + \angle 2 + \angle 3 = 180^\circ \]
Episode II

Teacher: "Well, students! We are given a triangle.

\[
\angle 1 + \angle 2 + \angle 3 = 180^\circ
\]

Through A, let us draw a line \( l \) parallel to BC.

The \( \angle 1 = \angle 5 \) (Alternate angles)

Also \( \angle 1 = \angle 4 \) (Alternate angles)

Thus, \( \angle 1 + \angle 3 + \angle 2 = \angle 5 + \angle 4 + \angle 2 \)

But, \( \angle 5 + \angle 2 + \angle 4 = 180^\circ \)

Thus \( \angle 1 + \angle 2 + \angle 3 = 180^\circ \)

It is clear that the teacher in episode I taught the subject effectively because he analysed the problem properly by analysing the different approaches for solving the problem.

Thus, from examples 4, 5 and 6 we get that the component 'Analysing the problem' has following subcomponents:

(a) Analysing the goals
(b) analysing the basic issues
(c) analysing the different approaches.
Now to know the subcomponents of 'Suggesting the Probable solution' let us study the following examples:

Example 7

Problem: There are 20 teachers who teach mathematics or physics in a school. Of these, 12 teach mathematics and 4 teach physics and mathematics. How many of them teach physics?

Episode I:

Teacher: If the set of teachers who teach mathematics is denoted as 'A' and set of teachers who teach physics be denoted as 'B' then we get the solution as $20 = 12 + n(B) - 4$

$\therefore n(B) = 12$

\therefore 12 teachers teach physics.

Episode II

Teacher: Let us try to find a solution to given problem. Try to recall which formula can be applied to solve it.

Aabha: sir, I fail to recall if we have done any formula for solving such problems.

Teacher: Try to recall. If the set of teachers of mathematics is denoted as set 'A' and that of physics teachers is denoted as 'B' then the set of teachers who teach mathematics or physics will be denoted by $A \cup B$. How should the set of teachers who teach both be denoted?
Archana: It should be denoted as $A\cap B$.

Aabha: Sir, now I can recall that the formula to solve this problem is
\[ n(A \cup B) = n(A) + n(B) - n(A \cap B) \]

In the above example, we observe that the teacher in episode I did not impart the instruction in an effective way. He made no attempt to suggest the probable solution by invoking recall of their previous knowledge, whereas the teacher in episode II did so.

Example 8:

Episode I:
Teacher: Students! Take note of the fact that the equation $(x + 5)^2 = x^2 + 10x + 25$ is true for all real values of $x$.

and the equation $(x + 5)^2 = x^2 + 8x + 10$ is true for only one value of $x$, whereas the equation $x^3 - 5x + 4 = 0$ is true for only two values of $x$.

Episode II:
Teacher: Well, students! For how many real value(s) of $x$ i.e. value(s) of $x$ in the set of real numbers are the following statements true?

(a) $(x + 5)^2 = x^2 + 10x + 25$

(b) $(x + 5)^2 = x^2 + 8x + 10$

(c) $x^3 - 5x + 4 = 0$
On reading the above two episodes it appeals to the reader the teacher in episode II has attempted to develop problem solving ability among his pupils by giving them an opportunity to guess the solution tentatively whereas the teacher in episode I asked them to note only.

Hence, from example 7 and 8 it is obvious that for developing the problem solving ability among the pupils a teacher should develop in himself the ability to suggest probable solution through: (a) invoking the recall of their previous knowledge and (b) letting them guess solutions tentatively.

Now let us recall that the fourth component of the skill is 'facilitating problem solving'. The following example will tell about its subcomponents.

Example 9

Episode I

Teacher: If A and B are any two sets, then the number of elements in the union set of A and B is equal to number of elements in the set A plus the number of elements in the set B minus the number of elements in the intersection set of A and B.
In mathematical language this can be written as $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

In the above example which of the two teachers is trying to develop the problem solving ability in his pupils more effectively? We are right, if our answer is teacher in episode II, because by making the pupils familiar with mathematical language a teacher develops among them the problem solving ability.

Example 10

Episode I

Teacher: Well, students! Today we shall learn the algebra of sets. The operations of union and interactions on sets have similarity with the operations of additions and multiplications on real number. We will note that many of the basic operations which hold for addition and multiplication of real numbers also hold for set operation. At the same time, there are some properties of operations that hold for real numbers but do not hold for sets and, of course, vice versa. We state below some of such relations. If $A$, $B$, $C$ are subsets of some set $Y$, then the following relations hold:

(1) Commutative Property
Episode II

Teacher: Students! Today we shall learn the algebra of sets. Note that it has following properties:

(1) (a) $A \cup B = B \cup A$
(b) $A \cap B = B \cap A$

(2) Associative property

(a) $(A \cup B) \cup C = A \cup (B \cup C)$
(b) $(A \cap B) \cap C = A \cap (B \cap C)$

In example 10 it is obvious that the teacher in episode I has tried to develop understanding of mathematical relationship so as to facilitate the problem solving.

Example 11.

Episode I

Teacher: Well, students! today we shall learn about 'similar figures'. Figures of 'same shape' but not necessarily of the 'same size' are said to be similar. It is obvious that two congruent figures are similar but the converse is not necessarily true, namely, two figures need not necessarily be congruent. Any two equilateral
triangles are similar, any two circles are similar, any
two squares are similar and any two segments are similar.

Episode II.

Teacher: Well, students! today we shall learn about
'similar figures'. Figure of 'same shape', but
not necessarily of the 'same size' are said to be
similar. It is obvious that two congruent
figures are similar but the converse is not
necessarily true, namely, two figures need not
necessarily be congruent. Any two equilateral
triangles are similar, any two circles are
similar, any two squares are similar and any two
segments are similar.

Let us place a lighted bulb at a point O
on the ceiling, and directly below it a table
in the room. Let us hold a plane figure, say a
triangle ABC, in a plane parallel to the plane
of the table and between the light and the
table. Then a shadow A'B'C' of ABC is cast on
the table. We note the shadow A'B'C' is a triangle.
In example 11 the teacher in episode II taught the lesson in more effective way than in episode I. Why? Because the teacher in episode I tried to develop problem solving ability among his pupil by facilitating the process of problem solving through the use of problem solving material.

Thus, the three points to be remembered for facilitating the process of problem solving are:

(a) development of mathematical language.
(b) development of understanding of mathematical relationship.
(c) using the problem solving material.

Now let us try to understand the last component of this skill that is 'checking the reasonableness of answer.'
Example 12.

Problem: The average speed of a car is 40 kilometers per hour and it takes six hours to travel from Chandigarh to Delhi. How far is Delhi from Chandigarh?

Episode I

Teacher: Solve the problem and let me know the answer.

Archana: Sir, the answer is 2,400 kms.

Teacher: No, you are wrong. The answer is 240 kms.

Episode II

Teacher: Try to solve the problem and let me know the answer.

Anil: Sir, the answer is 2,400 kms.

Teacher: Try to see if your answer is a reasonable answer.

Vibhu: Sir, how to check whether the answer is reasonable.

Teacher: If you go 40 kms. in an hour, you will go a little less than 100 kms. in two hours. Thus you will go less than 300 kms. in six hours. Now compare this rough estimate with your answer i.e. 2,400 kms. This comparison reveals the unreasonableness of your answer.

Thus, it is seen in the example 12 that the teacher in episode II has tried to develop problem solving-
ability among his pupils through checking the reasonableness of answer.

Thus, we have learnt various components and sub-components of the skill of developing problem solving ability among pupils. Now let us work on the exercises given below to test our understanding. If we want to check whether our answer is correct, we may refer back.

**EXERCISES**

1)  (a) What do you understand by the term problem?

   (b) What do you mean by problem solving?

   (c) What is use of developing problem solving ability among pupils?

2)  (a) What do you understand by 'creating right situation for problem solving'?

   (b) How will you create right situation for developing problem solving ability among your pupils? Give examples.

3)  What do you understand by analysing the problem? What are its sub-components?

4)  What is role of checking the reasonableness of answer in developing problem solving ability among pupils?

5. ) By what means the process of problem solving can be facilitated?
OBSERVATION PROFORMA FOR THE SKILL OF DEVELOPING PROBLEM SOLVING ABILITY

Name of the student teacher ___________ Roll No. ________
Topic ______________________________ Class _____________
Name of the supervisor ________________
Date ___________ Time duration ___________ Teach/Reteach_____

A glossary of the terms used in the observation schedule is given below:

Creating Right Situation:

Creating right situation means stating the problem in an interesting way, creating congenial atmosphere for problem solving and motivating the pupils to develop ability to solve problems.

Analysing The Problem:

The analysis of the problem refers to analysis of goal, analysis of basic issues underlying the problem, and analysis of different approaches for solving the problem.

Suggesting Probable Solution:

The probable solutions can be suggested by invoking recall of previous knowledge and guessing solution tentatively.
Facilitating problem Solving:

This involves developing mathematical language, developing understanding of mathematical relationships, and using problem solving material.

Checking The Reasonableness Of Answer:

This means to check the answer with the help of estimations and to see whether the answer is reasonable.

INSTRUCTIONS:

Mark the tallies in the appropriate cells as they occur during the lesson.

**COMPONENTS:**

Creating right situation
Analysing the problem
Suggesting probable solution
Facilitating problem solving
Checking reasonableness of answer.

**TALLIES:**

Comments (if any):
OBSERVATION PROFORMA FOR THE SKILL OF DEVELOPING PROBLEM SOLVING ABILITY:

Name of the student teacher..................Roll No........
Topic ..................................Class........
Name of the supervisor..........................
Date ............Time duration........Teach/Reteach

INSTRUCTIONS:

This proforma is meant to ascertain the extent to which the student teacher uses the skill, namely, developing problem solving ability. Judgments have to be given on a seven point scale for various aspects of skill. Indicate the extent of acquisition of various aspects of the skill by putting a cross (x) on the appropriate number you deem fit. The scale value '1' indicates that the student teacher did not use the concerned aspect (S) of the skill at all, whereas the scale value '7' means that the student teacher used/practised the skill aspect (S) very much. Keeping these two extremes in view, examine carefully the teacher behaviour related to the given aspects of the skill and rate them appropriately.
<table>
<thead>
<tr>
<th>COMPONENTS</th>
<th>NOT AT ALL</th>
<th>VERY MUCH</th>
</tr>
</thead>
<tbody>
<tr>
<td>The teacher created a right situation for problem solving by stating the problem in an interesting way, creating atmosphere cognenial to problem solving and by motivating them.</td>
<td>1 2 3 4 5 6 7</td>
<td></td>
</tr>
<tr>
<td>The teacher analysed the problem properly by way of analysing the goal to be achieved, analysing the basic issues underlying the problem and analysing the different approaches for solving the problem.</td>
<td>1 2 3 4 5 6 7</td>
<td></td>
</tr>
<tr>
<td>The teacher suggested probable solution by invoking recall of their previous knowledge and guessing solution tentatively.</td>
<td>1 2 3 4 5 6 7</td>
<td></td>
</tr>
<tr>
<td>The teacher facilitated the problem solving by his attempt to develop mathematical language, develop mathematical relationships and using problem solving material.</td>
<td>1 2 3 4 5 6 7</td>
<td></td>
</tr>
<tr>
<td>The teacher checked the reasonableness of answer.</td>
<td>1 2 3 4 5 6 7</td>
<td></td>
</tr>
</tbody>
</table>

Comments (if any).
MODEL LESSON FOR THE SKILL OF
DEVELOPING PROBLEM SOLVING ABILITY

Subject: Mathematics
Class: IX

Topic: Two Simultaneous Linear Equations

Problem: In a two-digit number, the sum of its digit is 8. If the number is subtracted from the one obtained by interchanging the digit, the result is 18. What is the number?

Teacher: Well, students! The given problem can be re-stated in an interesting way as: The age of a man is a two digit number such that the sum of these two digit is 8. The age of his father is obtained by interchanging the two digits. The difference between his father's age and his age is equal to his son's age. Find the age of the man if his son is 18.

Now let us try to solve the problem. Who will help me.

Anil: Sir, a two-digit number has a unit's place and a ten's place.

Annu: Let, \( x \) represent the digit in the unit's place and \( y \) represents the digit in the ten's place.

Then the first sentence translates as
\[
y + x = 8 \quad (1)
\]

The value of the number is
\[
10y + x
\]
Teacher: How?
Vibhu: Sir, because y is at ten's place so its every unit is equal to ten units at unit's place.
Teacher: Very good. Right you are.
Vandana: Sir, if we interchange the digits, x becomes the digit in ten's place and y the digit in the unit's place. The value of the number then is $10x + y$.
Arun: Sir, the second sentence translates as $10x + y - (10y + 8) = 18$
Or $9x - 9y = 18$
Or $x - y = 2$ (2)
Teacher: Thus, we have two simultaneous equations in two unknowns, namely, (1) and (2). Hence the whole problem reduces to solving these two equations simultaneously.
Anjali: From (2) we get $x = y + 2$ (3)
Putting this value of $x$ in (1) we have $y + y + 2 = 8$
Or $2y + 2 = 8$
Or $2y = 6$
Or $y = 3$
Now from (3) we get

\[ x = 3 + 2 \]
\[ = 5 \]

Amit: Sir, the age of the man, therefore, is 8.

Teacher: Amit! Just check whether your answer is a reasonable answer. Is your answer a 'two digit number'.

Amit: No, Sir. It is one digit number

Teacher: Then what should be the right answer.

Amit: Sorry, sir. The answer should be 35 and not 8.

Teacher: Now you are right. This answer seems to be a reasonable answer.

Teacher: Let us check the answer.

By putting the value of \( x \) and \( y \) in (1) we get

\[ 3 + 5 = 8 \]
\[ 53 - 35 = 18 \]

Both these results are in conformity with the conditions given in the problem. Hence the result is alright.
CONCLUDING REMARKS

Now it is hoped that you have understood the meaning of the skill of developing problem solving ability. Examples, exercises, observation proforma and model lessons are provided. This should help you to prepare your microlesson for practice teaching. It is hoped that you would gain competence in the skill of developing problem solving ability. Try.
SKILL OF FORMULATING MATHEMATICAL MODELS

Introduction:

Formulation of Mathematical Models is an important technique to be mastered by pupils of Modern Mathematics. In order to be an effective teacher of Modern Mathematics one must learn the skill of Formulating Mathematical Models. The working definition of mathematical model has been taken as "The Consistent Collection of statements dealing with space, time, quantity and relationship is called a mathematical model".

The main components of the skill are given below:

1 Formulation
   (a) Mathematical correctness
   (b) Conciseness
   (c) Relevancy to purpose
   and (d) Specificity
2 Fluency in Formulation
   (a) Clarity about the Fundamentals
   (b) Clarity about the Goal.
3 Evaluating Results
   Let us study these components in detail

1 Formulation:

"Formulation" refers to the process part of the mathematical model taken together with purpose of the formulation of the model. Various criteria for a well
formulate mathematical model are (i) mathematical correctness (ii) Conciseness, (iii) Relevancy, and (iv) Specificity. Each criterion is discussed below:

(a) Mathematical Correctness

It is desirable that the teacher should formulate mathematical models which strictly adhere to the principles of Modern Mathematics. If the formulation of models is based on principles which do not adhere to the principles of Modern Mathematics, the teacher may arrive at wrong and contradictory conclusions. This will create confusion in the mind of pupils. It, therefore, necessitates that the teacher should be careful while formulating mathematical models. Example for mathematically wrong model is given below:

An empty set is a set having no element. Because '0' has no value; therefore, the set comprising only 0 and denoted as \{0\} is an example of empty set.

The above model can be formulated correctly as follows:

An empty set is a set having no element. The example of an empty set is \{\}. The set \{0\} can't be called an empty set because it has one member, viz., 0.

(b) Conciseness

Conciseness refers to the size of the mathematical model. A model is said to be concise when it does not have redundant words or statements. It should consist of direct and necessary statements only. The example of a
A mathematical model, which is not concise, is given below:

There are 40 teachers in a government school called Govt. Hr. Sec. School, Charkhi Dadri. Out of these 40 teachers in that school 20 teachers teach Mathematics or Physics. Out of these 20 teachers, 12 teach Mathematics and 4 teach Physics and Mathematics. Out of 20 how many of them teach Physics?

The concise model can be:

There are 20 teachers, in a school, who teach Mathematics or Physics. Of these, 12 teach Mathematics and 4 teach both Physics and Mathematics. How many of them teach Physics?

Thus, we find that in the first Model the teacher used many extra words and statements. The attention of students is diverted and much of the time is wasted in speaking and writing these extra words. These words and sentences do not convey much meaning. Time could have been saved by framing statements which are short and to the point. This would increase the fluency of formulation of models. Therefore, the teacher should frame concise models.

(C) Relevancy

An irrelevant model is that which is not related to the topic in hand or to the purpose for which the model is originally meant for. A model is also irrelevant when it contains concepts which have not been established earlier.
(Teacher is teaching a lesson on Mapping)

Teacher: Let $A$ be a set of boys in a class. With each boy we may associate his age in years. We thus have a correspondence or mapping between the set $A$ of boys and set $B$ of numbers. We abbreviate this as:

$$f : A \rightarrow B \quad f(a) = \text{age of } a, \quad a \in A \quad f(a) \in B.$$  

This correspondence $f : A \rightarrow B$ is called function.

When we draw the map of a country, we, in fact, define a mapping of the cities of the country to certain points on the map. In roster method of representing a set we list the members of the set, separate them by commas and enclose them within braces.

The last part in which a reference has been made to roster method is irrelevant because it is neither related to the topic in hand nor to the purpose for which it was originally meant for. Therefore, such models should be formulated which contain concepts which have already been established. The models should contain matter related to the topic and they should be according to mental horizon of the pupils for whom the models are formulated.
(d) Specificity

Another criterion for a well formulated model is that it is specific. A specific model calls for a single concept at a time. It increases the clarity and fluency in formulation of models. Example for this type of model is given below:

\[ x \in (A')' \Rightarrow x \notin A \quad (i) \text{(By def. of complementary set)} \]

\[ \Rightarrow x \in A \quad (ii) \text{(By definition of complementary set)} \]

\[ \therefore (A')' \subseteq A \quad (iii) \text{(From i and ii)} \]

Also \[ x \notin A \Rightarrow x \notin (A')(iv) \text{(Complementary set)} \]

\[ \Rightarrow x \in (A')' \quad (v) \text{(By def. of complementary set)} \]

\[ \therefore A \subseteq (A')' \quad (vi) \text{(From iv and v)} \]

Hence \((A')' = A \quad (From iiii & vii)\)

Here, we observe that the model centres around one idea— the 'complementary sets' and thus fulfills the criteria of calling for a single concept or idea at a time. On the other hand, if the teacher formulates models which are general in nature, they will take longer time to formulate. Example for a general model is given below:

A well defined collection of objects is called a set. A relation with three properties, viz., reflexivity, symmetry and transitivity is called an equivalence relation. Set consisting of finite numbers of elements is called a finite set.

In the above model, it is not clear what specifically the teacher wants to convey to his pupils. This model, thus seems to be a general or vague model.
2 Fluency in Formulation

By 'fluency in formulation' we mean the rate of formulation of models per unit of time. This component of the skill of formulating Mathematical Model in turn has the following sub components:

(a) Clarity about the fundamentals
(b) Clarity about the Goal

Now, let us study these components one by one:

(a) Clarity about the Fundamentals:

This refers to clarifying the fundamentals underlying a mathematical model. Below is given an example in which the teacher clarifies the fundamentals underlying the formulation of a mathematical model:

**Topic: Cartesian product of Sets**

Teacher: The set consisting of all ordered pairs \((a,b)\) when \(a \in A, b \in B\) is called the Cartesian product of the sets \(A\) and \(B\) and is denoted by \(A \times B\).

(After a brief pause, he enquires from the students whether they have followed. There is a positive response to his enquiry. Then he asks the pupils to formulate a mathematical model of relationship between \(n(A \times B)\) and \(n(B \times A)\) where \(n(A \times B)\) and \(n(B \times A)\) stand for the number of elements in Cartesian product \(A \times B\) and \(B \times A\) respectively).

He, then, gives sufficient time to pupils to think and then points towards Vibhu to know the mathematical model she has formulated.
Vibhu: The number of elements in cartesian product $AXB$ is same as the number of elements in $BXA$ because the change in order does not affect the number of elements.

(b) Clarity about the Goal

Clarity about the goal means the knowledge given to the pupils by the teacher about the nature and type of mathematical model to be formulated. The clarity about the goal helps in increasing the frequency of formulation of mathematical models. If the pupils are not clear what type of product they are required to produce, their output will naturally be very slow. Let us view the following situation:

Teacher: Well, students! Formulate a mathematical model of set Theory.

Pupils: Look towards the face of the teacher strangely. They fail to formulate any mathematical model.

This was expected to happen because the teacher did not make the students clear about the goal of the model. To get a correct model from the pupils the teacher should have told the students to formulate a mathematical model of union of two sets or intersection of two sets or any other specific item.
3 Evaluating Result

The term 'Evaluating Result' refers to evaluating the competence gained by the pupils in formulating mathematical models. Many a time it happens that nothing is wrong with formulation and frequency part of the skill, still the competence gained by the pupils in formulating mathematical models is not notable. Several reasons may be given for such a situation. Some of the important reasons are enumerated below:

1. The pupils may not be taking interest in the lesson.
2. They may lack the previous knowledge.
3. They may not be intelligent enough to formulate mathematical models independently.
4. The rapport between the teacher and pupils may be lacking.
5. The above reasons when combined together might result in making them inattentive in the class.

In such situations, the teacher should re-examine and find underlying causes for poor gain in competence by the pupils in formulating Mathematical Models. Once he has diagnosed rightly the cause, he can find out the remedies for that particular type of behaviour of the pupils.

We have discussed above the three aspects of formulation of mathematical models, viz., 1. Formulation,
2. Fluency in Formulation, and 3. Evaluating Results. But the third component i.e. Evaluating Results is not observed as sometimes this component is governed by phenomenon beyond the control of the teacher. Therefore, the observation proforma comprises only the first two components, viz., formulation and Fluency in Formulation.

Exercises:
1. Give two examples for each of the following:
   
   (a) Mathematically correct models
   Your response (i)
   (ii)

   (b) Concise Mathematical models
   Your response (i)
   (ii)

   (c) Relevant mathematical models
   Your response (i)
   (ii)
(d) Specific Models
Your response (i)

(ii)

Evaluate the following mathematical models from the viewpoint of 'Formulation'.

(a) Define congruence of two geometrical figures

Your response

(b) To divide a number by another, we divide the logarithm of the divisor by the logarithm of the dividend and find the antilogarithm of the result.

Your response

(c) (The teacher is teaching the topic 'Cartesian product of Sets')
If A, B, C are subsets of some set X, then the following results hold:

1 $n(AXB) = n(BXA)$
2 $AUB = BUA$
3 $(AUB) UC = AU(BUC)$

Your response
(d) If \(A, B\) are two disjoint sets, then
\[ n(A \cup B) = n(A) + n(B) - n(A \cap B) \]

Your response

(e) Give mathematical model of sets and mapping.

Your response

(f) Define Logarithm of a number.

Your response

3 Evaluate the following situations from the point of view of 'Fluency in Formulation' situation I.

Teacher: Well, students! In order to draw the graph, it is enough to determine two distinct points that satisfy the given equation. As a precautionary measure, however, we always determine an additional ordered pair that satisfies the given equation and see if the corresponding point lies on the line. After talking this much the teacher asks the students to draw a graph of \(x=1\).

Vivek: The equation is clearly satisfied by the ordered pairs in which the first entry is fixed to be 1 and the second is any real number, for instance \((1,0),(1,1),(1,2),(1,-1),(1,-2)\) etc. The graph
Situation II
Teacher: Prepare some graphs
Alice: Graph of what type, sir?
Suman: Sir, trigonometrical graphs or algebraic graphs?
Vivek: What about the data, sir? Is it to be hypothetical or real?

Your response _______________________

Situation III
Teacher: Draw a mathematical model of the statement—An exterior angle of a triangle is always greater than either of the interior opposite angles.
Vibhu: Sir, you never told us what this exterior angle is?
Neeru: Sir, what is the sum of the three angles of a triangle?

Your response _______________________

\[ Y \text{, therefore, is } 0^* \]

\[ \{ I \} \]

\[ i \]

\[ \text{Your response } \] \[ \]
OBSERVATION SCHEDULE FOR THE SKILL OF FORMULATING
MATHEMATICAL MODELS

Name of the student teacher .................. Roll No. ...........
Topic ........................................ Class ...............
Name of the Supervisor ..............................
Date........... Time duration ........ Teach/reteach ............

A glossary of the key terms used in the schedule
is given below:

Mathematical Model

The collection of statements expressing consistent
ideas dealing with space, time, quantity and relationship
is called a mathematical model.

Formulation

'Formulation' skill refers to the process part
of formulation of mathematical model taken together with
the purpose of its formulation.

Various criteria for a well formulated model are:

(a) Mathematical Correctness: It refers to for­
mulation of mathematical model which strictly
adheres to the principles of Modern Mathe­
matics.

(b) Conciseness: A model is said to be concise when
it does not have redundant words or sentences.

(c) Relevancy: A model is said to be relevant when
it is related to the topic in hand or to the
Fluency in Formulation

By 'Fluency in Formulation' we mean the rate of formulation of models per unit of time. This component of the skill of 'Formulating Mathematical Model' in turn has the following sub-components:

(a) Clarity about the Fundamentals- This refers to clarifying the fundamentals underlying a mathematical model.

(b) Clarity about the Goal- This refers to knowing the nature and type of mathematical model to be formulated.

Instructions: Mark the tallies in the appropriate cells as they occur during the lesson.
<table>
<thead>
<tr>
<th>Components</th>
<th>Tallies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematical correctness</td>
<td></td>
</tr>
<tr>
<td>Conciseness</td>
<td></td>
</tr>
<tr>
<td>Relevancy</td>
<td></td>
</tr>
<tr>
<td>Specificity</td>
<td></td>
</tr>
<tr>
<td>Clarity about the fundamentals</td>
<td></td>
</tr>
<tr>
<td>Clarity about the goal</td>
<td></td>
</tr>
</tbody>
</table>

Comments (if any)
OBSERVATION SCHEDULE FOR THE SKILL OF FORMULATING
MATHEMATICAL MODELS

Name of the student teacher.................. Roll No........
Topic........................................ Class..................
Name of the supervisor............................
Date................ Time duration........ Teach/ Reteach.....

Instructions

This proforma is meant to ascertain the extent to which the student teacher exhibits or uses the skill, namely, skill of formulating mathematical models. Judgments have to be given on a seven-point scale for various aspects of the skill. Indicate the extent of acquisition of the various aspects of the skill by putting (X) under the appropriate number you deem fit. The scale value '1' indicates that the student teacher did not use the concerned aspect(s) of the skill at all, whereas the scale value '7' means that the student teacher used/practised the skill aspect(s) very much. Examine carefully the teacher behaviour related to the various given aspects and put (X) under the appropriate scale value.
<table>
<thead>
<tr>
<th>Components</th>
<th>Not at all</th>
<th>Very much</th>
</tr>
</thead>
<tbody>
<tr>
<td>Models formulated are mathematically correct</td>
<td>1 2 3 4 5 6 7</td>
<td></td>
</tr>
<tr>
<td>Models formulated are concise</td>
<td>1 2 3 4 5 6 7</td>
<td></td>
</tr>
<tr>
<td>Models formulated are relevant</td>
<td>1 2 3 4 5 6 7</td>
<td></td>
</tr>
<tr>
<td>Models formulated are specific</td>
<td>1 2 3 4 5 6 7</td>
<td></td>
</tr>
<tr>
<td>Fundamentals were clarified properly</td>
<td>1 2 3 4 5 6 7</td>
<td></td>
</tr>
<tr>
<td>Goal was clarified properly</td>
<td>1 2 3 4 5 6 7</td>
<td></td>
</tr>
</tbody>
</table>

Comments (if any)
Teacher: If \( A = (a, b, c) \) and \( B = (1, 2) \), then the set consisting of all ordered pairs whose first element belongs to set \( A \) and second element belongs to set \( B \) is called the cartesian product of \( A \) and \( B \) and is denoted by \( A \times B \). In the present example \( A \times B = (a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2) \).

A sub-set of set of cartesian product is called a 'Relation'. In the above example all the sub-sets of \( A \times B \) are examples of 'Relation'. In simple words, from a relation we mean 'connecting the members of one set to the members of another set', e.g. mutually perpendicular.

If \( l, m, n \) are three lines perpendicular to \( p \), then \( (l, p), (m, p), (n, p) \) is such a set in which the first element of an ordered pair is connected with the second by the rule "mutually perpendicular".

The symbol \( R \) is used to represent 'Relation Set' using the characteristic method of writing sets the above written set can be written as:
More examples are:

Example I. say, $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3, 8\}$

If $R = \{(x, y) : x \in A \land y \in B \land x < y\}$

then in roster method of writing:

$R = \{(1,2), (1,3), (2,3), (1,8), (2,8), (3,8), (4,8), (5,8)\}$

This is clear that $R$ is a sub-set of $A \times B$ and it contains only those members which satisfy the condition $x < y$, so $R \subset A \times B$

Example II. say, $A$ is a set of men: $A = \{M, N, P\}$ and $B$ is a set of children:

$B = \{m_1, m_2, n_1, p_1, p_2, p_3\}$ ; $M$ is father of $m_1, m_2$, $N$ is father of $n_1$, $P$ is father of $p_1, p_2, p_3$.

If $R = \{(x, y) : x \in A \land y \in B \land x$ is father of $y\}$

then in roster method of writing

$R = \{(M, m_1), (M, m_2), (N, n_1), (P, p_1), (P, p_2), (P, p_3)\}$

In this example also it is very much clear that $A \times B$ contains more ordered pairs than contained in $R$. 