In first chapter we have seen different general convergent series. But the question arises whether it is possible to accelerate the convergence of series. Its answer is affirmative; we can accelerate the convergence of series by using PMS.

P. A. Amore and R. Saenz, (see [4] & [5]) used PMS in perturbation theory. In addition, they used it to accelerate the convergence of some special series. We discuss the question if the PMS can be used to accelerate the general convergent series and other series that arise indifferent fields like the perturbation theory, biomathematics etc.

In this chapter, we deal with the problem of accelerating the convergence of slowly convergent series where a large number of terms are need to reach desired accuracy. This problem has many applications in physical problems. At this point, we know that perturbation theory often gives series, which converge very slowly or do not converge at all.
Therefore, to solve the problem we use PMS to accelerate some class of mathematical series, which do not rely on perturbative approach, which is an expansion in some small natural parameter. The method works by introducing an artificial dependence in the formulas upon an arbitrary parameter. And using this artificial dependence we can optimize faster rate of convergence.

Here, we will show that such a technique can be used to obtain exponentially convergent series for some mathematical function. A proof of convergence of the method is also provided. In this chapter, we denote the sum of the series \( \sum_{n=1}^{\infty} \left( \frac{a}{an-b} - \frac{c}{cn-d} \right) \) [27] by \( K(a,b,c,d) \).

As we know, many mathematical constants like \( \pi, e \), \( G \) – Catalan constant can be express infinite sums. In many cases, such a series converge very slowly and huge number of terms has to calculate before reaching desired accuracy. In mathematics many series sums to \( \pi \). Here we see an example of infinite series which sums to \( \pi \).

First, of all we consider the series of [27],

\[
S = 8 \sum_{n=1}^{\infty} \left[ \frac{1}{8n-6} - \frac{1}{8n-2} \right]
\]

, which converge very slowly to \( K(8,6;8,2) = \pi \) using [27].

Flajolet and Vardi [13] have shown that it is possible to convert series such as in above equation (1) into rapidly converging ones. Here, we generalize the method of Flajolet and Vardi. We introduce an arbitrary parameter in the series. Such a parameter, then tuned to accelerate the convergence of the series it-self by using Principle of Minimal Sensitivity (PMS) [4], as in following results.

Moreover, in this chapter we generalize Lerch Zeta function as G-Lerch Zeta function and H-Lerch zeta function. Then, check it is convergent and give its relation with other functions. Afterwards, derive its some results and see
ACCELERATION OF SOME SERIES

special case. At last, we will see some application of acceleration of series in the different field.

First, we start with the series

\[
S = \sum_{n=1}^{\infty} \left( \frac{1}{8n-6} - \frac{1}{8n-2} \right)
\]

Which is converges to \(K(8,6;8,2)\)

For the sake of completeness, we will introduce an arbitrary parameter \(\lambda\) and obtain its parametric representation in the next theorem [4].

2.1 Main Results on Acceleration of Series

Theorem 2.1.1:

The series

\[
S = 8 \sum_{m=1}^{\infty} \left( \frac{1}{1+\lambda} \right)^{m+1} \sum_{k=1}^{m} \binom{m}{k} \lambda^{m-k} \frac{6^k - 2^k}{8^{k+1}} \zeta(k+1)
\]

, converge to \(\pi\) for \(\lambda > -1\), with \(\lambda\) real.

Proof:

We can rewrite the series of equation (1) in the equivalent form,

\[
S = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{1+\lambda} \left( \frac{1}{1 - \frac{6/8n + \lambda}{1 + \lambda}} - \frac{1}{1 - \frac{2/8n + \lambda}{1 + \lambda}} \right)
\]

, where \(\lambda \neq -1\) is an arbitrary parameter.

Here, provided that \(\left|\frac{6/8n + \lambda}{1 + \lambda}\right| < 1\) and \(\left|\frac{2/8n + \lambda}{1 + \lambda}\right| < 1\) for all \(n \geq 1\).

This implies, \(\lambda > -1\) for all \(n \geq 1\).
Therefore, we can expand equation (2) as follows for \( \lambda > -1 \),

\[
S = 8 \sum_{n=1}^{\infty} \frac{1}{8n} \sum_{m=1}^{\infty} \left( \frac{1}{1 + \lambda} \right)^{m+1} \sum_{k=1}^{m} \binom{m}{k} \lambda^{m-k} \left[ \left( \frac{6}{8n} \right)^k - \left( \frac{2}{8n} \right)^k \right]
\]

As the above series (3) in \( m \) and \( n \) contains only positive terms, we can perform the series over \( n \) and obtain the following result,

\[
S = 8 \sum_{m=1}^{\infty} \left( \frac{1}{1 + \lambda} \right)^{m+1} \sum_{k=1}^{m} \frac{m}{k} \lambda^{m-k} \frac{6^k - 2^k}{8^{k+1}} \zeta(k+1).
\]

Now, we can write our series in equation (1) as

\[
S = 8 \sum_{n=1}^{\infty} \left[ \frac{1}{8n-6} - \frac{1}{8n-2} \right] = 4 \sum_{n=1}^{\infty} \left[ \frac{1}{4n-3} - \frac{1}{4n-1} \right]
\]

But in Vardi’s book [41, p. 159] gives an example Gregory’s series for \( \pi \),

\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \sum_{n=1}^{\infty} \left[ \frac{1}{4n-3} - \frac{1}{4n-1} \right]
\]

However, the rearrangement of the above series gives following result [41, p. 159].

\[
\pi = \sum_{m=1}^{\infty} \frac{3^m}{4^m} - 1 \zeta(m+1)
\]

**Remark 2.1.2:**

The series of equation (2) for \( \lambda = 0 \) coincides with the result of Flajolet and Vardi [16],

\[
S^{(FV)} = \sum_{m=1}^{\infty} \frac{3^m}{4^m} - 1 \zeta(m+1)
\]

Moreover, the series in above equation (5) converges to \( \pi \).
Here, we observe that equation (2) define a family of series converging to $\pi$ with $\lambda > -1$. Clearly, the dependence upon $\lambda$ in equation (2) is artificial and shows up when a finite number of the terms is considered. However, we know that the series in equation (4) is the equivalent form of series in equation (2). So, we can set $S_N(\lambda)$ to be the partial series of equation (4). Then the dependence upon $\lambda$ in $S_N(\lambda)$ disappears in the $\lim N \to \infty$.

For, fixed $N$ we evaluate the partial sum at the point where,

$$\frac{d}{d\lambda}(S_N(\lambda)) = 0,$$

, where $S_N(\lambda) = 8\sum_{m=1}^{N} \left( \frac{1}{1+\lambda} \right) \sum_{k=1}^{m} \binom{m}{k} \lambda^{m-k} \frac{6^k - 2^k}{8^{m-k}} \zeta(k+1)$

, since there the expression is less sensitive to change of the arbitrary parameter $\lambda$, a property which shares with whole series (4). This called Principle of Minimal Sensitivity (PMS). This provides an equation, which ones solved at the given order, provide an optimal value of $\lambda$ for a fixed partial sum $S_N(\lambda)$.

For $N = 2$, we obtain $S_2(\hat{\lambda})$

$$\lambda_i = -3 \frac{3}{\pi^3} \zeta(3) \approx -0.365381 > -1.$$  

**Remark 2.1.3:**

The series of equation (4) converges geometrically to $\pi$. We can estimate the rate of convergence by approximating the $m^{th}$ term in the series with

$$S_m = \left( \frac{1}{1+\lambda} \right) \sum_{k=1}^{m} \binom{m}{k} \lambda^{m-k} \frac{6^k - 2^k}{8^{m-k}} \zeta(k+1)$$
Therefore,

\[ S_m = \frac{1}{(1+\lambda)^{m+1}} \left[ \left( \lambda + \frac{6}{8} \right)^m - \left( \lambda + \frac{2}{8} \right)^m \right] \approx m \left( \lambda + \frac{6}{8} \right) \frac{6}{1+\lambda} \]  

Now, using the PMS value if equation (6) we obtain \( S_m \approx 1.65^{-m} \). This improves the rate \( S_m \approx 1.33^{-m} \) of the series of Flajolet and Vardi (5).

In Fig-1 we display first 100 terms of \( m^{th} \) term \( S_m \) when \( S_m \approx 1.33^{-m} \) (dot) for \( \lambda = 0 \) and \( S_m \approx 1.65^{-m} \) \( \lambda \) for \( \lambda = -0.365381 \). Then, in Fig-2, we plot...
partial sums of equation (4) up to 100 terms for $\lambda$ (dot) and $\lambda_i$ (line). In above Figure the vertical lines turn out that $\lambda_i$ is a very good approximation to the minimum of partial sum even for large values of terms.

![Fig-3](image1)

Next, in Fig.-3 we plot the error $\delta$ obtained in each terms up to 100 terms of equation (4) with $\lambda$ given by equation (7) and by using formula of Flajolet and Vardi, equation (6). Moreover, in last Fig-4, we display the error $\delta$ obtained in each partial sums up to 100 terms of equation (4) with $\lambda$ given by equation (7) and by using formula of Flajolet and Vardi, equation (6). Our series converges exponentially more rapidly than series in equation (6).
Now, we consider another series as follows:

(8) \[ S = \sum_{n=1}^{\infty} \left( \frac{1}{(8n-6)^2} - \frac{1}{(8n-2)^2} \right) \]

, which is slowly converge to \( \frac{G}{4} \), where \( G \) is the Catalan constant \( G \approx 0.9159656 \).

Theorem 2.1.4:
The series defined as

(9) \[ S = \sum_{m=1}^{\infty} \left( \frac{1}{1 + \lambda} \right) \sum_{k=1}^{m} \lambda^{k-1} \frac{6^{k-1} - 2^{k-1}}{8^k} k \zeta(k+1) \]

Converges to \( \frac{G}{4} \), where \( G \) is the Catalan Constant, \( G \approx 0.9159656 \) for any \( \lambda > -\frac{1}{2} \), with \( \lambda \) real.

The above result easily derived from the result of P. A. Amore of [4].

Here, note that the series of Theorem 4 reduces to the formula for \( \lambda = 0 \) as

(10) \[ S^{(FR)} = \sum_{m=1}^{\infty} \left( \frac{3^{m-1} - 1}{4^{m-1}} \right) m \zeta(m+1). \]

Next, we generalize Theorem 1 for any integer \( a > 3 \),

(11) \[ S = a \sum_{n=1}^{\infty} \left[ \frac{1}{an-3} - \frac{1}{an-1} \right] \]

, which converge to \( K(a,3;a,1) \) of [27].

In addition, the parametric representation of above series (11) obtained in next theorem.
Theorem 2.1.5:
The series
\[ S = \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{1}{1 + \lambda} \binom{m}{k} \left( \frac{3^k - 1}{a^{k+1}} \right) \zeta(k+1) \]

Converge to \( K(a,3;a,1) \) for \( \lambda > -\frac{1}{2} \), with \( \lambda \) real.

We can prove this theorem similar way as above theorem 2.1.1. Then after, in next remark we will see a special case of the series given by (12).

Remark 2.1.6:
For \( \lambda = 0 \), the above Theorem 2.5 reduces to the series
\[ S = \sum_{n=1}^{\infty} \frac{3^n - 1}{a^n} \zeta(m+1) \]
which is convergent for \( a > 3 \).

For any integers \( a, b \) and \( c \) with conditions \( a > b, \ a > c \) and \( b \neq c \), we can generalize the series in (11) as
\[ S = a \sum_{n=1}^{\infty} \left[ \frac{1}{an-b} - \frac{1}{an-c} \right] \]
which converge to \( K(a,b;a,c) \) of [27].

Therefore, in next theorem gives generalization of theorem 2.5.

Theorem 2.1.7:
The series
\[ S = \sum_{n=1}^{\infty} \sum_{k=0}^{n} \left( \frac{1}{1 + \lambda} \right) \binom{m}{k} \left( \frac{b^k - c^k}{a^{k+1}} \right) \zeta(k+1) \]
converge to \( K(a,b;a,c) \) for \( \lambda > -\frac{1}{2} \), with \( \lambda \) real.
Remark 2.1.8:
For \( \lambda = 0 \), the above Theorem 2.1.7 reduces to the series

\[
S = \sum_{n=1}^{\infty} \frac{b^n - c^n}{a^n} \zeta(m+1)
\]

, which is convergent for any integers \( a, b \) and \( c \) with conditions \( a > b, \ a > c \) and \( b \neq c \).

Remark 2.1.9:
Here, note that the Theorem 2.1.4 can generalized to sums of the form

\[
S_m = \sum_{n=1}^{\infty} \left[ \frac{1}{(an-b)^m} - \frac{1}{(an-c)^m} \right]
\]

, for any integers \( a, b \) and \( c \) with conditions \( a > b, \ a > c \) and \( b \neq c \). Any integers \( a, b \) and \( c \) with conditions \( a > b, \ a > c \).

2.2 Some Results on Lerch Zeta function and Its Generalization

Now, we apply the same strategy outlined above to the calculation of Lerch zeta function \( \Phi(z, s, x) \) is defined [20]as follows:

\[
\Phi(z, s, x) = \sum_{n=0}^{\infty} \frac{z^n}{(n+x)^s}, \text{ where } 0 < x \leq 1 \text{ and } \text{Re}(s) > 1
\]

We have,

\[
\frac{e^{-y(s-1)}}{e^y - z} = \sum_{n=0}^{\infty} z^n e^{-y(s+n)}
\]

And

\[
\int_0^1 e^{-y(s+n)} y^{s-1} dy = \frac{1}{(x+n)^s} \int_0^1 e^{-t} t^{s-1} dt = \frac{\Gamma(s)}{(x+n)^s}
\]

Moreover, we have the integral representation [20]

\[
\Phi(z, s, x) = \frac{1}{\Gamma(s)} \int_0^1 \frac{y^{s-1} e^{-y(s-1)}}{e^y - z} \ dy
\]
Now, we will return to Lerch Zeta function. In the next theorem we will obtain its parametric representation.

**Theorem 2.2.1:**
A series for the Lerch zeta function, which is valid for $0 < x \leq 1$, $\operatorname{Re}(s) > 1$ and $\lambda > -\frac{1}{2}$ given by

$$
\Phi(z, s, x) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \left( m \right) \lambda^{m-j} \cdot z^j \frac{1}{(1 + \lambda)^{m+1}} \frac{1}{(x + j)^s}
$$

**Proof:**
We use the integral representation (19)

$$
\Phi(z, s, x) = \frac{1}{\Gamma(s)} \int_0^{\gamma} \frac{y^{s-1} e^{-y(z-1)}}{e^y - z} dy
$$

Now, take change of variable $t = e^{-y}$ in above integral and introducing arbitrary parameter $\lambda$ by hand in equation we obtain

$$
\Phi(z, s, x) = \frac{1}{\Gamma(s)} \left( \frac{1}{1 + \lambda} \right)^s \int_0^{\infty} \frac{\log^{s-1} \left( \frac{1}{t} \right) \cdot t^{s-1}}{1 - \frac{zt + \lambda}{1 + \lambda}} dt
$$

Here, the condition $\left| \frac{zt + \lambda}{1 + \lambda} \right| < 1$ fulfilled uniformly for all $t \in [0, 1]$ provided that $|zt| < 1$ and $\lambda > -\frac{1}{2}$. In this case, we can expand the denominator in powers of $\left( \frac{zt + \lambda}{1 + \lambda} \right)$ and obtain

$$
\Phi(z, s, x) = \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} \sum_{j=0}^{m} \left( m \right) \lambda^{m-j} \cdot z^j \frac{1}{(1 + \lambda)^{m+1}} \int_0^{\infty} \log^{s-1} \left( \frac{1}{t} \right) \cdot t^{s-j-1} dt
$$
Again, take change of variable $t = e^{-q}$ in above equation (22) we obtain

\[
\Phi(z, s, x) = \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \left( \lambda^{m-j} \cdot \frac{z^j}{(1 + \lambda)^{m+j}} \right) \int_0^\infty e^{-q(s+j)} \cdot q^{s-1} \, dq
\]

Now using the result $\frac{\Gamma(n)}{a^n} = \int_0^\infty e^{-at} t^{n-1} \, dt$ of gamma function in (23), we get the desired result.

Now, we discuss the some properties of above series (20). The series (20) is independent of the arbitrary parameter $\lambda$, although the arbitrary parameter $\lambda$ appears explicitly in the series expression. This causes because the series in (18) is independent of $\lambda$ and in above theorem it has been just proved that the series in (20) converges to series in (18) provided that $|zt| < 1$ and $\lambda > -\frac{1}{2}$.

It means, we can say that the equation (20) gives the family of series, each corresponding to different value of $\lambda$ and each series in (20) converging to the same series (18).

Now, consider the partial sum

\[
\Phi^N(z, s, x) = \sum_{n=0}^{N} \sum_{j=0}^{\infty} \left( \lambda^{m-j} \cdot \frac{z^j}{(1 + \lambda)^{m+j}} \right) \frac{1}{(x+j)^s}
\]

Here, note that the above equation (24) obtained by restricting the infinite sum to the first $N+1$ terms. Clearly, the partial sum $\Phi^N(z, s, x)$ depends upon $\lambda$ as we are neglecting infinite numbers of terms. We can use this feature to our benefit and fix $\lambda$, therefore the convergence rate is maximal.
To obtain the proper values of \( \lambda \), apply PMS to the equation (24), so we have

\[
\frac{d}{d\lambda} \Phi^N(z, s, x) = 0
\]

For the lowest order, this corresponds to selecting \( N = 1 \) we obtain the values

\[
\lambda^i_{\text{PMS}} = \frac{1}{2} \left( 1 + \frac{z \cdot x}{(1+x)^{\lambda}} \right)
\]

Then, for \( s = 2, 3, 4, \ldots \) and fix value of \( x \) we can find \( \lambda^i_{\text{PMS}} \). And using the same strategy we can obtain higher order \( \lambda_{\text{PMS}} \). By using these values we can obtain graph and see that the rate of convergence of the series greatly improved by applying PMS. Next, we see some special cases of above theorem for different values of \( x, \lambda \) and \( z \).

**Corollary 2.2.2:**

Taking \( x = \frac{b}{a} \) in the series (20), where \( a \) and \( b \) are positive integers with \( b < a \).

Then, the series (26) expressed as

\[
a^i \cdot \Phi(z, s, x) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \left( m \right) \frac{\lambda^{m-j} \cdot z^j}{(1+\lambda)^{m+1}} \frac{1}{(a+j)^{\lambda}}, \text{ for } \lambda > 0.
\]

In next corollary, we will see another special case of series (26) for \( \lambda = 1 \).

**Corollary 2.2.3:**

Let \( \lambda = 1 \) in the series (20), then

\[
\Phi(z, s, x) = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \left( m \right) \frac{z^j}{2^{m+1}} \frac{1}{(x+j)^{\lambda}}.
\]
Here, we note that in the definition of Lerch zeta function \( \Phi(z, s, x) \) if we take \( z=1 \) then it is Hurwitz Zeta function \( \zeta(s, x) \). We use this fact in next corollary.

**Corollary 2.2.4:**

Let \( z=\lambda = 1 \) in the series (20), then we have

\[
(29) \quad \Phi(1, s, x) = \zeta(s, x) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \left( \frac{m}{2^{m+1}(x+j)} \right)
\]

As we generalize many series in before chapters, we think that it is possible to generalize the Leach Zeta function or not. Also, here arises a question that it is convergent or not.

So, we generalize the Lerch zeta function as in below definition and called it *H- Lerch zeta* function.

**Definition 2.2.5: [H-LERCH ZETA FUNCTION]**

For \( 0 < x \leq 1, s > 1 \) and \( u \in N \)

\[
(30) \quad \Phi(z, s, x, u) = \sum_{n=0}^{\infty} \frac{z^n}{(n^u + x)^s}
\]

Again a question that what is the relation of *H- Lerch zeta* functions with other mathematical function. Form them some of its relation with other mathematical functions which we will give below.

The Lerch Zeta function

\[
(31) \quad \Phi(z, s, x) = \sum_{n=0}^{\infty} \frac{z^n}{(n + x)^s} = \Phi(z, s, x, 1)
\]

The Riemann Zeta Function,

\[
(32) \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1, s, 1, 1)
\]
The Hurwitz Zeta Function.

\[ \zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^s} = \Phi(1, s, x, l) \]

The alternating zeta function, also it is known as Dirichlet eta function \( \eta(s) \)

\[ \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \Phi(-1, s, l, l) \]

The Dirichlet beta function

\[ \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} = 2^{-s} \cdot \Phi\left(-1, s, \frac{1}{2}, l\right) \]

The Legendre Chi Function,

\[ \chi(s) = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)^s} = 2^{-s} \cdot z \cdot \Phi\left(z^2, s, \frac{1}{2}, l\right) \]

The Polylogarithm Function,

\[ Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} = z \cdot \Phi(z, k, l, l) \]

Moreover, from Pedro’s result in [22] the polylogarithm function given by the integral

\[ z \cdot \Phi(z, k, l, l) = Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} = \frac{z}{\Gamma(k+1)} \left[ \frac{\log^k (1/ x)}{(1- zx)^2} \right] dx \]

After seeing its relations with other functions we think about our second question convergent of \( H\)-Lerch zeta function. The next theorem will give out questions answer.
Theorem 2.2.6:
The \( H \)-Lerch zeta function

\[
\Phi(z, s, x, u) = \sum_{n=0}^{\infty} \frac{z^n}{(n^u + x)^s}
\]

, where \( z \) is a real number, \( u \) is any positive integer, \( s > 1 \) and \( 0 < x \leq 1 \)

Then, the Generalized Lerch Zeta Function is convergent for \( |z| \leq 1 \) and divergent for \( |z| > 1 \).

Proof:

To prove this result, first of we consider the general term of above series (30) is

\[
a_n = \frac{z^n}{(n^u + x)^s}.
\]

Now, using the Ratio test for this series (34), we have \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = |z| \)

Therefore, it is convergent for \( |z| < 1 \) and divergent for \( |z| > 1 \).

Now, for \( z = 1 \) the series in (34) given by

\[
(39) \quad \Phi(1, s, x, u) = \sum_{n=0}^{\infty} \frac{1}{(n^u + x)^s}
\]

In addition, for all integers \( n \geq 0 \), we have \( \frac{1}{(n^u + x)^s} \leq \frac{1}{(n + x)^s} \)

So, \( \Phi(1, s, x, u) = \sum_{n=0}^{\infty} \frac{1}{(n^u + x)^s} = O(\zeta(s, x)) \)

As \( \zeta(s, x) \) is convergent for \( s > 1 \), the series in (30) is convergent for \( z = 1 \).

Therefore, it is obviously convergent when \( z = -1 \).
After seeing its convergent we will see some special representation of $H$-Lerch zeta function in next corollary.

Corollary 2.2.7:

For the $H$-Lerch zeta function $\Phi(z, s, x, u) = \sum_{n=0}^{\infty} \frac{z^n}{(n^u + x)^s}$, where $u$ is any positive integer, $s > 1$ and $0 < x \leq 1$, we have

1. $\Phi(1, s, x, u) + \Phi(-1, s, x, u) = \sum_{n=0}^{\infty} \frac{1}{((2n)^u + x)^s}$
2. $\Phi(1, s, x, u) - \Phi(-1, s, x, u) = \sum_{n=0}^{\infty} \frac{1}{((2n-1)^u + x)^s}$
3. $\Phi(z, s, x, u) = -\frac{1}{s} \frac{\partial}{\partial x} \Phi(z, s+1, x, u)$

Now, again we return to the main point of this chapter that is acceleration. Also, we will apply the same strategy outlined above to the calculation of $H$-Lerch zeta function $\Phi(z, s, x, u)$ by using PMS.

Theorem 2.2.8:

A series for the $H$-Lerch zeta function, which is valid for $0 < x \leq 1$ and $s > 1$ given by

\[ \Phi(z, s, x, u) = \frac{1}{x^s} + \sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{m}{j} \frac{\Gamma(m + s) (-1)^{m+j} \lambda^{2(m-j)} x^j}{\Gamma(s) \cdot m! (1 + \lambda^2)^{m+j}} \zeta(u(s + j)) \]

for $\lambda^2 > \frac{x-1}{2}$ and $x > 0$.

Proof:

We can represent the series in (30) as

\[ \Phi(z, s, x, u) = \frac{1}{x^s} + \sum_{n=0}^{\infty} \frac{z^n}{(n^u + x)^s} \]
We can rewrite the above series (45) as follows:

\[
\Phi(z, s, x, u) = \frac{1}{x^s} + \sum_{n=1}^{\infty} \frac{z^n}{n^s} \left( \frac{1}{(1 + \lambda^2)^s} \right) \left( \frac{1}{1 + A(n)} \right)^x
\]

where \( A(n) = \frac{n^s}{1 + \lambda^2} \) and \( \lambda \) is arbitrary constant introduced by hand.

Here, for all \( n \geq 1 \) the condition \( |A(n)| = \left| \frac{x^s - \lambda^2}{n^s} \right| < 1 \) provided that \( \lambda^2 > \frac{x-1}{2} \)

and \( x > 0 \).

Now, from the Binomial theorem we have

\[
\sum_{m=0}^{\infty} \frac{\Gamma(m+s)}{\Gamma(s) \cdot m!} (-A(n))^m
\]

By substituting the value of \( A(n) \) in equation (43), we get

\[
\frac{1}{(1 + A(n))^x} = \sum_{m=0}^{\infty} \frac{\Gamma(m+s)}{\Gamma(s) \cdot m!} \left( \frac{x^s - \lambda^2}{n^s} \right)^m
\]

Again using binomial theorem for last term of the above equation (44), so we obtain

\[
\frac{1}{(1 + A(n))^x} = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{m}{j} \frac{\Gamma(m+s)}{\Gamma(s) \cdot m!} \left( -1 \right)^{m-j} \lambda^{2(m-j)} \frac{x^j}{n^m}
\]

Hence, from equations (42) and (45)

\[
\Phi(z, s, x, u) = \frac{1}{x^s} + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{m}{j} \frac{\Gamma(m+s)}{\Gamma(s) \cdot m!} \left( -1 \right)^{m-j} \lambda^{2(m-j)} \frac{x^j}{(1 + \lambda^2)^m} \frac{n^s}{n^s (z^n)^{m+j}}
\]
As the summation runs over positive integers \( n \), the above equation given by

\[
\Phi(z, s, x, u) = \frac{1}{x^z} + \sum_{m=0}^{\infty} \sum_{j=0}^{m} \left( \Gamma(m+s) \frac{(-1)^{m+j} \lambda^{2(m-j)} x^j \cdot z^n}{\Gamma(s) \cdot m! \cdot (1 + \lambda^2)^{m+j}} \zeta(u(s+j)) \right)
\]

This completes the proof.

After proving the main result, equation (40), we now discuss some properties of the new series in (40). We can easily see that the series in (40) is independent of the arbitrary parameter \( \lambda \), although \( \lambda \) appear in the expression of series in (40). This happens because the series in equation (30) is independent of arbitrary parameter \( \lambda \) and it is easy to see that the new series in (40) converges to the series in (30) for \( \lambda^2 > \frac{x-1}{2} \) and \( x > 0 \). This means the series in (40) describes a family of series, each corresponding to a different value of \( \lambda \) and each series converging to the same series in (30).

Now, consider the partial sum of the series (46)

\[
\Phi_N(z, s, x, u) = \frac{1}{x^z} + \sum_{m=0}^{N} \sum_{j=0}^{m} \left( \Gamma(m+s) \frac{(-1)^{m+j} \lambda^{2(m-j)} x^j \cdot z^n}{\Gamma(s) \cdot m! \cdot (1 + \lambda^2)^{m+j}} \zeta(u(s+j)) \right)
\]

It obtained by restricting the infinite sum to the first \( N+1 \) terms. Clearly, the partial sum \( \Phi_N(z, s, x, u) \) depends upon the arbitrary parameter \( \lambda \) as we neglecting infinite number of terms. Here, we can use this feature to our benefit and fix \( \lambda \) therefore the convergence rate of the series is maximal. To accelerate the convergence of this series (30), apply the same strategy using PMS, which we apply to before series in this paper.

Now, we look toward one more step to go up. That means, we more generalize the H- Lerch Zeta function as G- Lerch zeta function, which is defined in below definition.
Definition 2.2.9: \textit{[G-LERCH ZETA FUNCTION]} 

For any real number $z$ and positive integer $u$ and $a$, the \textit{G-Lerch Zeta} function is define by the series 

\begin{equation}
\Phi(z,s,u,a) = \sum_{n=0}^{\infty} \frac{z^{n+a}}{(n^u + x)^s},
\end{equation}

where $0 < x \leq 1$.

Now, we will see some special cases and its relation with other mathematical function.

For $a = 1$, the G-Lerch Zeta function is H-Lerch Zeta function,

\begin{equation}
\Phi(z,s,x,u) = \sum_{n=0}^{\infty} \frac{z^n}{(n^x + x)^s} = \Phi(z,s,x,u,1)
\end{equation}

Take, $u = a = 1$ in (49), then the Lerch Zeta function,

\begin{equation}
\Phi(z,s,x) = \sum_{n=0}^{\infty} \frac{z^n}{(n + x)^s} = \Phi(z,s,x,1,1)
\end{equation}

If, we take $z = u = a = 1$ in G-Lerch Zeta function then it co-inside with the Hurwitz Zeta Function.

\begin{equation}
\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^s} = \Phi(1,s,x,1,1)
\end{equation}

Moreover, when $x = z = u = a = 1$, it becomes Riemann Zeta Function,

\begin{equation}
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1,s,1,1,1)
\end{equation}

Next, the alternating zeta function, Also it is known as Dirichlet eta function $\eta(s)$

\begin{equation}
\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \Phi(-1,s,1,1,1)
\end{equation}
The Dirichlet beta function

\[
\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} = 2^{-s} \cdot \Phi\left(-1, s, \frac{1}{2}, 1, 1\right)
\]

The Legendre Chi Function,

\[
\chi_s(z) = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n+1)^s} = 2^{-s} \cdot z \cdot \Phi\left(z^2, s, \frac{1}{2}, 1, 1\right)
\]

The Poly-logarithm Function,

\[
Li_k(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^k} = z \cdot \Phi(z, k, 1, 1, 1)
\]

From the results in (38) and (57) we get,

\[
\Phi(z, k, 1, 1, 1) = \Phi(z, k, 1, 1) = \frac{1}{\Gamma(k+1)} \int_0^1 \frac{\log^k (1/x)}{(1-zx)^2} dx
\]

Similar question again arise that the G- Lerch Zeta function is convergent or not. That we will see in the next theorem.

**Theorem 2.2.10:**

The G- Lerch Zeta function

\[
\Phi(z, s, u, a) = \Phi(z, k, 1, 1, 1) = \frac{1}{\Gamma(k+1)} \int_0^1 \frac{\log^k (1/x)}{(1-zx)^2} dx
\]

Then, the G- Lerch Zeta Function is convergent for \(|z| \leq 1\) and divergent for \(|z| > 1\).

**Proof:**

To prove this result, first of we consider the general term of above series (49) is

\[
a_s = \frac{z^{u+a}}{(n^u + x)^s}
\]
Now, using the Ratio test for this series (49), we have \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = |z| \)

Therefore, the series in (49) is convergent for \( |z| < 1 \) and divergent for \( |z| > 1 \).

Now, for \( z = 1 \) the series in (49) given by \( \Phi(1, s, x, u, a) = \sum_{n=0}^{\infty} \frac{1}{(n^u + x)^s} \)

And for all integers \( n \geq 0 \)

\[
\frac{1}{(n^u + x)^s} \leq \frac{1}{(n + x)^s}
\]

Therefore, \( \Phi(1, s, x, u, a) = \sum_{n=0}^{\infty} \frac{1}{(n^u + x)^s} = O(\zeta(s, x)) \)

As \( \zeta(s, x) \) is convergent for \( s > 1 \), the series in (49) is convergent when \( z = 1 \).

In addition, we can easily see that the series in (49) is convergent for \( z = -1 \).

Now, we apply the same strategy to G-Lerch Zeta function as we apply to H-Lerch Zeta function, obtain its parametric representation as given following theorem, and accelerate its convergence by applying PMS. Moreover, similar way we can prove the following theorem.

**Theorem 2.2.11:**

A series for the G-Lerch zeta function, which is valid for \( 0 < x \leq 1 \) and \( s > 1 \) given by

\[
\Phi(z, s, x, u, a) = \frac{1}{x^s} + \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} \binom{m}{j} \frac{\Gamma(m+s)}{\Gamma(s) \cdot m!} \left( \frac{-1}{\lambda^2} \right)^j \frac{x^{j-1}}{(1 + \lambda^2)^{m+s}} \zeta(u(s+j)) \right)
\]

for \( \lambda^2 > \frac{x-1}{2} \) and \( x > 0 \).

Moreover, we can apply the same strategy to other general series \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(ak-b)} \),

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(bc-ad)}{(ak-b)(ck-d)} \]. As we know that from chapter 1, the series \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(ak-b)} \)
converges to $J^*(a,b)$. We can obtain its parametric representation as shown in the following theorem.

**Theorem 2.2.12**

The series

$$S = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \left( \frac{1}{1 + \lambda} \right)^{m+1} \binom{m}{k} \lambda^{m-k} \frac{b^k}{a^{k+1}} \eta(k+1)$$

Converge to $J^*(a,b)$ for $\lambda > -\frac{1}{2}$, with $\lambda$ real.

Similarly, we can obtain

**Theorem 2.2.13:**

The series

$$S = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \left( \frac{1}{1 + \lambda} \right)^{m+1} \binom{m}{k} \lambda^{m-k} \frac{b^k - c^k}{a^{k+1}} \eta(k+1)$$

Converge to $K^*(a,b;a,c)$ for $\lambda > -\frac{1}{2}$, with $\lambda$ real.

### 2.3 Applications of Acceleration of infinite series in different field

Today, theoretical description of biological process is unthinkable without statistical analysis, and concurrently, fields like “bioinformatics” and “computational biology” are emerging. Many important mathematical functions needed in the theory of statistical distributions represented by slowly convergent series, and acceleration of series can benefit in this computations. For example, the *Discrete Zipf-related Distributions* whose probability mass function are represented by the terms of infinite series defining Riemann Zeta, Generalized Zeta and Poly-logarithm functions and whose total probability is calculated with these functions. This distribution are used in statistical analysis of folds of biological sequences of *RNA, DNA, and protein molecules* and occurrence analysis of folds of proteins [21]. Moreover the generalized
representation of these distributions was shown in the form of Lerch
distributional family, which requires calculations of Lerch’s $\Phi$ transcendent
[21] . The transcendent is given by the following power series,

$$\Phi(z,s,v) = \sum_{n=0}^{\infty} \frac{z^n}{(n+s)^v}$$

for $|z|<1, |z|\approx 1$, the power series is very slowly convergent. We can accelerate the convergence of power series as shown in above theorem 2.10.

In addition, various long-standing problems in theoretical physics can solved by using computational methods based on convergence acceleration techniques. The calculations are of importance for the test of fundamental quantum theories and for the determination of fundamental physical constants. Further application includes the perturbation theory, as said in introduction of this chapter. Moreover, the rather general applicability of the convergence acceleration methods makes them very attractive tool in scientific computing.