We are familiar with the Cantor ternary set. It was introduced by German mathematician Georg Cantor in 1883, but discovered in 1875 by Henry John Stephen Smith. The cantor set is the set of all points of the interval [0, 1] whose ternary expansion is without digit 1. In this chapter first we see a generalization of cantor set.

It was Mohsen Soltanifar, see [37] who defined a Cantor Fractal set of middle $q^s$ and of the order $s$ as, for any positive integer $s$

$$\Gamma(s) = \left\{ x \in [0, 1] \mid x = \sum_{n=1}^{\infty} \frac{a_n}{q^n}, \quad q = 2s + 1, \quad a_n = 0, \quad 2, \ldots, \quad (q-1) \right\}.$$  Moreover, he proved its properties.

In this chapter, we proved one main theorem. However, to prove the main result first we recall some important concepts. First, we are familiar with Cantor
ternary set, which is the set of all number of the interval $[0, 1]$ whose ternary expansion is without digit 1. It constructed by removing middle $3^{rd}$ of the interval $[0, 1]$. In this chapter, we will see a generalization of Cantor set. In first section of this chapter, we will define a generalization of cantor set as Cantor Fractal set of middle $q^a$ and of the order $s$. Then in second section we will define repunit prime and give some different representation of repunit prime. In addition, in the last section I will define Cantor Fractal Prime and then discuss a property of pertaining to base-$q$ primes. That is, those which satisfy an equation of the form

$$tp + 1 = q'$$

, where $r$ is a prime and $q = 2s + 1$ with $p > q$ for $s$ is any positive integer and $t \in \{2, 4, \ldots, (q-1)\}$

### 4.1 Preliminary

In this section of this chapter, we will see a generalization of cantor set and some results of it from [37]. Now, we define a Cantor like set as follows.

For any positive integer $s$, 

$$\Gamma(s) = \left\{ x \in [0, 1] \mid x = \sum_{n=0}^{\infty} \frac{a_n}{q^n}, \quad q = 2s + 1, \quad a_n = 0, 2, \ldots, (q-1) \right\}$$

We call the above set in (1), a Cantor Fractal set of middle $q^a$ and of the order $s$.

Here, note that for $s = 1$, it the Cantor ternary set. Next, we construct the $\Gamma(s)$ - the Cantor Fractal set of middle $q^a$ and of the order $s$.

We define $\Gamma(s)$ in the unit interval $[0, 1]$. First, consider the $q = 2s + 1$, where $s$ is any positive integer.
Then, let us consider a map

\[ T_{i,j} = \frac{1}{q} (x) + \frac{2i}{q}, \text{ where } i = 0, 1, 2, \ldots, s \]

Now, let \( I_{s,0} = [0, 1] \) and \( n \in N \).

Then, we define \( I_{s,n} \) inductively as follows

\[ I_{s,n} = \bigcup_{i=0}^{s} T_{i,j} (I_{s,n-1}) \]

Since \( T_{i,j} \)'s are continuous closed maps, the take each closed sub-interval
of \([0, 1]\) in to a closed sub-interval of \([0, 1]\) each map \( T_{i,j} \) sends \((s+1)^{n-1}\) disjoint
closed intervals to \((s+1)^n\) disjoint closed intervals.

Hence, their union- which is \( I_{s,n} \), the disjoint union of \((s+1)^n\) disjoint closed
intervals \( I_{s,n} = [a_{s,n}, b_{s,n}] \). Each \( I_{s,n} = [a_{s,n}, b_{s,n}] \) has length \( \left( \frac{1}{q} \right)^n \). So, the total
length of \( I_{s,n} = \left( \frac{s+1}{q} \right)^n \).

By constructive \( I_{s,n} \)'s for each \( n = 1, 2, 3, \ldots \) and \( i = 1, 2, 3, \ldots n \), we have

\[ (T_{i,j} (I_{s,n-1}) \subset I_{s,n-1} \]

Let \( a \in I_{s,n} \) for each \( n = 1, 2, 3, \ldots \) and \( i = 1, 2, 3, \ldots n \),

Then, by definition of \( I_{s,n} \) we have

\[ a \in I_{s,n} = \bigcup_{i=0}^{s} T_{i,j} (I_{s,n-1}) \]
This implies,
\[ a \in T_{s,i}\left(I_{s,n=1}\right), \quad \text{for at least one } i = \{0, 1, 2, \ldots, s\} \]

So, using (4) in above, we have

(5) \[ I_{s,n} = \bigcup_{j=0}^{s} T_{j,2}\left(I_{s,n=1}\right) \subset I_{s,n=1} \]

This implies,

(6) \[ I_{s,0} \supseteq I_{s,1} \supseteq I_{s,2} \supseteq I_{s,3} \ldots \ldots \]

And each \( I_{s,n} \) is a close set and thus a compact subset of \([0, 1]\).

Since the collection has finite intersection property and \([0, 1]\) is compact, they have a non-empty intersection. Then, \( \Gamma(s) \) - the Cantor Fractal set of middle \( q^n \) and of the order \( s \) is defined as follows

(7) \[ \Gamma(s) = \bigcap_{n=0}^{\infty} I_{s,n} \]

Since, \( \Gamma(s) \) - the Cantor Fractal set of middle \( q^n \) and of the order \( s \) is the intersection of closed subset of \([0, 1]\). So, it is closed and bounded subset of \([0, 1]\).

Thus, by using Heine-Borel theorem in \( \mathbb{R} \), \( \Gamma(s) \) - the Cantor Fractal set of middle \( q^n \) and of the order \( s \) is compact for every positive integer \( s \).

**Theorem 4.1.1.** [37]

For every positive integer \( s \), \( \Gamma(s) \) - the Cantor Fractal set of middle \( q^n \) and of the order \( s \) given by

(8) \[ \Gamma(s) = [0, 1] - \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{q^n-1} \left( \frac{qk + (2r - 1)}{q^n}, \frac{qk + 2r}{q^n} \right) \]
Proof:
Before we introduce two new lemmas that are needed to prove this theorem, let us have the following notation,

\[ I_{s,n}^* = I_{s,0} - I_{s,n} \text{, for } n = 0,1,2,3 \ldots \text{ and } s = 1,2,3 \ldots \]

Lemma 4.1.2:
With the above definition and notations, the equality

\[ I_{s,n}^* = I_{s,1} \cup (\bigcup_{i=0}^{s} T_{s,i}(I_{s,n-1}^*)) \]

, holds for every \( n = 0,1,2,3 \ldots \text{ and } s = 1,2,3 \ldots \)

Proof of Lemma 4.1.2:
By definition of \( I_{s,n}^* \):

\[
I_{s,n}^* = I_{s,0} - I_{s,n} = I_{s,0} - \left( \bigcup_{i=0}^{s} T_{s,i}(I_{s,n-1}) \right) = \bigcap_{i=0}^{s} \left( I_{s,0} - T_{s,i}(I_{s,n-1}) \right)
\]

\[
= \bigcap_{i=0}^{s} \left( I_{s,0} - T_{s,i}(I_{s,0} - I_{s,n-1}^*) \right) = \bigcap_{i=0}^{s} \left( I_{s,0} - (T_{s,i}(I_{s,0}) - T_{s,i}(I_{s,n-1}^*)) \right)
\]

\[
= \bigcap_{i=0}^{s} \left( (I_{s,0} - T_{s,i}(I_{s,0})) \cup T_{s,i}(I_{s,n-1}^*) \right) = I_{s,0} \cup \left( \bigcup_{i=0}^{s} T_{s,i}(I_{s,n-1}^*) \right)
\]

Lemma 4.1.3:
With the above definition and notations, the equality

\[ I_{s,n}^* = \bigcup_{m=1}^{n} \bigcup_{k=0}^{q^m-1} \left( \frac{qk + (2r - 1)}{q^m}, \frac{qk + 2r}{q^m} \right) \]

, holds for every \( n = 0,1,2,3 \ldots \text{ and } s = 1,2,3 \ldots \)

Proof of Lemma 4.1.3:
We prove the assertion by induction. It is clear that it holds for \( n = 1 \). Let it hold for positive integer \( n - 1 \), then:
\[ I_{*,n}^* = I_{*,1}^* \cup \left( \bigcup_{i=0}^{s} T_{*,i}(I_{*,n-1}^*) \right) \]
\[ = I_{*,1}^* \cup \left( \bigcup_{i=0}^{s} \bigcup_{m=1}^{2^m+1} \bigcup_{k=0}^{r-1} \left( \frac{q_k + (2r-1)}{q_{m+1}}, \frac{q_k + 2r}{q_{m+1}} \right) \right) \]
\[ = I_{*,1}^* \cup \left( \bigcup_{i=0}^{s} \bigcup_{m=1}^{2^m+1} \bigcup_{k=0}^{r-1} \left( \frac{q_k + (2r-1)}{q_{m+1}}, \frac{q_k + 2r}{q_{m+1}} \right) \right) \]

Hence,
\[ I_{*,n}^* = \bigcup_{m=1}^{n} \bigcup_{k=0}^{r-1} \left( \frac{q_k + (2r-1)}{q_{m+1}}, \frac{q_k + 2r}{q_{m+1}} \right) \]

### 4.2 Repunit Prime

We are familiar with the prime numbers. In this section, we will discuss about the repunit prime. First time A. H. Beiler [11] to indicate that numbers that can be expresses by using repeateds units defined the term repunit.

**Definition 4.2.1:**

A prime number \( p \) called a base- \( N \) repunit prime if it satisfies an equation of the form

\[ (N-1)p + 1 = N^r \]

where \( N \) is any positive integer grater than 1 and \( r \) is a prime.

Such primes have the following property:

\[ p = \frac{N^r - 1}{N-1} = \sum_{k=1}^{r} N^{r-k} \]

So, they are given by a contiguous sequence of 1’s in base- \( N \).
For example, \( p = 31 \), for \( N = 2 \) and \( q = 5 \) it satisfies the equation (9). In addition, it can express as \( 11111 \) in base- \( 2 \).

The reciprocal of any such prime is an infinite series of the form

\[
\frac{1}{p} = \frac{N - 1}{N^r - 1} = \sum_{k=1}^{\infty} \frac{N - 1}{N^{r-k}}
\]

The above equation (11) shows that \( \frac{1}{p} \) can give by in base- \( N \) using only zero and digit \( N - 1 \). This single non-zero digit will appear periodically in base- \( N \) representation of \( \frac{1}{p} \) at positions, which are multipliers of \( r \).

### 4.3 Cantor Fractal prime

In this section, we discuss such property of pertaining to base- \( q \) primes. That is, those which satisfy an equation of the form

\[
 tp + 1 = q'
\]

where \( r \) is a prime and \( q = 2s + 1 \) with \( p > q \) for \( s \) is any positive integer and \( t \in \{2, 4, \ldots, (q-1)\} \)

For this first, we define Cantor Fractal prime for \( \Gamma(s) \) - the Cantor Fractal set of middle \( q^s \) and of the order \( s \) as follows:

**Definition 4.3.1:**

A prime number \( p \) called a Cantor Fractal prime if \( \frac{1}{p} \in \Gamma(s) \) -the Cantor Fractal set of middle \( q^s \) and of the order \( s \).
Theorem 4.3.2:
A prime number $p$ is a Cantor Fractal prime if and only if it satisfies the equation of the form $tp + 1 = q^r$, where $r$ is a prime and $q = 2s + 1$ with $p > q$ for $s$ is any positive integer and $t \in \{2, 4, \ldots, (q-1)\}$.

Proof:
To prove this theorem it is necessary to consider the nature of $\Gamma(s)$ - the Cantor Fractal set of middle $q^o$ and of the order $s$ briefly. The set $\Gamma(s)$ constructed as shown previously in section-1.

The construction of $\Gamma(s)$ suggest some simple condition which a prime number must satisfy in order to be Cantor Fractal prime.

Let a prime number $p > q$ be a Cantor Fractal prime then, the first non-zero $a_i$ in $q$-base expansion of $\frac{1}{p}$ must be $t$, where $t \in \{2, 4, \ldots, (q-1)\}$.

It means that, for some $k \in N$, prime $p$ must satisfy

\[(12)\ldots \frac{t}{q^k} < \frac{1}{p} < \frac{1}{q^{k+1}}\]

In addition, we can express above inequality as

\[(13)\ldots q^i \in \{tp, \quad qp\}\]

Now, prime numbers are that, for which there is no power of $q$ in the interval $(tp, \quad qp)$ can excluded immediately from further consideration.

If the next non-zero digit after $a_i$ is to be another $t$, where $t \in \{2, 4, \ldots, (q-1)\}$ then, it must be case for some $n \in N$. 
We can express above inequality as

\[(15) \quad q^k \in \left( \frac{tp}{q^k - tp}, \frac{qp}{q^k - tp} \right)\]

Thus, any prime number which satisfy the condition (13) but which primes is the no power of \(q\) in the interval \(\left( \frac{tp}{q^k - tp}, \frac{qp}{q^k - tp} \right)\), can be again excluded.

At this point, the problem is that the \(q\)-base expansion under these consideration are all non-terminating. Therefore, here we can easily see that it is an endless sequence of test like these would be applicable to ensure that \(a_k \neq 2s + 1\), for \(k, s \in N\).

However, conditions (13) and (15) do not give us all the required information to prove that if prime \(p\) is a Cantor Fractal prime then, it satisfy the equation of the form \(tp + 1 = q^r\), where \(r\) is a prime and \(q = 2s + 1\) with \(p > q\) for \(s\) is any positive integer and \(t \in \{2, 4, \ldots, (q - 1)\}\).

To prove this, let us consider prime \(p\) be a Cantor Fractal prime. In addition, assume that \(q^r\) be the smallest power of \(q\) that exceeds \(tp\).

Then, conditions (13) and (15) both satisfy for \(k = r\).

Now, putting \(k = r\) in (15) and multiplying it both sides by \(q^{-n}\), we obtain

\[(16) \quad q^r = q^{-n} \cdot q^n \in \left( \frac{q^{-n} \cdot tp}{q^r - tp}, \frac{q^{-n} \cdot qp}{q^r - tp} \right)\]

It gives that, \(q^r \in (tp, qp)\).
There is the only one way in which the interval in (16) is to be equal to the interval 
\((tp, qp)\) if,

\[(17)\ldots q^{-n} = q' - tp\]

But, this is impossible unless \(r = n\). It proves that if prime \(p\) is a Cantor Fractal prime then, it satisfy equation of the form \(tp = q' - 1\), where \(q = 2s + 1\) with \(p > q\) for \(s\) is any positive integer and \(t \in \{2, 4, \ldots, (q-1)\}\) as we claimed.

Here, it remains to prove that \(r\) is a prime. To prove that \(r\) is a prime, let us assume that \(r\) is a composite.

So, for any positive integers \(c\) and \(d\) we have

\[r = cd\]

Therefore, we can obtain algebraic factorization of \(tp = q' - 1\) as follows:

\[(18)\ldots q'^{-1} = (q')^{-t} = (q'^{-1})(q'^{t-1} + q'^{t-2} + \ldots + 1)\]

From above we would have,

\[(19)\ldots p = \frac{q' - 1}{t} = \frac{(q')^{-t} - 1}{t} = \frac{(q'^{-1})(q'^{t-1} + q'^{t-2} + \ldots + 1)}{t} + \ldots + 1\]

Since, we have \(tp = q' - 1\). So, from equation (19) we have \((t \mid q' - 1)\), that means \(p\) is a composite, which is a contradiction.

So, \(r\) must be a prime.

Conversely, we prove that if \(p\) satisfies an equation of the form \(tp = q' - 1\), where \(q = 2s + 1\) with \(p > q\) for \(s\) is any positive integer and \(t \in \{2, 4, \ldots, (q-1)\}\) then, \(p\) must be a Cantor Fractal prime.
Here, we have \( p = \frac{q' - 1}{t} \)

Taking reciprocal of above, we obtain

\[
(20) \quad \frac{1}{p} = \frac{t}{q' - 1}
\]

In (20), expand \( \frac{1}{q' - 1} \) as geometric series, then we have

\[
(21) \quad \frac{1}{p} = \sum_{k=1}^{\infty} \frac{t}{q'^k}
\]

Equation (21) shows that \( \frac{1}{p} \) can be expressed in base-\( q \) using \( t \), where 
\( t \in \{2, 4, \ldots, (q-1)\} \).

Since, only 0 and \( t \), where \( t \in \{2, 4, \ldots, (q-1)\} \) appear in \( q \)-base representation of \( \frac{1}{p} \). Therefore, \( \frac{1}{p} \) is never remove in construction of \( \Gamma(s) \) - the Cantor Fractal set of middle \( q'^s \) and of the order \( s \). Therefore, prime \( p \) must be a Cantor Fractal prime.