Appendix I

ELECTRON SCATTERING ON LIGHT NUCLEI

In recent years the scattering of high energy electrons by atomic nuclei has been used extensively and successfully to elicit the details of the charge distribution. This has been made possible by the availability of intense mono-energetic beams of electrons of sufficiently high energy (\(>100\) MeV). It is well known that the selection of a high energy electron over a nucleon of similar energy lies in the fact that both these interactions measure quite different properties of a nucleus. The nucleon-nucleon scattering gives the information regarding the nucleonic potential whereas the electron scattering informs only about the distribution of protons in nuclei and the distribution of the neutrons (if one looks for the density of mass distribution) can be inferred only by extrapolation.

In studying the nuclear structure problems by means of high energy electrons as projectiles, many theoretical models for the distribution of charge have been suggested. As such it is difficult to correlate the parameters obtained in such different distributions. However, it is now well established that one can distinguish between the different proposed types of charge distribution, e.g. uniform, gaussian, exponential, Fermi etc., on the basis of the experimental results. Several excellent reviews\(^1\) are now available on the subject and we give briefly some of the important relevant results.

The distribution that is now most generally used to fit the experimental data is the Fermi distribution (or Saxon-
Woods type)
\[ \phi(\kappa) = \phi_0 \left[ 1 + \exp \left( \frac{\kappa - c}{s} \right) \right]^{-1} \quad \cdots (1.1) \]

which is characterised by two parameters, \( c \) the half density radius, and \( s \) (which is related to \( z \) by \( s = 4.4 z \)) the surface thickness, the latter being defined as the distance over which the density decreases from 90 percent to 10 percent of its maximum value. Other distributions such as trapazoidal or Helm's distribution\(^2\) etc. which give equally good fit to the data are, when plotted, found to be almost indistinguishable from the Fermi distribution. It then appears that for heavy nuclei \( A > 16 \), the charge distribution is characterised by a central region where the density is almost constant, and a surface region in which the density rapidly falls off. The surface thickness \( s \) is found to be almost constant for all nuclei and has the value \( s = 2.4 \pm 0.3 \) fm (1 fm = 10\(^{-13}\) cm.). The parameter \( c \), on the other hand, is mass dependent, and is rather well described by \( c = 1.07 A^{1/3} \) fm. Thus the charge distribution for heavy nuclei is now quite satisfactorily described in terms of two parameters \( c \) and \( s \). There remains, however, some uncertainty regarding the charge distribution at the centre ( \( r = 0 \)).

For light nuclei \( 6 < A < 16 \), the situation is not so simple. Again several different types of charge distributions have been employed and the most popular distribution, one with some theoretical basis, is the oscillator distribution
\[ \phi(\kappa) = \phi_0 \left[ 1 + \frac{2 - 2z}{a} \left( \frac{a}{a_0} \right)^2 \right] \exp \left( -\frac{a^2}{a_0^2} \right) \quad \cdots (1.2) \]

wherein it is assumed that the extension of the s-shell wave function is the same as that of the p-shell i.e. \( a_s = a_p = a_0 \).
It has, however, been shown by Burleson and Hofstadter\(^3\) and
by Elton\(^4\) for Li\(^6\) and by Waghmare \& Pandya\(^5\) for Be\(^9\) that
\(a_s \neq a_p\). We extend these calculations in section 2 of this
appendix.

To systematise the results for electron scattering on
all nuclei, it would also be desirable to carry out the calcula-
tions for light nuclei in terms of Fermi distribution, although
the oscillator distribution has been found to be very successful.
Thus Elton, Hiley and Price\(^6\) analysed the results for C\(^{12}\) on
the basis of Fermi distribution. Such calculations were also
made for O\(^{16}\). The conclusion from C\(^{12}\) and O\(^{16}\) was that these
light nuclei are almost all surface, and the surface thickness
is somewhat smaller viz. \(s = 1.85\) fm. Elton et.al. suggested that
it would be very desirable to carry out similar calculations for
other p-shell nuclei. Such calculations of high energy electrons
with Saxon-Woods charge distribution (Fermi type), for light
nuclei having large quadrupole moments viz. Be\(^9\), B\(^{11}\) and N\(^{14}\),
have been made in section 1 and the variation of the parameters
studied. This work is an extension of the preliminary calcula-
tions of Waghmare\(^3\) on the subject. We show that the contribution
due to quadrupole scattering forms a sizable part of the total
elastic scattering cross section.

I.1 Saxon-Woods Charge Distribution (Quadrupole Scatterings)

\(\S 1.1\) Analysis

The differential scattering cross section for a high
energy electron incident on a massive nucleus is given\(^1\) in
Born Approximation by

\[
\frac{d\sigma}{d\Omega} = \left( \frac{Ze^2}{2E} \right)^2 \frac{\cos^2 \frac{1}{2} \theta}{\cos^4 \frac{1}{2} \theta} |F(q)|^2,
\]

\[\text{---(I.3)}\]
where the form factor $F(q)$ is expressed as

$$F(q) = \int_0^\infty e^{iqz} \varphi(z) dz$$

in terms of the four-momentum transfer $q$, with

$$kq = \frac{2E}{c} \sin \frac{1}{2} \theta$$

In considering the deformation of a nucleus, we expand the charge distribution $\varphi(r)$ in the form of a series as

$$\varphi(r) = \varphi_0(r) + \frac{\partial \varphi_0(r)}{\partial \theta} \cos \theta + \frac{\partial^2 \varphi_0(r)}{\partial \theta^2} \cos 2\theta + \cdots$$

The form factor in this case is then expressed as

$$|F|^2 = |F_0|^2 + |F_2|^2$$

where

$$F_0(q) = 4\pi \int \varphi_0(r) \hat{j}_0(qr) r^2 dr$$

$$F_2(q) = -\frac{2\pi}{20} \int \varphi_2(r) \hat{j}_2(qr) r^2 dr$$

The quadrupole moment is defined as

$$Q = \frac{16\pi}{5} \int \varphi_2(r) r^4 dr$$

For the charge distribution $\varphi_0(r)$ we assume the Saxon-Woods type (Fermi) distribution defined in eq. (I.1).

Now the analysis of the experimental scattering cross section yields the form factor $F(q)$ from eq. (I.3) and this in turn can be analysed in different ways to yield some information on the charge distribution $\varphi(r)$. It should be mentioned that the Born approximation treatment and hence equation (I.3) is valid only for light nuclei ($Z$-small) and small angle $\theta$.

Particularly near the diffraction minimum in the scattering cross section, Born Approximation gives unreliable results. For heavy nuclei, an exact calculation of the cross section using lengthy
and involved phase shift analysis must be carried out. In this case, one must assume a charge distribution \( S(r) \); calculate the scattering cross section and adjust the parameters of \( S(r) \) to obtain a good agreement with the observed data.

§ 1.2 Results and Discussion

The results for scattering of 300 MeV electrons by \( \text{Be}^9 \) have been discussed earlier\(^3\), and we shall just quote them briefly. Although the monopole part of the charge distribution (corresponding to a spherical nucleus) gives satisfactory results with the parameters \( c = 1.4 \) fm and \( z = 0.6 \) fm up to an angle of \( 80^\circ \), the calculations showed a deviation from the experimental results for the higher values of \( \theta \) as is usually the case with Born approximation. However, the quadrupole part of the form factor gives, for the deformation parameter \( \varepsilon = 0.256 \), calculated by assuming the quadrupole moment of \( \text{Be}^9 \) to be 2.0 fm\(^2\), the required cancelling of the discrepancy. For these values of the parameters, the root mean square radius was obtained to be 2.48 fm. A comparison with the parameters given by other authors is made in table I.1. Fig.I.1 shows the difference in charge distribution densities between our analysis and that of Meyer-Berkhout et al.\(^9\). The latter is slightly more extended towards the surface than ours.

In B\(^11\) it is interesting to note (fig.I.2) that even for smaller angles the quadrupole part plays an important role. The parameters obtained\(^3\) from the B\(^11\) analysis were \( c = 1.6 \) fm, \( z = 0.7 \) fm and \( \varepsilon = 0.458 \). The root mean square radius obtained was 2.40 fm.

Next, the results of elastic scattering\(^3\) of 420 MeV electrons by \( \text{N}^{14} \) were analysed, and it was found that a satisfactory
Table I.1
Variation of parameters of Saxon-Woods distribution

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>c(fm)</th>
<th>z(fm)</th>
<th>s(fm)</th>
<th>Q(fm²)</th>
<th>ε</th>
<th>&lt;r²&gt;₁/₂(fm)</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Be⁹</td>
<td>1.40</td>
<td>0.60</td>
<td>2.64</td>
<td>2.00</td>
<td>0.253</td>
<td>2.48</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.90</td>
<td>0.79</td>
<td>2.48</td>
<td></td>
<td></td>
<td>2.99</td>
<td>9)</td>
</tr>
<tr>
<td></td>
<td>1.80</td>
<td>0.45</td>
<td>2.00</td>
<td></td>
<td></td>
<td>2.25</td>
<td>10)</td>
</tr>
<tr>
<td>B¹¹</td>
<td>1.60</td>
<td>0.70</td>
<td>2.06</td>
<td>3.55</td>
<td>0.458</td>
<td>2.40</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.00</td>
<td>0.45</td>
<td>2.00</td>
<td></td>
<td></td>
<td>2.23</td>
<td>10)</td>
</tr>
<tr>
<td>N¹⁴</td>
<td>2.60</td>
<td>0.72</td>
<td>3.17</td>
<td>2.00</td>
<td>0.140</td>
<td>3.38</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.30</td>
<td>0.50</td>
<td>2.20</td>
<td></td>
<td></td>
<td></td>
<td>9)</td>
</tr>
<tr>
<td></td>
<td>2.40</td>
<td>0.42</td>
<td>1.85</td>
<td></td>
<td></td>
<td>2.46</td>
<td>10)</td>
</tr>
</tbody>
</table>

The quantity s is the "surface thickness" as conventionally defined for the Saxon-Woods distribution.
Fig. 1.1

$P(r)$

$\text{fm}^3$

$\rho(r)$

$\text{fm}^3$

$\gamma$ (fm)

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Be

A — Present

B — Meyer-Berkhout et al.\textsuperscript{9)}

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$B^\perp$ $300$ MeV

- **Monopole**
- **Quadrupole**
- **Experimental**

**Fig. I.2**
agreement can be obtained with the parameters $c = 2.60 \text{ fm}$, $z = 0.72$ and $\epsilon = 0.138$, giving the value of the root mean square radius as 3.38 fm. It should be remarked that a still better fit for this nucleus could be obtained by a slight variation of the parameters, which, however, would not change the value of the deformation parameter $\epsilon$ to any considerable extent.

It is interesting to note that the consideration of the deformation of a nucleus in calculating the charge density does not affect the root mean square radius by any sizable amount. However, it does affect the values of $c$ and $z$ appreciably and gives a proper contribution to filling up the diffraction minimum.

The value of the parameter $\epsilon$ is larger for $^{11}_{\text{B}}$ than for $^{9}_{\text{Be}}$ or $^{14}_{\text{N}}$: this is expected as $^{11}_{\text{B}}$ lies in the middle of the $p$-shell while the $^{9}_{\text{Be}}$ nucleus lies before $^{11}_{\text{B}}$ and $^{14}_{\text{N}}$ almost at the end of the $p$-shell, which also suggests that $\epsilon_{^{14}_{\text{N}}} < \epsilon_{^{9}_{\text{Be}}}$. I.2 Harmonic Oscillator Charge Distribution.

The experimental data for the elastic scattering of high energy electrons from most of the $p$-shell nuclei may be fitted by using the harmonic oscillator wave functions of the shell model assuming the same length of the radial extension parameters for both $s$- and $p$-shell nucleons\(^1\)\. However, Burleson and Hofstadter\(^3\) found that for 426 MeV electron scattering on $^{6}_{\text{Li}}$ such a fit is possible only if the $s$- and $p$-protons are assumed to move in different potentials; for this case the radial extension parameter $a_s$ for $s$-protons is greater than $a_p$ for $p$-protons. On the other hand Elton\(^4\) has shown that by considering the oscillator distribution to be 'modified' due to the finite
size of the proton and the centre of mass motion in the shell model given by Tassie and Barker\textsuperscript{12}, the best fit for the same 426 MeV electron scattering data for Li\textsuperscript{6} is obtained by taking \( a_p > a_s \). This result has been confirmed by the calculations of Jackson\textsuperscript{13} for the same experimental data for Li\textsuperscript{6} by using a smoothed, finite oscillator potential with identical parameters for both s- and p-nucleons such that this smoothing leads to a more extended distribution for p-shell nucleons than for s-shell nucleons and also by Waghmare and Pandya\textsuperscript{5}) for the elastic as well as inelastic scattering of 190 MeV electrons on Be\textsuperscript{9} by using an intermediate coupling shell model. Here we extend the application of the 'modified oscillator distribution' to the other p-shell nuclei viz. Be\textsuperscript{9} and B\textsuperscript{11}.

For the 'modified oscillator distribution' the nuclear form factor \( F(q) \) for the p-shell nuclei is given by the expression\textsuperscript{12,14}

\[
F(q) = \frac{2}{Z} \exp\left(-\frac{q^2a_{sc}^2}{4}\right) + \frac{Z-2}{Z} \left(1 - \frac{q^2a_{pc}^2}{6}\right) \exp\left(-\frac{q^2a_{pc}^2}{4}\right)
\]  

\text{---(I.10)}

where \( a_{sc} \) and \( a_{pc} \), the length parameters corrected for the finite size of proton and the centre of mass motion in the shell model\textsuperscript{12}, are connected to the radial extension parameters \( a_s \) and \( a_p \) through the relations:

\[
a_{sc}^2 = (1 - 1/A) a_s^2 + a_p^2
\]

and

\[
a_{pc}^2 = (1 - 1/A) a_p^2 + a_s^2
\]  

\text{---(I.11)}

\( a_p^2 \) being 2/3 of the mean square radius of proton.

Using these expressions for the form factor with \( a_s = a_p = a_0 \) Mayer-Berkhout et al.\textsuperscript{9} studied the elastic scattering of electrons of different energies on Be\textsuperscript{9}, B\textsuperscript{11} and
and obtained the best possible fits for the length parameter $a_0$ of these nuclei. They used the 160, 300 and 420 MeV electrons and measured the scattering cross-sections for the scattering angles varying from $30^\circ$ to $100^\circ$. Here we study these cases by taking $a_s$ and $a_p$ different.

For Be$^9$, the best fit was obtained$^9$ by taking the length parameter $a_0 = 1.60$ fm. For 160 MeV electron scattering (fig. I.3, dotted line) we see that this parameter explains the whole range of the available data. The fit is exact and complete. However, for 300 MeV, this length parameter gives a fit only up to $\theta \approx 55^\circ$ (fig. I.4, dotted line) whereas experimental data extends up to $90^\circ$. After $\theta \approx 55^\circ$ the theoretical curve drops down more rapidly than the experimental points. If we consider $a_s \neq a_p$ we find that an exact fit with the experimental data can be obtained by taking the length parameter $a_s = 1.40$ fm and $a_p = 1.90$ fm as shown by the solid curve in fig. I.4. It is clear from the figure that the fit is exact and complete for the whole range. Thus, in agreement with the previous results, a more extended potential well is needed for p-shell than for s-shell protons.

Similar calculations have been carried out for the 300 and 420 MeV electron scattering on B$^{11}$. For $a_s = a_p = a_0$ the best possible fit was obtained$^9$ for $a_0 = 1.60$ fm. Consideration of $a_s \neq a_p$ shows that for the case of elastic scattering of 300 MeV electrons the best fit is given for $a_s = 1.30$ fm and $a_p = 1.70$ fm and for 420 MeV the parameters are $a_s = 1.29$ fm and $a_p = 1.80$ fm. However, such a result is very striking. It is surprising that the radial extension parameters for a particular nucleus are different for different energies of the incident electron. In
Fig. I.3

$\frac{d\sigma}{d\eta}$

$^{9}\text{Be}$

160 MeV

$\alpha_o = 1.60 \text{ fm}$

$\alpha_s = 1.40 \text{ fm}$

$\alpha_p = 1.90 \text{ fm}$
$^9\text{Be}$

300 MeV

\[
\frac{d\sigma}{d\Omega} = 1.40 \text{ fm}.
\]

\[
\frac{d\sigma}{d\Omega} = 1.90 \text{ fm}.
\]

\[
\frac{d\sigma}{d\Omega} = 1.60 \text{ fm}.
\]

Fig. 14
this connection it may be mentioned that a higher energy
electron (say 420 MeV) will see the charge distribution more
clearly than a lower energy electron (say 300 MeV). Thus, this
energy dependence of the radial extension parameter should dis-
appear when we take these parameters $a_n$ and $a_p$ for a particular
nucleus to be the one for the higher energy electron scattering
data. Calculations have been made in the light of such a
suggestion. For Be$^9$ it is found that the parameters $a_n = 1.40$
and $a_p = 1.90$ fm obtained for 300 MeV electron scattering data
when applied to 160 MeV electron scattering data gives an exactly
the same fit as is obtained for $a_p = 1.80$ fm. This is shown by
solid curve in fig. 1.3. Similarly for Bi$^{11}$, the parameters
obtained for the 420 MeV electron scattering data give essentially
an exact fit for 300 MeV data.

Thus we conclude that for the proper explanation of the
elastic scattering data of p-shell nuclei for all the lower and
higher energy electrons we need to assume s- and p-nucleons to
move in different potentials. In short, other words, a two parameter
description i.e., $a_n \neq a_p$ with $a_p > a_s$ seems to be more appropriate.
Such a result is in agreement with Waghmare and Pandya$^5$ for the
intermediate coupling model calculations on Be$^9$. It may be further
concluded that 'modified' harmonic oscillator charge distribution
with $a_p > a_s$ is quite adequate for the study of elastic scattering
of electrons for p-shell nuclei.
References.