Chapter 3

Competitive Equilibrium Theory

In the capitalist market, vital regulatory functions such as ensuring, stability, competency, and fairness are relegated to pricing mechanisms. Thus, competitive equilibrium theory of equilibrium prices acquired a prominent place in mathematical economics. With the advent of internet, there has been an extensive research done at the boundary across computer science and economic theory over the past few years. We discuss in this chapter about the competitive equilibrium theory and its applications to computer science.

3.1. Introduction

Competitive equilibrium theory can be opined as a specialized branch of game theory that deals with decision making in large markets. It is extensively used in the analysis of economic activities dealing with fiscal or tax policy, in finance for analysis of stock markets and commodity markets, to study interest, and exchange rates, and other prices. It serves as a yardstick for efficiency in economic analysis. It relies on the assumption of competitive market, where each trader decides upon a quantity that is so small compared to the total
quantity traded in the market, such that their individual transactions have no influence on the prices. Competitive markets are an ideal, and a standard that other market structures are evaluated by [22].

Competitive equilibrium is a state of market, characterized by a set of prices and an allocation of commodities such that at equilibrium prices, each agent maximizes his objective function subject to his technological limitations and resource constraints, and the market clears i.e., the aggregate supply and demand for the commodities traded are equal.

At competitive equilibrium, the allocation is pareto-optimal which stipulates an important social justification for this theory i.e., there is no other feasible allocation that can increase the benefit of at least one agent in the economy without reducing the benefit of some other agent.

In section 3.2 we discuss on the history of competitive equilibrium theory. In sections 3.3 and 3.4, we describe two market models- Walrasian market model and Fisher’s market model respectively. Section 3.5 presents dynamic case of Fisher’s market model and recursive competitive equilibrium.
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Section 3.6 deals with some important definitions and theorems related to competitive equilibrium theory. Section 3.7 presents applications of competitive equilibrium in computer science and we conclude with section 3.8.

3.2. History of Competitive Equilibrium Theory
In the capitalist market, vital regulatory functions such as ensuring stability, competency, and fairness are relegated to pricing mechanisms. The chief principal in pricing mechanisms is that prices be such that demand equals supply; that is the economy should function at equilibrium. Since the nineteenth century, economists have presented models that represent the concept of market equilibrium. In 1874, Walras published the “Elements of Pure Economics”, in which he introduced a model that indicates the state of an economic system in terms of demand and supply, and expresses the supply equal demand equilibrium conditions [55]. He also introduced a price-adjustment process called tâtonment process performed by a fictitious auctioneer. The prices are announced by auctioneer, and agents report their demands at these prices. No transactions and no production can happen at disequilibrium prices. Instead, prices are lowered for goods with non negative prices and excess supply
and prices are raised for goods with excess demand. The question to be answered is, under what conditions such a process will terminate in equilibrium in which demand equals supply for goods with non-negative prices, and demand does not exceed supply for goods with a zero price? Walras was not able to give a conclusive answer to this question.

The first proof for the existence of equilibrium was given by Wald in 1936, although under severe restrictions [10]. Later in 1954, Arrow and Debreu jointly gave the proof for the existence of equilibrium under much milder assumptions [44]. Also, they have validated using The First Fundamental Welfare Theorem that such equilibrium is Pareto optimal [44].

In 1891 Irving Fisher [94] independently modeled a market. Fisher and Walrasian market models have been studied extensively in mathematical economics. Two techniques have been used mostly for computing equilibrium for these models- the primal-dual scheme [64] and an auction-based approach [76].

3.3. **Walrasian Market Model**
The Walrasian model is also known as the Arrow and Debreu model or the exchange model. In Walrasian model the market
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consists of \( m \) agents and \( n \) divisible goods. Each agent \( i \) have
an initial endowment of goods \( \mathbf{w}_i = (w_{i1}, \ldots, w_{in}) \in \mathbb{R}^n_+ \), and
preferences for consuming goods described by utility
function \( u_i: \mathbb{R}^n_+ \rightarrow \mathbb{R}_+ \). At given prices, each agent \( i \) sells their
initial endowment and then uses the income to buy a bundle
of goods \( \mathbf{x}_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n_+ \), such that their utility \( u_i(\mathbf{x}) \) is
maximized.

The problem is to find prices \( \mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n_+ \), for the
goods, such that if each agent sells her initial endowment at
these prices and buys her optimal bundle, the market clears;
i.e., there is no shortage or surplus of any good. In other
words, an equilibrium is a set of prices \( p \), such that

i. For each agent \( i \), the vector \( \mathbf{x}_i \) maximizes \( u_i(\mathbf{x}) \) subject to

the constraints \( \sum_{i=1}^{n} p_i \cdot x_{ji} \leq \sum_{i=1}^{n} p_i \cdot w_{ji} \) for all \( j = 1, \ldots, n \)

ii. For each good \( j \), \( \sum_{i=1}^{n} x_{ji} = \sum_{i=1}^{n} w_{ji} \)

3.4. Fisher’s Market Model
Fisher’s model is a market of \( n \) goods and \( m \) utility
maximizing buyers of the goods. Each buyer \( i \), is endowed with
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money $e_i$ and a concave utility function $u_i : R_+^n \rightarrow R_+$ for specifying the preference for consuming various goods. Each good $j$ has an initial endowment $b_j$ of the good. Equilibrium is then a non-negative vector of prices $p = (p_1, \ldots, p_n) \in R_+^n$ at which there is a bundle $x_i = (x_{i1}, \ldots, x_{in}) \in R_+^n$ of goods for each trader $i$ such that the following two conditions hold:

i. For each buyer $i$, the vector $x_i$ maximizes $u_i(x_i)$ subject to the constraint $\sum_{k=1}^n p_k * x_{ik} \leq e_i$

ii. For each good $j$, $\sum_{k=1}^m x_{kj} = b_j$

It can easily be observed that the Fisher’s model is a special case of Walras model, when money is considered a good.

3.5 Recursive Competitive Equilibrium Theory

Recursive Competitive Equilibrium Theory was first developed by Mehra and Prescott [56] and further refined in Prescott and Mehra [69] which establish the existence of, and asserts the Pareto optimality of recursive competitive equilibrium. Recursive Competitive Equilibrium approach is one way of
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modeling uncertain dynamic phenomena to search for optimal actions.

Now, we define the dynamic case of Fisher’s market model which consists of a set of $m$ buyers and a set of $n$ divisible goods. Let $e_i(t)$ be the amount of money agent $i$ has at time $t$, and $x_i(t) = x_{i1}(t), \ldots, x_{in}(t)$ be the consumption vector of agent $i$ at time $t$, where $x_{ij}(t)$ is consumption of good $j$ by $i$ at time $t$. Let $b_j(t)$ be the amount of good $j$ present at time $t$, and $p_j(t)$ be the price of good $j$ at time $t$. Let the preference of agent $i$, relative to the consumption $x_i(t)$ at time $t$ be denoted by the utility function $u_i(t, x_i(t))$.

At the start of each period $t$, the amount $e_i(t)$ is determined from the previous period $t-1$ as follows

$$e_i(t) = e_i(t-1) - \sum_{k=1}^{n} p_k(t-1) \cdot x_{ik}(t-1)$$

and the amount $b_j(t)$ of good $j$ at time $t$ is determined as follows

$$b_j(t) = b_j(t-1) - \sum_{k=1}^{m} x_{kj}(t)$$
Next, in each period, equilibrium prices \( p = (p_1(t), \ldots, p_n(t)) \) are determined such that there is a bundle of goods \( x_i(t) = x_{i1}(t), \ldots, x_{in}(t) \) and the following conditions hold:

i. For each buyer \( i \), the vector \( x_i(t) \) maximizes \( u_i(t, x_i(t)) \) subject to the constraints \( \sum_{k=1}^{n} p_k \cdot x_{ik}(t) \leq e_i(t) \)

ii. For each good \( j \), \( \sum_{k=1}^{m} x_{kj}(t) \leq b_j(t) \)

Note that, the period prices depend only on the state variables in that period.

3.6. Properties and Characterization of general equilibrium

The most important issues in general equilibrium analysis are concerned with the conditions under which equilibrium will be efficient, which efficient equilibrium can be achieved, and when equilibrium is guaranteed to exist.

Let \( S \) be a commodity space, where each vector represents a basket of commodities. Let there be \( I \) consumers, \( 1,2,\ldots,I \). Each consumer \( i \) chooses a basket of commodities in the set \( x_i \subseteq S \), giving each commodity point (that is, basket of commodities) a value called the utility function \( u_i : X_i \rightarrow \mathbb{R} \). Let there be \( J \) firms, \( 1,2,\ldots,J \). Each firm \( j \) produces a basket of
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commodities in the set $Y_j \subseteq S$, subject to its technological limitations.

The firms and consumers, interact with each other. Firms demand factors of production and supply final goods, whereas consumers supply factors of production (for example, in the form of labor) and demand final goods.

The selections made by firms and consumers are constrained such that the market clears i.e., $\sum_i x_i - \sum_j y_j = 0$. This means that all produced goods are consumed, and there is no unemployment. The pair $(x_i, y_j)$ called the allocation describes the consumption $x_i$ of each consumer and the production $y_j$ of each firm. The allocation is feasible if $x_i \in X_i$, for all $i$ and $y_j \in Y_j$, for all $j$, and $\sum_i x_i - \sum_j y_j = 0$. Finally, the price system is a continuous linear function $\Phi: S \rightarrow R$. This means that a vector in $S$ representing a basket of commodities has an associated price $p_i$ for each commodity and can be assigned a single number which is the expenditure of purchasing that basket of commodities for consumers.
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**Definition 3.1** An allocation \( (\mathbf{x}_i, \mathbf{y}_j) \) is Pareto Optimal if it is feasible and if there is no other feasible allocation \( (\mathbf{x}'_i, \mathbf{y}'_j) \) such that \( u_i(\mathbf{x}'_i) \geq u_i(\mathbf{x}_i) \) for all \( i \) and \( u_i(\mathbf{x}'_i) > u_i(\mathbf{x}_i) \) for some \( i \). That is, an allocation is Pareto Optimal if no individual can improve his utility without decreasing the utility of others.

**Definition 3.2** An allocation \( (\mathbf{x}_i^0, \mathbf{y}_j^0) \) together with a price system \( \Phi : S \to R \) is a competitive equilibrium if the following three conditions are satisfied.

(C1) The allocation \( (\mathbf{x}_i^0, \mathbf{y}_j^0) \) is feasible.

(C2) For each \( i \), \( \mathbf{x} \in \mathbf{X}_i \) and \( \Phi(\mathbf{x}) \leq \Phi(\mathbf{x}_i^0) \) implies \( u_i(\mathbf{x}_i) \leq u_i(\mathbf{x}_i^0) \).

Equivalently, \( u_i(\mathbf{x}_i) > u_i(\mathbf{x}_i^0) \) implies \( \Phi(\mathbf{x}) > \Phi(\mathbf{x}_i^0) \).

(C3) For each \( j \), \( \mathbf{y} \in \mathbf{Y}_j \) implies \( \Phi(\mathbf{y}) \leq \Phi(\mathbf{y}_j^0) \).

We note from conditions (C3) and (C2) that the allocation is profit maximizing for each firm and, the allocation is utility maximizing for each consumer respectively at the given price system.

**Definition 3.3** The local nonsatiation condition for consumers is satisfied if for each consumer \( i \) and each \( \mathbf{x} \in \mathbf{X}_i \), and for
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every $\epsilon > 0$, there exists $x' \in X_i$ such that $\|x' - x\| \leq \epsilon$ and $u_i(x') > u_i(x)$. That is, for any basket of commodities there is another basket of commodities arbitrarily nearby that is strictly preferred to it.

**Lemma 3.1 (Existence)** If in an economy $X_i$ is a closed convex set bounded from below and utility function $u_i : X_i \to \mathbb{R}$ is continuous and concave, then there exists equilibrium.

Generally the proofs of the existence theorem depend upon Brouwer's fixed point theorem for functions, and Kakutani's fixed point theorem for set-valued functions. There are other methods of proof that use Sard's lemma and Baire category.

**Theorem 3.1 (First Welfare Theorem)** If the local nonsatiation condition for consumers is satisfied and $(x^0_i, y^0_j, \Phi)$ is a competitive equilibrium, then the allocation $(x^0_i, y^0_j)$ is Pareto optimal.

**Proof.** For the sake of contradiction, let the initial allocation $(x^0_i, y^0_j, \Phi)$ be not Pareto optimal. Then there is another feasible allocation $(x'_i, y'_j)$ such that
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\( u_i(x_i') \geq u_i(x_i^0) \) for all \( i \), with strict inequality for some \( i \). Employing condition (C2), this strict inequality \( u_i(x_i') > u_i(x_i^0) \) implies \( \Phi(x_i') > \Phi(x_i^0) \).

Since \( \Phi \) is linear, summing across all \( i \) gives

\[
\sum_i \Phi(x_i') > \sum_i \Phi(x_i^0) .
\]

Since \( \sum_i x_i - \sum_j y_j = 0 \) we have

\[
\sum_j \Phi(y_j') = \sum_i \Phi(x_i') > \sum_j \Phi(y_j^0) = \sum_i \Phi(x_i^0) \]

and this contradicts condition (C3)

**Theorem 3.2 (Second Welfare Theorem)** Assume the economy satisfies the following conditions

1. Fore each consumer \( i=1,\ldots,m \)
   
   a. \( x_i \) is non empty and convex
   
   b. \( u_i(x_i) \) is continuous, locally nonsatiated, and convex

2. For each producer \( j=1,\ldots,n \)
   
   a. \( y_j \) is non empty and convex

Let \( (x_i^0, y_j^0) \) be an efficient allocation. Then there is non-zero price system \( \Phi : S \rightarrow \mathbb{R} \) such that
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(S1) For each consumer $i=1,\ldots,m$, $x \in X_i$ and $u_i(x) \geq u_i(x^0_i)$

Implies $\Phi(x) > \Phi(x^0_i)$

(S2) For each $j$, $y \in Y_j$, implies $\Phi(y) \leq \Phi(y^0_j)$

According to definition of competitive equilibrium conditions (S2) and (C3) are identical since both satisfy the profit maximizing condition for firms. Condition (S1) is slightly weaker one because it only minimizes the expenditure for consumers, rather than maximizing utility as in condition (C2). However, it is possible to improve (S1) if we impose conditions that the consumption set be convex, and the utility function $u_i : X_i \rightarrow \mathbb{R}$ be continuous, and locally non-satiated.

**Proof:** Since $\left(\left(x^0_i\right)\left(y^0_j\right)\right)$ is efficient, it is impossible to make every one better off. Let us define the set $V$ as

$$V = \sum_i x_i$$

and set $A$ by $A = \sum_j y_j$

By Pareto optimality $A \cap V = \emptyset$. Suppose $x \in A \cap V$, since $x \in V$, we can write $x = \sum_i x_i$, where each $x_i \in X_i$. Since $y \in A$, we can write $y = \sum_j y_j$,
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each \( y_j \in Y_j \). But then \( (x_i, y_j) \) is an allocation, and
\[ u_i(x_i) > u_i(x_i^0) \] for each \( i \), contradicting the efficiency of \( (x_i^0, y_j^0) \).

\( V \) is open and convex since each summand is, and is nonempty by local nonsatiation. Similarly \( A \) is convex. Thus by the separating hyperplane theorem, there is a non-zero price system \( \Phi: S \to \mathbb{R} \) satisfying \( \Phi(x) > \Phi(y) \) for each and \( x \in V \) and \( y \in A \).

Each \( x_i^0 \) belongs to the closure of \( X_i \), so \( \sum_i x_i^0 \) belongs to the closure of \( V \). Now \( \sum_i x_i = \sum_j y_j \) so it belongs to \( A \).

It follows that \( \Phi(x) \geq \Phi(\sum_i x_i) = \Phi(\sum y_j) \geq \Phi(y) \) for each \( x \in V \), \( y \in A \). From the summation principle we have
\[ \Phi(x_i^0) \leq \Phi(x) \text{ for all } x \in X_i \]
and
\[ \Phi(y_j^0) \geq \Phi(y) \text{ for all } y \in Y_j \]
3.7. Applications of Competitive Equilibrium

Over the past few years, there has been an extensive research done in applying economic solutions to the problems of computer science. We list below some of the domains in computer science for which competitive equilibrium solutions are proposed.

**Congestion Control and Avoidance**

Over the years computer networks have seen an explosive growth. While available bandwidth is increasing, the demand for bandwidth is growing even more. Congestion control is therefore an important engineering topic. A good congestion control algorithm should consider the utilities of users while being fair. Competitive equilibrium approach for congestion control and avoidance is studied in [8],[9],[50],[78].

**Information Security and Network Security**

With the advent of internet, an increasing amount of information is available on-line and cyber crime is becoming one of the most profitable criminal activities. Technology based solutions for security are available, but they fail because incentives are wrong. For instance, the people who should guard the system are not the one who suffers the full costs of failure. Consequently, they make less effort than
would be socially optimal. The concepts of competitive equilibrium theory are becoming just as important as the mathematics of cryptography to the security problems ([1],[24],[66],[67])

**Distributed Computing**

Distributed applications generate a large number of tasks/jobs that need resources scattered on various sites that are connected to each other by a network. Since resources are distributed and used by many users having different requirements, users are likely to behave in a selfish manner. Load balancing is a technique, that distributes workload evenly across the resources, such that resource utilization and throughput are maximized, and response time and overload are minimized. Competitive equilibrium approach for load balancing is considered in [46],[47],[48], which simultaneously minimizes the mean response time of all jobs, and the response time of each job individually.

**Video Streaming**

In direct broadcast satellite, cable TV, video-on-demand service, and video surveillance, multiple video steams are transmitted simultaneously through a shared channel.
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Generally, while multiplexing several video streams on a single channel, the quality of high motion videos is improved at the expense of reduction in the quality of low motion videos. Competitive equilibrium approach for simultaneously improving the quality of all video streams is studied in [60],[61].

Spectrum Allocation
With the explosion of various radio devices and services, multiple wireless systems sharing a communication spectrum must co-exist. In such a multi-user system, each user's performance, measured by Shannon utility function, depends on not only the power allocation of its own, but also of other users in the system. A decentralized method for spectrum allocation management and optimization using competitive equilibrium approach is discussed in [57],[70] which maximize each user's utility simultaneously.

3.8. Conclusion
Competitive equilibrium theory of a market, is a powerful tool for modeling, analyzing, and understanding the situations where there are multiple self-interested agents who interact with each other while making their decisions. Each agent
strives to maximize its utility by choosing their strategies depending on the strategies chosen by other agents. Most problems in computer science can be modeled as a market. Therefore, competitive equilibrium theory concepts can be applied for many problems in computer science and provide a useful insight. In this paper an introductory overview of basic concepts of competitive equilibrium theory and their applications in networking, security, and many more are discussed.