Chapter 2

Game Theory

The nature of computing has changed with the advent of internet and the revolution in Information Technology. With the progress in technologies, the components such as network, computing resources, storage sites, and software have been commoditized. In the new paradigm, there are several objects (hardware, software agents, protocols etc.) that work on behalf of various independent entities (such as a user, a business etc.) and deliver services to other similar entities. Internet has made it possible for many such geographically spread entities to interact with each other and deliver various services. These entities will work on behalf of their owners to achieve their individual objectives (i.e., maximize their individual payoffs) in contrast to obtaining a system optima (that is social optimal solution). This results in a computing paradigm where the "work" is done in absolutely distributed/decentralized manner by different entities where the chief objective of each entity is to maximize its own objective. Consequently, it is essential to analyze conventional computer science concepts such as algorithm design, protocols, and performance optimization under a game-theoretic model.
2.1. Introduction

Game theory is a branch of applied mathematics used in varied fields such as social sciences, biology, engineering, computer science, and particularly in economics [23]. Game theory attempts to mathematically capture interactions among individuals in a competitive environment, where an individual’s success in decision-making depends on the choices of others. It was initially developed to analyze interactions among individuals in parlor games such as Poker or Bridge, and has been now expanded to analyze a wide class of interactions, which are classified according to several criteria.

There are two main branches of game theory: cooperative and non-cooperative game theory. In cooperative games the players can impose binding commitments and the players can choose the strategies by a consensus decision making process; whereas in non-cooperative games the players make decisions independently. Another distinguishing feature is that the communication among players is permitted in cooperative games, but not in non-cooperative games. Cooperative games focus on the games at large, but in contrast non-cooperative games model situations to the finest details, producing accurate results. Hybrid games contain the features of both cooperative and
non-cooperative games, where the coalitions of players are formed as in a cooperative game, but they play in a non-cooperative fashion.

Although games have been studied by many scholars, the credit for conceptualizing the games into a mathematical theory of strategy was made in 1921 by Émile Borel, and was fostered further by the mathematician John von Neumann in 1928, when he proved the minimax theorem, the fundamental theorem of games of strategy. Until the publication of “Theory of Games and Economic Behavior” by von Neumann and the economist Oskar Morgenstern, game theory was not recognized as a discipline in its own right. In the 1950s and the 1960s, game theory was advanced to a great extent by many researchers, and applied to problems of war and politics. Subsequently in 1970s, it brought a great revolution in economic theory. In addition, it also found applications in sociology and psychology, and found links with evolution and biology. It received special attention when eight game theorists were awarded Nobel Prizes in economic sciences.

This chapter aims to provide an insight into various game-theoretic concepts and its applications in different domains. The rest of the chapter is organized as follows.
Section 2.2 presents definitions of games, strategies, costs and payoffs. In section 2.3, different kinds of equilibrium are discussed. In section 2.4, proof for the existence of Nash equilibrium is given, and then strategy for obtaining equilibrium and complexity of finding equilibrium are discussed. Section 2.5 presents some of the most important applications of game theory in computer science and we conclude with section 2.6.

2.2. Games, Strategies, Costs and Payoffs

Usually, a game is characterized by three elements: the set of players, \( i \in I \) where \( I=1 \) to \( n \), the pure strategy space \( S_i \) for each player \( i \in I \), and the utility function \( U_i : S_1 \times S_2 \times \ldots \times S_n \rightarrow \mathbb{R} \) for each player \( i \in I \).

In order to play the game, each player \( i \) must select a strategy \( s_i \in S_i \). Let \( s = s_1, \ldots, s_n \) denote the vector of strategies selected by the players and \( S = S_1, \ldots, S_n \) denote the set of all possible ways in which players can choose strategies.

The outcome of each player is determined by the vector of strategies \( s \in S \) chosen by all the players, and will usually be different for each player. In order to specify the game, we need to provide for each player, a preference ordering on the outcomes by giving a complete, transitive, reflexive binary
relation on the set of all strategy vectors \( S \). That is, given two strategies of \( S \), the relation for player \( i \), should specify, which of these two strategies \( i \) prefers.

One way of specifying preferences is by assigning for each player, a value to each outcome. The values can be thought of as the *payoffs* to players or, the *costs* incurred by players. Let the functions \( p_i : S \to \mathbb{R} \) and \( c_i : S \to \mathbb{R} \) denote payoff and cost respectively. Evidently, costs and payoffs can be used interchangeably, since \( p_i(S) = c_i(S) \).

### 2.3. Basic Solution Concepts

We will start the discussion here by describing what are perhaps the most well-known and well-studied games.

**Example 2.1 (Prisoner’s Dilemma)** Two prisoners are on a trial for a crime and each one has a choice of either confessing to the crime or remaining silent. If they both remain silent, both will be sentenced for a prison of 2 years. If one of them confesses and the other remains silent, the defector’s term will be reduced to 1 year and the silent accomplice will get a sentence of 5 years. Finally, if they both confess, both will have to serve the prison for 4 years each. Clearly, there are four total outcomes depending on the choices made by each of the two prisoners; we can
concisely summarize the costs incurred by each player for the four outcomes by means of the following 2x2 matrix.

\[
\begin{array}{cc}
\text{Prisoner 1} & \text{Confess} & \text{Silent} \\
\text{Confess} & 4 & 5 \\
\text{Silent} & 1 & 2 \\
\end{array}
\]

The only stable solution in this game is that both prisoners confess, because in each of the other three cases, at least one of the players can change from “silence” to “confession” and improve his payoff.

**Example 2.2 (Battle of Sexes)** Two players, a boy and a girl have two choices for spending evening: going to a baseball game or going to a softball game. The boy prefers baseball game, while the girl prefers softball game. We can express the players’ preferences by means of a payoff matrix as follows.

\[
\begin{array}{cc}
\text{Boy} & \text{Baseball} & \text{Softball} \\
\text{Girl} & 6 & 1 \\
\text{Baseball} & 5 & 2 \\
\text{Softball} & 1 & 5 \\
\end{array}
\]
Undoubtedly, the two solutions where the two players choose differently are not stable because in each case, either of the two players can improve their own payoff by changing their decision. On the other hand, the two remaining options of both choosing the same game, whether it is softball or baseball, are both stable solutions. The girl prefers the first, while the boy prefers the second.

Example 2.3 (Matching Pennies) Two players, each having a penny have two choices- heads (H) and Tails (T). Player 1 wins if the two pennies match, while player 2 wins if they do not match, as shown by the following pay off matrix, where 1 indicates win and -1 indicates loss.

\[
\begin{array}{c|cc}
   & \text{Player 1} & \text{Player 2} \\
\hline
\text{Player 1} & -1 & 1 \\
1 & -1 & 1 \\
-1 & 1 & -1 \\
\end{array}
\]

Clearly this game has no stable solution. As an alternative, it appears to be best for the players to randomize in order to impede the strategy of the other player.
2.3.1. Dominant Strategy Solution
We say that a game has a dominant strategy solution, if each player has a unique best strategy, independent of the strategies played by the other players.

For a strategy vector \( \mathbf{s} \in \mathcal{S} \), let \( \mathbf{s}_i \) denote the strategy played by player \( i \), and \( \mathbf{s}_{-i} \) denote the (n-1) dimensional vector of the strategies played by all other players. Also let \( u_i(\mathbf{s}_i, \mathbf{s}_{-i}) \) denote the utility incurred by player \( i \). Then the vector \( \mathbf{s} \in \mathcal{S} \) is a dominant strategy solution, if for each player \( i \), and each alternate strategy vector \( \mathbf{s}' \in \mathcal{S} \), we have

\[
u_i(\mathbf{s}_i, \mathbf{s}_{-i}) \geq u_i(\mathbf{s}'_i, \mathbf{s}'_{-i})
\]

One can notice that a dominant strategy solution may not result in an optimal payoff to any of the players. The outcome of Prisoner’s dilemma indicating mutual defection is a dominant strategy solution.

2.3.2. Pure Strategy Nash Equilibrium
Games rarely possess dominant strategy solutions; therefore a less stringent and more broadly applicable solution is needed. An attractive game theoretic solution is one in which each player acts in accordance with their incentives, maximizing their own payoff. This solution concept is best captured by the notion of a Nash equilibrium, which has
emerged as the predominant solution concept in game theory, with enormously varied applications.

A strategy vector $s \in S$ is said to be Nash equilibrium, if for each player $i$ and each alternate strategy $s'_{i} \in S_{i}$, we have that

$$u_{i}(s_{i}, s_{-i}) \geq u_{i}(s'_{i}, s_{-i})$$

Alternatively, no player can change his chosen strategy from $s_{i}$ to $s'_{i}$ and consequently improve his payoff, when all other players stick to the strategies they have chosen in $s$.

2.3.3. Mixed Strategy Nash Equilibrium

As demonstrated by the Matching Pennies game, a game need not possess any pure strategy Nash equilibrium. However, one can obtain a stable solution for the matching pennies game, if the players are permitted to randomize, and each player can choose each of his two strategies with probability $\frac{1}{2}$. Then the expected payoff of each player will be 0 and neither player can improve on this by choosing a different randomization.

In randomized strategies each player can pick a probability distribution over his set of possible strategies; such a choice is called a mixed strategy. A mixed strategy Nash equilibrium is then a mixed strategy profile with the characteristic that no single player can improve his
expected payoff by unilateral deviation. Expected payoffs must be taken into account because the outcome of the game may be random.

2.4. Finding Equilibria and Learning in Games
In this section we discuss about the conditions for existence of Nash Equilibrium in a game, complexity of finding the Nash Equilibrium, and a natural game playing strategy for arriving at Nash Equilibrium.

2.4.1. Existence of Nash Equilibrium
The proof for the existence of Nash equilibrium relies on one of the fixed point theorems- Kakutani’s fixed point theorem

**Theorem (Kakutani’s Fixed Point Theorem)** Let \( A \subset \mathbb{R}^n \) be a non empty, convex, compact set and \( f: A \to A \) be a upper hemi-continuous correspondence from \( A \) to itself, such that \( f(x) \in A \) is non-empty and convex for each \( x \in A \). Then \( f \) has a fixed point i.e., \( x \in f(x) \) for some \( x \in A \).

Let \( \sigma_i \) be a mixed strategy profile of all players except for player \( i \). Let us define \( b_i \) as a relation from the set of all probability distributions over opponent player profiles to a set of player \( i \)'s strategies, such that each element of \( b_i(\sigma_{-i}) \) is a best response to \( \sigma_{-i} \).
Nash’s Theorem: A game has Nash equilibrium if for all \( i \in I \), the set \( S_i \) are non-empty, convex and compact subset of Euclidean space and the utility function is continuous and quasiconcave in each \( S_i \).

Proof: Let us define \( b(\sigma) = b_1(\sigma_{-1}) \times b_2(\sigma_{-2}) \times \ldots \times b_n(\sigma_{-n}) \). The function \( b_i \) is non-empty, convex-valued and upper hemi-continuous, since each \( S_i \) is non-empty, compact, and convex, and the utility function \( u_i(s_i, s_{-i}) \) is continuous and quasiconcave in each of the variable \( S_i \).

As all the conditions for kakutani’s fixed point theorem are satisfied by function \( b_i \), there exists a fixed point for this correspondence, say \( s \in S \), such that \( s \in b(s) \). Clearly \( s \) is Nash equilibrium since each \( s_i \in b_i(s_{-i}) \).

2.4.2. Complexity of Finding Equilibrium
Note that the proof in the previous section only shows that Nash equilibrium in mixed strategies exists, but does not show how to find that equilibrium.

Usually, NP-completeness is used to characterize computational problems which, just like NASH and
SATIFIABILITY, are computationally hard. However, NASH is a very different kind of intractable problem, one for which NP-completeness is not suitable concept of complexity. The reason is that every game is guaranteed to have NASH equilibrium. On the other hand, in a typical NP-complete problem such as SATISFIABILITY, the required solution may or may not exist.

The proof that establishes that Nash equilibrium exists for mixed strategies relies on Kakutani’s fixed point theorem, which contains a parity argument. Therefore, we will use PPAD-completeness (Polynomial Parity Argument Directed) to characterize the complexity of finding Nash equilibrium. In fact, the problem of finding NASH equilibrium is PPAD-complete even for two-player game in standard form.

**2.4.3. Best Response and Learning in Games**

A game playing strategy should quickly lead players to either find the equilibrium or at least converge to equilibrium in the limit. The usual natural game playing strategy is the following “best response”. Consider a strategy vector $s$ and a player $i$. With the strategy vector $s$, player $i$ gets the value of utility $u_i(s)$. Switching from the strategy $s_i$ to some other strategy $s'_i$ the player can change his utility.
to \( u_i(s_i, s_{-i}) \) if all other players stick to their strategies in \( s_{-i} \). A change from strategy \( s_i \) to \( s'_i \) is an improving response for player \( i \) if \( u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}) \) and, best response if \( s'_i \) maximizes the players utility \( \max_{s_i \in S_i} u_i(s_i, s_{-i}) \). Playing a game by repetitively allowing some player to make an improving or a best response move is possibly the most natural game play.

In some games, such as the prisoner’s dilemma, this natural game playing strategy leads the players to Nash equilibrium in a few steps. In other games, the players may not reach the equilibrium in a finite number of steps, but the strategy vector will converge to the equilibrium. In still other games, the play may cycle and not converge.

### 2.5. Applications

Over the past few years, there has been an extensive research done at the interface of computer science and game theory. In this section a survey on some of the important applications of game theory in computer science are presented.

**Resource Allocation**

In the current computing paradigm, resources like storage repositories, computing servers, and network bandwidth are
used by many users having different requirements. Usually users on the network compete with each other in order to use the resources. Their competition can be modeled using the concepts of game theory. Moreover, because of the diverse ownership of these network resources, no centralized control of network usage is possible. The problem of resource allocation is to ensure fair sharing of network resources. Game theoretic models for resource allocation emphasize on maximizing each user's utility, and consequently ensure fair allocation. A game theoretic framework for bandwidth allocation is considered in [19],[98] and job allocation is considered in [28],[71].

**Artificial Intelligence**

Game theory and Artificial Intelligence are two research disciplines stemming from similar roots, but have taken different research directions over last 5 decades. However, the link between these areas is profound. There is an extensive work done at the intersection of these two research areas ([12],[58],[63],[75])

**Cryptography**

The cryptography and game theory domain appear to have an overlapping because both deal with communication between mutually distrustful parties which has some end
outcome. In cryptography the multiparty communication deals with a set of parties communicating for the purpose of evaluating a function on their inputs, where each party receives at the end some output of the evaluation. Similarly in a game, the participants receive payoff according to the joint actions of all the parties, while each participant wishes to maximize his own payoff. In the past few years, the relationship between these two areas has been investigated widely ([27],[59],[86]).

**Routing**

Routing was first studied in the perspective of transportation networks by Wardrop [92]. Routing in computer networks is more complex than transportation networks because of the presence of multiple domains belonging to diverse authorities. Selfishness of authorities or even the selfishness of the nodes themselves could result in significant performance degradation from the expected one. Therefore, we should consider all forms of manipulative behavior of selfish agents when designing the routing mechanism. This means that we must provide incentives that ensure selfish agents find it in their best interest to behave truthful. Use of incentives for preventing the manipulation by selfish agents was proposed in [34], [35].
Wireless Sensor Networks

Wireless sensor networks are tiny, power-constrained nodes used in a wide variety of environments like monitoring of environmental attributes, intrusion detection, and various military and civilian applications. Common performance criteria for wireless sensor networks is extending network lifetime while satisfying coverage and connectivity in the deployment region. Security is another important performance criterion in wireless sensor networks, where hostile environments pose various kinds of threats to reliable network operation. The application of wireless sensor networks in intruder detection environments lends itself to game-theoretic formulation of these environments, because pursuit-evasion games provide an appropriate framework to model detection, tracking and surveillance applications. The appropriateness of using game theory to analyze security and energy efficiency problems and pursuit-evasion scenarios using wireless sensor networks originates from the nature of strategic interactions between nodes. Thus, game theoretic approaches can be used to optimize node-level as well as network-wide performance by taking advantage of the distributed decision-making capabilities of wireless sensor networks and have been discussed in [4], [31], [37], [39].
**Datamining**
Datamining extracts useful knowledge from a huge amount of database. Applying datamining algorithm together with game theory poses a significant potential as a new way to analyze complex engineering systems. Game theory based datamining approach are studied in [99],[102].

**2.6. Conclusion**
Game theory deals with systems where there are multiple agents, facing uncertainty and having possibly different goals. Most problems in computer science can be modeled as a game involving two or more agents, who interact with each other, and each agent strives to maximize his own payoff by choosing a strategy depending on strategies chosen by other agents. Therefore, general methods and concepts presented by game theory, such as strategies and equilibrium can be applied and provide valuable insights in many applications of computer science.