in $L^2([0, \phi])$ and it is also compact and there exists a $M > 0$ such that $\left\| U(\cdot, \cdot) \right\| \leq M$. Define

$$K(t, s, x_s) = \int_0^t K(t, s, t - s, \xi, u(s, \xi)) \, ds$$

and

$$f(t, x_t) = f(t, t - s, \xi, u(s, \xi))$$

Clearly Now let $u_0 \in X$, $\int_0^t K(t, s, x_s)$ and $f(t, \cdot)$ is a continuous function. All conditions of theorem 6.3.1 are now fulfilled so we deduce that 6.4.1 has an integral solution. □
Chapter 7

EXISTENCE RESULTS FOR FUNCTIONAL FRACTIONAL FUZZY IMPULSIVE DIFFERENTIAL EQUATIONS

7.1 Introduction

The fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, Physics, economy and science. We can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [20; 31; 33; 43; 59; 60]. In recent years, there has been a significant development in fractional differential equations. One can see the monographs of Kilbas et al. [49], Miller and Ross [62], Podlubny [70], Lakshmikantham et al. [53], and the chapters on abstract fractional differential equations [2; 10; 12; 30; 69; 80] and the references therein.

The study of fuzzy differential equations has been initiated as an independent subject in conjunction with fuzzy valued analysis [18; 67] and set-valued differential equations [51]. Using the Hukuhara derivative of multi-valued functions, Puri and Ralescu [71] have introduced the concept of H-differentiability for fuzzy functions. This concept has been studied and applied in context of fuzzy differential
equations by Seikkala [77] and Kaleva [46] in time dependent form. Kaleva showed in [46; 47] that if continuous and satisfies the Lipschitz condition with respect to \( u \), then there exists a unique local solution for the fuzzy initial value problem
\[
\dot{u}(t) = f(t, u), \quad u(0) = u_0 \quad \text{on } (E^n, D).
\]
The existence theorem of solutions for fuzzy initial value problem under different sets of assumptions are given in Ding et.al [19] and Song et.al [78]. Park et.al [69] studied the approximate solutions of the fuzzy functional integral equations. The local existence and uniqueness and investigated by Balasubramaniam and Muralisankar [7] for nonlinear fuzzy neutral functional differential equations. Guo et.al [38] established some existence results for the fuzzy impulsive functional differential equations. Recently Lupulescu [58] has established the local and global existence and uniqueness results for fuzzy functional differential equations.

Our aim in this chapter is to study the existence and the uniqueness of the solution for functional fractional fuzzy impulsive differential equations.

\[
D^\alpha_I x(t) = A(t)x(t) + f(t, x(t)), \quad \text{for each } t \in [0, \theta], \quad t = t_k, \quad k = 1, 2, \ldots, m
\]

\[
\Delta x(t)/t = I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, \ldots, m
\]

\[
x(0) = x_0,
\]

(7.1.1)

where \( 0 < \alpha < 1 \) and \( \{A(t), t \in J\} \) is a family of closed densely defined linear unbounded operators on \( X \). \( D^\alpha_I \) is the 
\( \alpha \) caputo fractional derivative, \( f: J \times C^\infty \to X \), is a given fuzzy function satisfying appropriate conditions, \( \phi \in C^\infty \), \( \phi(0) = 0 \) and \( C^\infty = C(( -\infty, E^1) \), \( x(t_k) = x(t_k^+) - x(t_k^-) \) where \( x(t_k^+), \) and \( x(t_k^-) \) are right and left limits of \( x(t_k) \) at \( t_k, k = 1, 2, \ldots, m. \)

The chapter is organized as follows: In section 7.2, we introduce the mild
solution of \((7.1.1)-(7.1.3)\) and recall some Lemmas which are used in the sequel.

In section 7.3, we study the existence and uniqueness of mild solution of system \((7.1.1)-(7.1.3)\) under some suitable conditions. An example to illustrate our results is given at last.

### 7.2 Preliminaries

In this section, we introduce notations, definitions and preliminaries facts which are used throughout this chapter.

By \(C(J, R)\) we denote the Banach space of all continuous functions from \(J\) into \(R\) with the norm

\[
\|y\|_\infty = \sup \{ \|y(t)\|_t \in J \}.
\]

Let \(K_c(R^n)\) denote the collection of all non empty compact convex subsets of \(R^n\).

We define the Hausdorff distance between \(A, B \in K_c(R^n)\) by

\[
d_H(A, B) = \max \{ \sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\| \}.
\]

Denote

\[
E^n = \{ u : R^n \to [0, 1] / u \text{ satisfies } (i)-(iv) \}
\]

(i) \(u\) is normal, that is, there exists an \(x_0 \in R^n\) such that \(u(x_0) = 1\),

(ii) \(u\) is fuzzy convex, that is, for \(x, y \in R^n\) and \(0 \leq \lambda \leq 1\),

\[
u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\};
\]

(iii) \(u\) is upper semicontinuous;

(iv) \([u]^0 = cl\{x \in R^n : u(x) > 0\}\) is compact.
For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$. Then from $(i) - (iv)$, it follows that the $\alpha$-level set $[u]^\alpha \in K_c((\mathbb{R}^1))$ for all $0 < \alpha \leq 1$. For later purpose, we define $\hat{0} \in E^n$ as $\hat{0}(x) = 1$ if $x = 0$ and $\hat{0}(x) = 0$ if $x = 0$. If we define

$$D[u, v] = \sup_{0 \leq \alpha \leq 1} d_\gamma([u]^\alpha, [v]^\alpha),$$

then it is well known that $D$ is a metric in $E^n$ and that $(E^n, D)$ is a complete metric space [46; 71]. We list the following properties of $D[u, v]$:

(i) $D[u + w, v + w] = D[u, v]$, $D[u, v] = D[v, u],$

(ii) $D[\lambda u, \lambda v] = |\lambda|D[u, v],$

(iii) $D[u, v] \leq D[u, w] + D[w, v]$

for all $u, v, w \in E^n$ and $\lambda \in \mathbb{R}$.

In the following we recall some main concepts and properties of differentiability and integrability for fuzzy functions [46; 71; 72].

If there exists $w \in E^n$ such that $u = v + w$, then $w$ is called the $H$-difference of $u$ and $v$ and is denoted by $u - v$. Let $I$ be an interval in $\mathbb{R}$. A mapping $F : I \to E^n$ is differentiable at $t_0 \in I$ if there exists a $F'(t_0) \in E^n$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t_0) - F(t_0 - h)}{h}$$

exist and equal to $F'(t_0)$. If $F : I \to E^1$ is a fuzzy function such that

$$[F(t)]^\alpha = [F^\alpha_1(t), F^\alpha_2(t)], \quad \alpha \in [0, 1],$$

and there exists $F'(t_0)$ for some $t_0 \in I$, then

$$[F'(t_0)]^\alpha = [(F^\alpha_1)'(t_0), (F^\alpha_2)'(t_0)], \quad \alpha \in [0, 1].$$

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The fuzzy function $F : I \rightarrow E^n$ is called strongly measurable if for each $\alpha \in [0, 1]$ the set-valued function $F_\alpha : I \rightarrow K_c(R^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable. A function $F : I \rightarrow E^n$ is called integrable bounded if there exists an integrable function $h : I \rightarrow R^+$ such that $D[F_0(t), \hat{0}] \leq h(t)$ for all $t \in I$. The integral of $F$ over $I$, denoted $\int_* F(t) dt$, is defined levelwise by the equation

$$\int_* F(t) dt^\alpha = \int_* F_\alpha(t) dt,$$

$$= \int_* f(t) dt^\alpha \quad \text{if} \quad f : I \rightarrow R^n \text{ is measurable selection for } F_\alpha \text{ for } 0 \leq \alpha \leq 1.$$

A strongly measurable and integrable function $F : I \rightarrow E^n$ is said to be integrable over $I$ if $\int_* F(t) dt \in E^n$.

**Proposition 7.2.1** If $F : I \rightarrow E^n$ is continuous, then it is integrable over $I$. Moreover, in this case, the function $G(t) = \int_{t_0}^t F(s) ds$, $t_0, t \in I$, is differentiable and $G(t) = F(t)$.

**Proposition 7.2.2** Let $F : I \rightarrow E^n$ be differentiable and assume that the derivative $F'(t)$ is integrable over $I$. Then, for each $t \in I$, we have

$$F(t) = F(t_0) + \int_{t_0}^t F'(s) ds, \quad t_0 \in I.$$

It is well known that $F, G : I \rightarrow E^n$ are integrable, then the followings hold [46]

(i) $\int_* (F(t) + G(t)) dt = \int_* F(t) dt + \int_* G(t) dt$;

(ii) $\int_* \lambda F(t) dt = \lambda \int_* F(t) dt, \quad \lambda \in R$;

(iii) $D[F, G]$ is integrable;
(iv) \( D^\alpha \left( \int_a \dot{F}(t)dt, \dot{G}(t)dt \right) \leq \int_a D[F(t), G(t)]dt \)

(v) \( \int_{t_0}^{t_2} \dot{F}(t)dt = \int_{t_0}^{t_1} \dot{F}(t)dt + \int_{t_1}^{t_2} \dot{F}(t)dt \) for \( t_0, t_1, t_2 \in I \).

If \( I \) is a compact interval of \( R \), then \( C(I, E^n) \) denotes the set of all fuzzy continuous functions from \( I \) into \( E^n \). On the space \( C(I, E^n) \) we consider the following metric:

\[
D(u, v) = \sup_{t \in I} D[u(t), v(t)].
\]

We denote by \( C_\infty \) the space \( C((-\infty, 0], E^1) \). Also, we denote by

\[
D_\infty(u, v) = \sup_{t \in (-\infty, 0]} D[u(t), v(t)]
\]

the metric on the space \( C_\infty \). Let \( u(.) \in C_\infty \). Then for each \( t \in [0, \bar{t}] \) we denote by \( u(t) \) the element of \( C_\infty \) defined by \( u(t) = u(t + s), s \in (-\infty, 0] \).

**Definition 7.2.1.** The fractional primitive of order \( \alpha > 0 \) of a function \( h : R^+ \to R \) is defined by

\[
\mathcal{I}_0^\alpha h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds
\]

provided the right side exists pointwise on \( R^+ \) and \( \Gamma \) is the gamma function.

**Definition 7.2.2.** The fractional derivative of order \( \alpha > 0 \) of a fuzzy continuous function \( h : R^+ \to E^1 \) is given by

\[
\frac{d^\alpha h(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} h(s)ds
\]

\[
= \frac{d}{dt} \mathcal{I}_0^{1-\alpha} h(t).
\]

**Definition 7.2.3.** Let \( f : R_+ \to R \). The Caputo fractional derivative of order \( \alpha \) of \( f \) is defined by

\[
\mathcal{D}^\alpha f(t) = \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(s)ds
\]

where \( \alpha \in (n-1, n), n \in N \).
\[ f(s) = \begin{cases} 
(t - s) & s > 0, \\
0 & s < 0. 
\end{cases} \]
Definition 7.2.4. Let $f: \mathbb{R}_+ \to \mathbb{R}$ be a continuous function on $\mathbb{R}^+$ and $\alpha \geq 0$. Then the expression
\[
\mathcal{J}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0
\]
is called the Riemann-Liouville integral of order $\alpha$.

In this chapter, we assume that the space $(C_\infty, D_\infty)$ is a metric space of fuzzy functions from $(-\infty, 0]$ to $E^1$ and satisfies the following axioms:

If $y: (-\infty, \beta] \to E^1$ and $y_0 \in C_\infty$, then for every $t \in [0, \beta]$ the following conditions hold:

(A1) $y_t$ is in $C_\infty$;

(A2) $D_\infty(y_t, \hat{0}) \leq K(t) \sup \{D[y(s), \hat{0}] | 0 \leq s \leq t\} + M(t) D_\infty(y_0, \hat{0})$;

Lemma 7.2.1. (1) $U(t, 0) \in L_b(X)$ for $0 \leq \theta \leq t \leq T$. Denotes $M = \sup_{t \in J} \|U(t, s)\|_{L_b(X)}$, which is a finite number.

(2) $U(t, r)U(r, \theta) = U(t, \theta)$ for $0 \leq \theta \leq r \leq t \leq T$.

(3) $U(\cdot, x) \in C(\Lambda, X)$ for $x \in X, \Lambda = \{(t,0) \in J \times J | 0 \leq \theta \leq t \leq T\}$.

(4) For $0 \leq \theta \leq t \leq T$, $U(t, \theta): X \to D$ and $t \to U(t, \theta)$ is strongly differentiable on $X$. The derivative $\frac{\partial}{\partial t} U(t, \theta) \in X$ and it is strongly continuous on $0 \leq \theta \leq t \leq T$. Moreover,

$$\frac{\partial}{\partial t} U(t, \theta) = -A(t) U(t, \theta) \quad \text{for} \quad 0 \leq \theta \leq t \leq T.$$
\|A(t)U(t, \theta)A(\theta)^{-1}\|_X \leq C \quad \text{for} \quad 0 \leq \theta \leq t \leq T.
(5) For every \( v \in D \) and \( t \in (0, T] \), \( U(t, \theta)v \) is differentiable with respect to \( \theta \) on \( 0 \leq \theta \leq t \leq T \)

\[
\frac{\partial}{\partial \theta}U(t, \theta)v = U(t, \theta)A(\theta)v.
\]

(6) \( U(t, \theta) \) is compact operator for \( 0 \leq \theta \leq t \leq b \). And, for each \( x_0 \in X \), the Cauchy problem has a unique classical solution \( x \in C^1(J, X) \) given by

\[
x(t) = U(t, 0)x_0, \quad t \in J.
\]

**Definition 7.2.5.** By a PC-mild solution of the system (7.1.1) we mean the function \( x \in PC(J, X) \) which satisfies

\[
x(t) = U(t, 0)x_0 + \frac{1}{\Gamma(\alpha)} \int_{0< \theta < t}^t (t - s)^{\alpha - 1} U(t, s)f(s, x(s))ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t - s)^{\alpha - 1} U(t, s)f(s, x(s))ds + \int_{0< \theta < t}^t U(t, \theta)I_k(x(\theta)).
\]

**7.3 Existence results**

In this section, we will derive the existence result concerning the PC-mild solution for the system (7.1.1) under some easily checked conditions.

We make the following assumptions.

(H1) \( f : J \times X \to X \) is strongly measurable with respect to \( t \) on \( J \) and for any \( x, y \in X \) satisfying \( ||x'||, ||y'|| \leq \rho \) there exists a positive constant \( L_f(\rho) > 0 \) such that

\[
D[f(t, x), f(t, y)] \leq L_f(\rho)||x - y'||.
\]

(H2) There exists a positive constant \( M_f > 0 \) such that

\[
||f(t, x)|| \leq M_f(1 + ||x||^m) \quad \text{for all} \quad t \in J, \text{some} \quad m > 1.
\]
(H3) (1) The nonlinear map $I_k : X \rightarrow X$, $I_k(X)$ is a bounded subset of $X$, $k = 1, 2, \ldots, \delta$.

(2) There exist constants $h_k > 0$, such that

$$D[I_k(x), I_k(y)] \leq h_k |x - y|, \text{ for all } x, y \in X, k = 1, 2, \ldots, \delta.$$ 

**Theorem 7.3.1.** *Under the assumptions (H1)–(H3), system (7.1.1) has at least a PC-mild solution on $J$.***

**Proof.** Let $x_0 \in X$ be fixed. Define an operator $H$ on $PC(J, X)$ which is given by

$$(Hx)(t) = U(t, 0)x_0 + \frac{1}{\Gamma(\alpha)} \int_{0 < t_k < t} (t_k - s)^{\alpha - 1} U(t, s)f(s, x(s))ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha - 1} U(t, s)f(s, x(s))ds$$

$$+ U(t, t_k)I_k(x(t_k)). \quad (7.3.1)$$

Using (H1) and (H2), one can verify that $H$ is a continuous mapping from $PC(J, X)$ to $PC(J, X)$ for $x \in PC(J, X)$. In fact, for $0 < \tau < t \leq t_1$, it comes from (H1) and the following inequality

$$D[(Hx)(t), (Hx)(\tau)] \leq \left| U(t, 0)x_0 - U(\tau, 0)x_0 \right|$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{0 < t_k < t} (t - s)^{\alpha - 1} U(t, s)D[f(s, x(s)), f(s, x'(s))]ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha - 1} U(t, s) - U(\tau, s) \cdot D[f(s, x(s)), f(s, x'(s))]ds$$

$$\leq M \left| U(t, \tau)x_0 - x_0 \right| + \frac{M \cdot f_{PC}}{\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha - 1} ds$$

$$+ \frac{M \cdot f_{PC}}{\Gamma(\alpha)} \int_{0}^{\tau} (t - s)^{\alpha - 1} ds$$

$$\leq M \left| U(t, \tau)x_0 - x_0 \right| + \frac{M \cdot f_{PC}}{\Gamma(\alpha + 1)} (t - \tau)^{\alpha}$$

$$+ \frac{M \cdot f_{PC}}{\Gamma(\alpha + 1)} t^\alpha - (t - \tau)^{\alpha} \cdot \left| U(t, \tau) - I \right|$$

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that \( Hx \in C([0, t_1], X) \). With analogous arguments we can obtain \( Hx \in C([t_k, t_{k+1}], X), k = 0, 1, 2, \ldots, \delta \). That is \( Hx \in PC(J, X) \).

Step 1. \( H \) is a continuous operator on \( PC(J, X) \). Let \( x_1, x_2 \in PC(J, X) \) and

\[
\|x_1 - x_2\|_{PC} \leq 1, \text{ then } \|x_2\|_{PC} \leq 1 + \|x_1\|_{PC} = \rho. \]

By assumptions \((H1) - (H3)\), we obtain

\[
D((Hx_1)(t), (Hx_2)(t)) \leq \frac{1}{\Gamma(\alpha)} \left[ \int_{0<t_k<t} (t_k - s)^{\alpha-1} U(t, s) \cdot f(s, X_1(s)), f(s, X_2(s)) \, ds \\
+ \int_{0<t_k<t} (t - s)^{\alpha-1} U(t, s) \cdot f(s, X_1(s)), f(s, X_2(s)) \, ds \\
+ \int_{0<t_k<t} U(t, t_k) \cdot I_k(x_1(t_k)) - I_k(x_2(t_k)) \right] \\
\leq \frac{ML_f(\rho)}{\Gamma(\alpha)} \left[ \int_{0<t_k<t} (t_k - s)^{\alpha-1} \|x_1(s) - x_2(s)\|ds \\
+ \int_{0<t_k<t} (t - s)^{\alpha-1} \|x_1(s) - x_2(s)\|ds \\
+ M \int_{0<t_k<t} h_k \|x_1(t_k) - x_2(t_k)\| \right] \\
\leq \int_{0}^{t} (t - s)^{\alpha-1} ds \cdot \frac{ML_f(\rho)}{\Gamma(\alpha)} + M \int_{0<t_k<t} h_k \|x_1 - x_2\|_{PC}. \\
\]

One can deduce that

\[
D[Hx_1, Hx_2]_{PC} \leq \frac{\Gamma(\alpha) \cdot \Gamma(n + 1)}{\Gamma(\alpha + n + 1)} h^{\alpha+n} \cdot \frac{ML_f(\rho)}{\Gamma(\alpha)} + M \int_{0<t_k<t} h_k \|x_1 - x_2\|_{PC}. \\
\leq L \|x_1 - x_2\|_{PC}. \\
\]

Where

\[
L = Mh^{\alpha+n} \frac{\Gamma(\alpha) \cdot \Gamma(n + 1)}{\Gamma(\alpha + n + 1)} h_f(\rho) + M \int_{0<t_k<t} h_k. \\
\]

Step 2. \( H \) is a compact operator on \( PC(J, X) \).

Let \( B \) be a bounded subset of \( PC(J, X) \), there exists a constant \( \mu > 0 \) such that \( \|x\|_{PC} \leq \mu \) for all \( x \in B \). Using \((H3)\), there exists a constant \( N \) such that
\[ I_k(x(t)) \leq N_k \] for all \( x \in B, \ t \in J, \ k = 1, 2, \ldots, \delta. \) Also using \((H2)\), there exists a constant \( \omega \) such that \( |f(t, x(t))| \leq M \rho (1 + t x^m) \leq M (1 + \mu^m) = \omega \) for all \( x \in B, \ t \in J. \) Further, \( HB \) is a bounded subset of \( PC(J, X) \) in fact, let \( x \in B, \) we have

\[
|Hx(t)| \leq M |x_0| + \frac{M \omega}{\Gamma(\alpha)} \sum_{0 < t_k < t} (t_k - s)^{\alpha - 1} ds + \frac{M \omega}{\Gamma(\alpha)} t_k (t - s)^{\alpha - 1} ds
\]

\[ + M \sum_{0 < t_k < t} N_k \]

\[ \leq M |x_0| + MN_k \delta + \frac{M \omega}{\Gamma(\alpha)} t_n (t - s)^{\alpha - 1} ds = 0 \]

\[ \leq M |x_0| + MN_k \delta + \frac{M \omega}{\Gamma(\alpha + n + 1)} (t - s)^{\alpha + n}. \]

Hence \( HB \) is bounded.

Define \( \Pi = HB \) and \( \Pi(t) = \{ (Hx(t)) | x \in B \} \) for \( t \in J. \)

Clearly, \( \Pi(0) = \{ x_0 \} \) is compact, hence, it is only necessary to check that \( \Pi(t) = \{ (Hx(t)) | x \in B \} \) for \( t \in (0, b] \) is also compact. For \( 0 < s < t \leq b, \) define

\[
\Pi_s(t) \equiv H_s B(t) = \{ (Hx(t)) | x \in B \}, \quad (7.3.2)
\]

and the operator \( H_s \) is defined by

\[
(H_s x)(t) = U(s) \left[ U(t-s)x_0 + \int_0^t \frac{1}{\Gamma(\alpha)} (t_k - s)^{\alpha - 1} U(t - s - s) f(s, x(s)) ds \right]
\]

\[ + U(s) \left[ \frac{1}{\Gamma(\alpha)} (t_k - s)^{\alpha - 1} U(t - s - s) f(s, x(s)) ds \right]
\]

\[ + U(s) \left[ \int_0^{t_k} (t - s)^{\alpha - 1} U(t - s - s) f(s, x(s)) ds \right]
\]

\[ + \frac{1}{\Gamma(\alpha)} t_k \left[ \frac{1}{t_k} (t - s)^{\alpha - 1} U(t - s) f(s, x(s)) ds \right]
\]

\[ + \frac{1}{\Gamma(\alpha)} t_k \left[ (t - s)^{\alpha - 1} U(t - s) f(s, x(s)) ds \right]
\]

\[ + \int_0^{t_k} \frac{1}{\Gamma(\alpha)} (t_k - s)^{\alpha - 1} U(t - s) f(s, x(s)) ds \]

\[ + \int_0^{t_k} \frac{1}{\Gamma(\alpha)} (t_k - s)^{\alpha - 1} U(t - s) f(s, x(s)) ds \]

\[ + U(t - t_k) I_k(x(t_k)), \]

\[ 96. \]
from which implies that \( \Pi_s(t) \) is relatively compact for \( t \in (s, \delta] \) due to \( \{U(t), t \leq 0\} \) is a compact evolution system.

For interval \((0, t_1]\), (7.3.2) reduces to

\[
\Pi_s(t) \equiv (H_s \mathcal{B})(t) = \{(H_s \mathcal{x})(t) | x \in \mathcal{B}\}
\]

Combine with above, we can deduce

\[
D[(Hx)(t) - (H_s x)(t)] \leq \frac{1}{\Gamma(\alpha)} \int_{t-s}^{t} (t-s)^{\alpha-1} U(t-s) D[\mathcal{f}(s, x(s)), \mathcal{f}(s, x'(s))] ds
\]

\[
\leq \frac{\gamma^2}{t-s} \left( t-s \right)^{\alpha-1} ds
\]

\[
\leq \frac{\gamma^2}{\Gamma(\alpha + 1)}
\]

Step 3. \( \Pi \) is equicontinuous on the interval \((t_k, t_{k+1}], k = 1, 2, \ldots, \delta \). For interval \((0, t_1)\), we note that for \( t_1 > h > 0 \)

\[
D[(Hx)(h), (Hx)(0)] \leq \left| U(h) - I_0^h x_0 \right| + \gamma^2 \int_{t-h}^{t} \left( t-h - s \right) \left( (t-h) - s \right)^{\alpha-1} ds
\]

and for \( t_1 \geq t + h \geq t \geq \gamma \geq 0, \gamma < h \) and \( x \in \mathcal{B} \).

\[
D[(Hx)(t + h), (Hx)(t)] = (U(t + h) - U(t)) x_0
\]

\[
\times \int_{t-h}^{t} \left( t+h - s \right)^{\alpha-1} \left( (t+h) - s \right)^{\alpha-1} \left( (t+h) - s \right)^{\alpha-1} ds + \frac{1}{\Gamma(\alpha)} \int_{t-h}^{t} \left( t+h - s \right)^{\alpha-1} \left( (t+h) - s \right)^{\alpha-1} \left( (t+h) - s \right)^{\alpha-1} ds
\]

\[
\left[ U(t+h) - U(t-s) \right] D[\mathcal{f}(s, x(s)), \mathcal{f}(s, x'(s))] ds
\]

\[
\int_{t-h}^{t} \left( t+h - s \right)^{\alpha-1} \left( (t+h) - s \right)^{\alpha-1} \left( (t+h) - s \right)^{\alpha-1} ds
\]

\[
\left[ U(t+h) - U(t-s) \right] D[\mathcal{f}(s, x(s)), \mathcal{f}(s, x'(s))] ds
\]

\[
\frac{1}{\Gamma(\alpha)} \int_{t-s}^{t} \left( t+h - s \right)^{\alpha-1} \left( (t+h) - s \right)^{\alpha-1} \left( (t+h) - s \right)^{\alpha-1} ds
\]

\[
\left[ U(t+h) - U(t-s) \right] D[\mathcal{f}(s, x(s)), \mathcal{f}(s, x'(s))] ds
\]

\[
\frac{1}{\Gamma(\alpha)} \int_{t-s}^{t} \left( t+h - s \right)^{\alpha-1} \left( (t+h) - s \right)^{\alpha-1} \left( (t+h) - s \right)^{\alpha-1} ds
\]

\[
\left[ U(t+h) - U(t-s) \right] D[\mathcal{f}(s, x(s)), \mathcal{f}(s, x'(s))] ds
\]

\[
\frac{1}{\Gamma(\alpha)} \int_{t-s}^{t} \left( t+h - s \right)^{\alpha-1} \left( (t+h) - s \right)^{\alpha-1} \left( (t+h) - s \right)^{\alpha-1} ds
\]

\[
\left[ U(t+h) - U(t-s) \right] D[\mathcal{f}(s, x(s)), \mathcal{f}(s, x'(s))] ds
\]

\[
\frac{1}{\Gamma(\alpha)} \int_{t-s}^{t} \left( t+h - s \right)^{\alpha-1} \left( (t+h) - s \right)^{\alpha-1} \left( (t+h) - s \right)^{\alpha-1} ds
\]

\[
\left[ U(t+h) - U(t-s) \right] D[\mathcal{f}(s, x(s)), \mathcal{f}(s, x'(s))] ds
\]

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Hence,

\[ D((Hx)(t + h), (Hx)(t)) \leq M \| U(h) - I \| x_0 + \frac{wM}{\Gamma(\alpha + 1)} h^\alpha \]

\[ + wM \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} ds + \int_0^t \left( \frac{(t + h)^{\alpha - 1}}{\Gamma(\alpha + 1)} \right) ds \]

Since \( \| U(h) - I \| \to 0 \), \( (t + h - s)^{\alpha - 1} - (t - s)^{\alpha - 1} \to 0 \) as \( h \to 0 \), thus the righthand side of the above can be made as small as desired by choosing \( h \) sufficiently small.

Hence, \( \Pi(t) \) is equicontinuous in interval \((0, t_1)\).

Step 4. \( H \) has a fixed point in \( PC(J, X) \). According to Leray-Schauder fixed point theorem, it suffices to show the following set

\[ \{ x \in PC(J, X) \mid x = \sigma Hx, \ \sigma \in [0, 1] \} \]

is a bounded subset of \( PC(J, X) \). In fact, let \( x \in \{ x \in PC(J, X) \mid x = \sigma Hx, \ \sigma \in [0, 1] \} \), we have

\[ \| x(t) \| = \| H(\sigma x(t)) \| \leq \| U(t) \| \sigma x_0 \]

\[ + \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha - 1} U(t - s) \sigma x(s) ds \]

\[ + \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t - s)^{\alpha - 1} U(t - s) \sigma x(s) ds \]

\[ + \left( \frac{\| U(t - t_k) \|}{\Gamma(\alpha)} \right) I_k(\sigma x(t_k)) \]

\[ \leq \sigma M \| x_0 \| + M M_r \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha - 1} \sigma \| x(s) \| ds \]

\[ + M \left( \| I_k(0) \| + \sigma h_k \| x(t_k) \| \right) \]