Then
\[ D(z, \hat{0}) \leq \tilde{M} \| I^{\alpha} q \| + \frac{M_{\tilde{b}^2} \| p \|_{\infty}}{\Gamma(\alpha + 1)} = \mathcal{M}. \]

Set
\[ U = \{ z \in C_{0}/D_{0}(z, \hat{0}) \leq \mathcal{M} + 1 \}. \]

\( N : \overline{U} \to C_{0} \) is continuous and completely continuous. From the choice of \( U \), there is no \( z \in \partial U \) such that \( z = \lambda N(z) \) for \( \lambda \in (0, 1) \). As a consequence of the nonlinear alternative of Leray-Schauder type [36], we deduce that \( N \) has a fixed point \( z \) in \( U \).

### 5.4 Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional differential equation
\[ D^2 u(t, x) = \frac{\partial}{\partial x^2} u(t, x) + f(t, t - s, \xi, u(s, \xi)) ds, \quad 0 < \alpha < 1, \quad (5.4.1) \]
\[ (t, x) \in [0, b] \times (0, \pi), \quad t = \frac{b}{2}, \]
\[ u(t, 0) = u(t, \pi), \quad t \in [0, b]. \quad (5.4.2) \]

Let \( X = L^{2}([0, \pi]) \) and consider the operator \( A : D(A) \subset X \to X \) defined by
\[ D(A) = \{ u \in X : \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \in X \text{ and } u(0) = u(\pi) = 0 \}, \]
\[ Au = \frac{\partial^2}{\partial x^2} u. \]

Clearly \( A \) is densely defined in \( X \) and is the infinitesimal generator of a strongly continuous Semigroup \( (T(t))_{t \geq 0} \) on \( X \). Now let \( u_0 \in X \), and \( f(t, .) \) is continuous function. All conditions of theorem 5.3.1 are now fulfilled so we deduce that \( (5.4.1) - (5.4.2) \) has an integral solution. \( \square \)
Chapter 6

EXISTENCE RESULTS FOR DELAY EVOLUTION FRACTIONAL FUZZY INTEGRODIFFERENTIAL EQUATIONS

6.1 Introduction

Differential equations of fractional order have recently proved valuable tools in the modelling of many physical phenomena [20; 31; 33; 59; 60; 74]. There has been a significant theoretical development in fractional differential equations in recent years; see the monographs of Kilbas et al [48], Miller and Ross [62], Podlubny [70], Samko et al [75], and the papers of Bai and Lu [8], Diethelm et al [20; 21; 22], El-Raheem [26], El-Sayed [27; 28; 29], Kilbas and Trujillo [48], Mainardi [59], Momani and Hadid [65], Momani et al [64], Nakhushev [66], and Yu and Gao [82; 83].

The study of fuzzy differential equations has been initiated as an independent subject in conjunction with fuzzy valued analysis [18; 67] and set-valued differential equations [51]. Using the Hukuhara derivative of muti-valued functions, Puri and Ralescu [72] have introduced the concept of H-differentiability for fuzzy functions. This concept has been studied and applied in context of fuzzy differ-
ential equations by Seikkala [77] and Kaleva [46] in time dependent form. Kaleva showed in [46; 47] that if \( f \) is continuous and satisfies the Lipschitz condition with respect to \( u \), then there exists a unique local solution for the fuzzy initial value problem \( u'(t) = f(t, u), \quad u(0) = u_0 \) on \( (E^n, D) \). The existence theorem of solutions for fuzzy initial value problem under different sets of assumptions are given in Ding et al [19] and Song et al [78]. Park et al [69] studied the approximate solutions of the fuzzy functional integral equations. The local existence and uniqueness are investigated by Balasubramaniam and Muralisankar [7] for nonlinear fuzzy neutral functional differential equations. Guo et al [38] established some existence results for the fuzzy impulsive functional differential equations. Recently Lupulescu [58] has established the local and global existence and uniqueness results for fuzzy functional differential equations. Our aim in this paper is to study the existence and the uniqueness of the solution for the delay evolution fractional fuzzy integrodifferential equation.

\[
D_+^\alpha x(t) = A(t)x(t) + \int_0^t K(t, s, x_s)ds + f(t, x_t), \quad \text{for each } t \in J = [0, b],
\]

\[
x(t) = \phi(t), \quad t \in (-\infty, 0],
\]

where \( 0 < \alpha < 1 \) and \( \{A(t), t \in J\} \) is a family of closed densely defined linear unbounded operators on \( X \). \( D_+^\alpha \) is the caputo fractional derivative, \( f : J \times C_\infty \rightarrow X, K : J \times J \times C_\infty \rightarrow X \) are given fuzzy functions satisfying appropriate conditions, \( \phi \in C_\infty, \phi(0) = 0 \) and \( C_\infty = C((-\infty, E^1)) \).

For any mapping \( x \) defined on \( (-\infty, b] \) and \( t \in J \), we denote by \( x_t \) the element of \( C_\infty \) defined by

\[
x_t(\theta) = x(t + \theta), \quad \theta \in (-\infty, 0].
\]

This chapter is organized as follows: In section 6.2, we introduce the mild solution of (6.1.1) – (6.1.3) and recall some Lemmas which are used in the sequel. In
section 6.3, we study the existence and uniqueness of mild solution of system (6.1.1) – (6.1.3) under some suitable conditions. An example to illustrate our results is given at last.

6.2 Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout this chapter.

By $C(J, R)$ we denote the Banach space of all continuous functions from $J$ into $R$ with the norm

$$||y||_{\infty} = \sup \{|y(t)| : t \in J\}.$$  

Let $K_c(R^n)$ denote the collection of all nonempty compact convex subsets of $R^n$. We define the Hausdorff distance between $A, B \in K_c(R^n)$ by

$$d_H(A, B) = \max \{\sup_{x \in A} \inf_{y \in B} ||x - y||, \sup_{y \in B} \inf_{x \in A} ||x - y||\}.$$  

Denote

$$E^n = \{u : R^n \to [0, 1] / u \text{ satisfies (i) – (iv)}\}.$$  

(i) $u$ is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(ii) $u$ is fuzzy convex, that is, for $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\};$$

(iii) $u$ is upper semicontinuous;

(iv) $[u]^0 = \text{cl}\{x \in R^n : u(x) > 0\}$ is compact.
For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{ x \in \mathbb{R}^n : u(x) \geq \alpha \}$. Then from $(i) - (iv)$, it follows that the $\alpha$-level set $[u]^\alpha \in \mathcal{K}_c((\mathbb{R}^l))$ for all $0 < \alpha \leq 1$. For later purpose, we define $\hat{0} \in \mathbb{E}^n$ as $\hat{0}(x) = 1$ if $x = 0$ and $\hat{0}(x) = 0$ if $x = 0$. If we define

$$D[u, \nu] = \sup_{0 \leq \alpha \leq 1} d_H([u]^\alpha, [\nu]^\alpha),$$

then it is well known that $D$ is a metric in $\mathbb{E}^n$ and that $(\mathbb{E}^n, D)$ is a complete metric space [46; 71]. We list the following properties of $D[u, \nu]$:

(i) $D[u + w, v + w] = D[u, v]$, $D[u, v] = D[v, u]$,

(ii) $D[\lambda u, \lambda v] = |\lambda|D[u, v]$,

(iii) $D[u, v] \leq D[u, w] + D[w, v]$

for all $u, v, w \in \mathbb{E}^n$ and $\lambda \in \mathbb{R}$.

In the following we recall some main concepts and properties of differentiability and integrability for fuzzy functions [46; 71; 72].

If there exists $w \in \mathbb{E}^n$ such that $u = v + w$, then $w$ is called the $H$-difference of $u$ and $v$ and is denoted by $u - v$. Let $I$ be an interval in $\mathbb{R}$. A mapping $F : I \rightarrow \mathbb{E}^n$ is differentiable at $t_0 \in I$ if there exists a $F'(t_0) \in \mathbb{E}^n$ such that the limits

$$\lim_{h \rightarrow 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{F(t_0) - F(t_0 - h)}{h}$$

exist and equal to $F'(t_0)$. If $F : I \rightarrow \mathbb{E}^l$ is a fuzzy function such that

$$[F(t)]^\alpha = [F_1^\alpha(t), F_2^\alpha(t)], \quad \alpha \in [0, 1],$$

and there exists $F'(t_0)$ for some $t_0 \in I$, then

$$[F'(t_0)]^\alpha = [(F_1^\alpha)'(t_0), (F_2^\alpha)'(t_0)], \quad \alpha \in [0, 1].$$
The fuzzy function $F : I \rightarrow E^n$ is called strongly measurable if for each $\alpha \in [0, 1]$ the set-valued function $F_{\alpha} : I \rightarrow K_c(R^n)$ defined by $F_{\alpha}(t) = [F(t)]^\alpha$ is Lebesgue measurable. A function $F : I \rightarrow E^n$ is called integrable bounded if there exists an integrable function $h : I \rightarrow R^+$ such that $D[F(t), \hat{0}] \leq h(t)$ for all $t \in I$. The integral of $F$ over $I$, denoted $\int_I F(t) dt$, is defined levelwise by the equation

$$\int_I F(t) dt^\alpha = \int_I F_{\alpha}(t) dt$$

$$= \int_I f(t) dt f : I \rightarrow R^n \text{ is measurable selection for } F_{\alpha}$$

for all $0 \leq \alpha \leq 1$. A strongly measurable and integrable function $F : I \rightarrow E^n$ is said to be integrable over $I$ if $\int_I F(t) dt \in E^n$.

**Proposition 6.2.1** If $F : I \rightarrow E^n$ is continuous, then it is integrable over $I$. Moreover, in this case, the function $G(t) = \int_{t_0}^{t} F(s) ds$, $t_0, t \in I$, is differentiable and $G(t) = F(t)$.

**Proposition 6.2.2** Let $F; I \rightarrow E^n$ be differentiable and assume that the derivative $F'(t)$ is integrable over $I$. Then, for each $t \in I$, we have

$$F(t) = F(t_0) + \int_{t_0}^{t} F'(s) ds, \ t_0 \in I$$

It is well known that $F, G : I \rightarrow E^n$ are integrable, then the followings hold [46]

(i) $\int_I (F(t) + G(t)) dt = \int_I F(t) dt + \int_I G(t) dt$;

(ii) $\int_I \lambda F(t) dt = \lambda \int_I F(t) dt, \ \lambda \in R$;

(iii) $D[F, G]$ is integrable;
(iv) \( D^\ast I, F(t)dt;^\ast I, G(t)dt \leq^\ast I D[F(t), G(t)]dt; \)

(v) \( ^\ast t_2 t_0 F(t)dt = ^\ast t_1 t_0 F(t)dt + ^\ast t_2 t_1 F(t)dt \) for \( t_0, t_1, t_2 \in I. \)

If \( I \) is a compact interval of \( R \), then \( C(I, E^1) \) denotes the set of all fuzzy continuous functions from \( I \) into \( E^1 \). On the space \( C(I, E^1) \) we consider the following metric:

\[
D(u, v) = \sup_{t \in I} D[u(t), v(t)].
\]

We denote by \( C_\infty \) the space \( C((-\infty, 0], E^1) \). Also, we denote by

\[
D_\infty(u, v) = \sup_{t \in (-\infty, 0]} D[u(t), v(t)]
\]

the metric on the space \( C_\infty \). Let \( u(.) \in C_\infty \). Then for each \( t \in [0, \hat{t}] \) we denote by \( u_t \) the element of \( C_\infty \) defined by \( u_t(s) = u(t+ s), s \in (-\infty, 0] \).

**Definition 6.2.1.** The fractional primitive of order \( \alpha > 0 \) of a function \( h : R^+ \rightarrow R \) is defined by

\[
I^\alpha_0 h(t) = \int_0^t \frac{(t-s)\alpha-1}{\Gamma(\alpha)} h(s)ds
\]

provided the right side exists pointwise on \( R^+ \) and \( \Gamma \) is the gamma function.

**Definition 6.2.2.** The fractional derivative of order \( \alpha > 0 \) of a fuzzy continuous function \( h : R^+ \rightarrow E^1 \) is given by

\[
\frac{d^\alpha h(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\infty (t-s)^{\alpha-1} h(s)ds
\]

\[
= \frac{d}{dt} \int_0^{t} (t-s)^{\alpha-1} h(s)ds.
\]

**Definition 6.2.3.** Let \( f : R_+ \rightarrow R \). The Caputo fractional derivative of order \( \alpha \) of \( f \) is defined by

\[
D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^{\infty} \frac{n-\alpha-1 (n)}{(t-x)^{n-\alpha}} f(x)dx
\]

where \( \alpha \in (n-1, n), n \in N. \)
\[ f(s) = \begin{cases} 0 & \text{if } s < t \leq 0, \\ (t - s) & \text{if } s \leq t > 0. \end{cases} \]
Definition 6.2.4. Let \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \) be a continuous function on \( \mathbb{R}^+ \) and \( \alpha \geq 0 \). Then the expression

\[
J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds, \quad t > 0
\]

is called the Riemann-Liouville integral of order \( \alpha \).

In this chapter, we assume that the space \((\mathcal{C}_\infty, D_\infty)\) is a metric space of fuzzy functions from \((-\infty, 0] \rightarrow \mathbb{E}_1\) and satisfies the following axioms:

If \( y : (-\infty, \theta] \rightarrow \mathbb{E}_1 \) and \( y_0 \in \mathcal{C}_\infty \), then for every \( t \in [0, \theta] \) the following conditions hold:

(A1) \( y_t \) is in \( \mathcal{C}_\infty \);

(A2) \( D_\infty(y_t, \hat{0}) \leq K(t) \sup \{ D[y(s), \hat{0}] \mid 0 \leq s \leq t \} + M(t)D_\infty(y_0, \hat{0}) \);

Lemma 6.2.1. (1) \( U(t, 0) \in L_b(X) \) for \( 0 \leq \theta \leq t \leq T \). Denotes \( M = \sup_{t \in J} \left| U(t, s) \right|_{L_b(X)} \), which is a finite number

(2) \( U(t, r)U(r, \theta) = U(t, \theta) \) for \( 0 \leq \theta \leq r \leq t \leq T \).

(3) \( U(\cdot, x) \in C(\Lambda, X) \) for \( x \in X, \Lambda = \{(t, 0) \in J \times J \mid 0 \leq \theta \leq t \leq T \} \).

(4) For \( 0 \leq \theta \leq t \leq T \), \( U(t, \theta) : X \rightarrow D \) and \( t \rightarrow U(t, \theta) \) is strongly differentiable on \( X \). The derivative \( \frac{\partial}{\partial t} U(t, \theta) \in X \) and it is strongly continuous on \( 0 \leq \theta \leq t \leq T \). Moreover,

\[
\frac{\partial}{\partial t} U(t, \theta) = -A(t)U(t, \theta) \quad \text{for} \quad 0 \leq \theta \leq t \leq T.
\]

\[
\frac{\partial}{\partial t} U(t, \theta) \mid_{x = \lambda} = \lambda A(t)U(t, \theta) \mid_{x = \lambda} \leq \frac{C}{t-\theta}.
\]
\( |A(t)U(t, \theta)A(\theta)^{-1}|_{\infty} \leq C \) for \( 0 \leq \theta \leq t \leq T \).
(5) For every \( v \in D \) and \( t \in (0, T] \), \( U(t, \theta)v \) is differentiable with respect to \( \theta \) on \( 0 \leq \theta \leq t \leq T \)
\[
\frac{\partial}{\partial \theta} U(t, \theta)v = U(t, \theta)A(\theta)v.
\]

(6) \( U(t, \theta) \) is compact operator for \( 0 \leq \theta \leq t \leq b \). And, for each \( x_0 \in X \), the Cauchy problem (6.1) has a unique classical solution \( x \in C^1(J, X) \) given by
\[
x(t) = U(t, 0)x_0, \quad t \in J.
\]

### 6.3 Existence Results

Let the space
\[
\Omega = \{ x : (-\infty, b) \to E^1/|x|_{(-\infty,0]} \in C_{\infty} \text{ and } x|_{[0,b]} \text{ is continuous} \}.
\]

**Definition 6.3.1.** A fuzzy function \( x \in \Omega \) is said to be a solution to be a solution of (6.1.1) if \( y \) satisfies the equation
\[
D^\alpha x(t) = A(t)x(t) + \int_0^t K(t, s, x)ds + f(t, x)
\]
on \([0, b]\) and the condition \( x(t) = \phi(t) \) on \((-\infty, 0]\).

**Theorem 6.3.1.** Let \( f : [0, b] \times C_{\infty} \to E_1 \). Assume that there exists \( l, l_1 > 0 \) such that
\[
D[f(t, \phi), f(t, \psi)] \leq lD_{\infty}(\phi, \psi)
\]
and
\[
D^{\frac{\alpha}{2}}K(t, \tau, \phi)dt \leq l_1 D_{\infty}(\phi, \psi),
\]
for \( t \in [0, b] \) and for every \( \phi, \psi \in C_{\infty} \). If
\[
\frac{M_{x_0}K_{b}}{\Gamma(\alpha+1)} + \frac{M_{f}K_{b}}{\Gamma(\alpha+1)} < 1,
\]
where
\[
K_{b} = \sup\{ |K(t)|, t \in [0, b] \},
\]
then there exists a unique solution for the fuzzy fractional integrodifferential equation (6.1.1) on the interval \((-\infty, b]\).
Proof. Transform the problem (6.1.1) into a fixed point problem. Consider the operator \( N : \Omega \to \Omega \) defined by
\[
\phi(t) \quad \text{if} \quad t \in (-\infty, 0],
\]
\[
(Nx)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) f(s,x_s)ds & \text{if} \quad t \in [0,b], \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) \int_0^s K(t,\tau,x_\tau)d\tau ds & \text{if} \quad t \in [0,b]. \end{cases}
\]

Let \( y(.): (-\infty, b] \to E^1 \) be the fuzzy function defined by
\[
y(t) = \begin{cases} 0 & \text{if} \quad t \in [0,b], \\ \phi(t) & \text{if} \quad t \in (-\infty, 0]. \end{cases}
\]

Then \( y_0(t) = y(t+0) = \phi(t) \) for every \( t \in (-\infty, 0] \). For each \( z \in C([0,b], E^1) \) with \( z(0) = 0 \), we denote by \( \bar{z} \) the fuzzy function defined by
\[
\bar{z}(t) = \begin{cases} z(t) & \text{if} \quad t \in [0,b] \\ 0 & \text{if} \quad t \in (-\infty, 0] \end{cases}
\]

If \( x(.) \) satisfies the fuzzy integral equation
\[
x(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) f(s,x_s)ds & \text{if} \quad t \in [0,b], \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) \int_0^s K(t,\tau,x_\tau)d\tau ds, \end{cases}
\]
we can decompose \( x(.) \) as \( x(t) = \bar{z}(t) + y(t), \quad 0 \leq t \leq b \), which implies \( x_t = \bar{z}_t + y_t, \quad 0 \leq t \leq b \) and the fuzzy function \( z(.) \) satisfies
\[
z(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) f(s,\bar{z}_s + y_s)ds & \text{if} \quad t \in [0,b], \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) \int_0^s K(t,\tau,\bar{z}_\tau + y_\tau)d\tau ds, \end{cases}
\]

Set
\[
C_0 = \{ z \in C([0,b], E^1) : z_0 = 0 \}.
\]
and let $D_0$ be the metric in $C_0$ defined by

$$D_0(z, \tilde{0}) = D_\infty(z_0, \tilde{0}) + \sup \{ D[z(t), \tilde{0}], \ 0 \leq t \leq b \}$$

$$= \sup \{ D[z(t), \tilde{0}], \ 0 \leq t \leq b \}, \quad z \in C_0.$$

Then the space $C_0$ is a complete metric space. Let the operator $N : C_0 \to C_0$ be defined by

$$(Nz)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) f(s, z_s + y_s) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) x_s \frac{d}{ds} (t-s)^{\alpha-1} U(t,s) ds$$

$$\times \int_0^s K(t, \tau, \tilde{z}_\tau + x_\tau) d\tau \ ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) D_\infty(\tilde{z}_s, \tilde{z}_s) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) J_1 D_\infty(\tilde{z}_s, \tilde{z}_s) ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) J_1(\sup_{s \in [0,t]} D_0(\tilde{z}(s), \tilde{z}'(s))) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) J_1(\sup_{s \in [0,t]} D_0(\tilde{z}(s), \tilde{z}'(s))) ds$$

$$\leq \frac{M_1 K_b}{\Gamma(\alpha)} D_0(z, z') \int_0^t (t-s)^{\alpha-1} ds$$

$$+ \frac{M_1 K_b}{\Gamma(\alpha)} D_0(z, z') \int_0^t (t-s)^{\alpha-1} ds$$

$$\leq \frac{M_1 K_b}{\Gamma(\alpha+1)} D_0(z, z') + \frac{M_1 K_b}{\Gamma(\alpha+1)} D_0(z, z')$$

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\[ D_0(pz, pz') \leq \frac{M_{pK} b \alpha}{\Gamma(\alpha + 1)} + \frac{M_{pK} b \alpha}{\Gamma(\alpha + 1)} D_0(z, z'). \]

By hypothesis, \( p \) is a contractive mapping. Thus \( p \) has a unique fixed point by Banach's contraction principle.

**Lemma 6.3.2.** [42] Let \( \nu : [0, b] \to [0, \infty) \) be a real function and \( w(\cdot) \) is a nonnegative, locally integrable function on \( [0, b] \) and there exist constants \( a > 0, \ 0 < \alpha < 1 \) such that

\[ \nu(t) \leq w(t) + a^\alpha \int_0^t \frac{\nu(s)}{(t-s)^\alpha} ds. \]

Then there exists a constant \( K = K(\alpha) \) such that

\[ \nu(t) \leq w(t) + K(\alpha) a^\alpha \int_0^t \frac{w(s)}{(t-s)^\alpha} ds. \]

for every \( t \in [0, b] \).

Assume that the following hypotheses hold:

1. \((H_1)\) \( f \) is a continuous function,

2. \((H_2)\) There exist \( p_1, q_1 \in C([0, b], R^+) \) such that

\[ D[f(t, \varphi), \hat{0}] \leq p_1(t) + q_1(t) D_\infty(\varphi, \hat{0}) \]

3. \((H_3)\) There exist \( p_2, q_2 \in C([0, b], R^+) \) such that

\[ D_{\alpha}^\alpha K(t, s, \varphi), \hat{0} \leq p_2(t) + q_2(t) D_\infty(\varphi, \hat{0}) \]

for \( t \in [0, b] \) and each \( \varphi \in C_{\infty} \) and \( \| I^\alpha p \| < +\infty \). Then the fuzzy fractional integrodifferential equation (6.1.1) has at least one solution on \( (-\infty, b] \).
Proof. Let $\rho : C_0 \to C_0$. We shall show that the operator $\rho$ is continuous and completely continuous.

Step 1. $\rho$ is continuous. Let $\{z_n\}$ be a sequence such that $z_n \to z$ in $C_0$ as $n \to \infty$. Then

$$D[\rho z_n(t), \rho z(t)] \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t, s) D[f(s, z_{n,s} + x_s), f(s, z + x_s)] ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t, s) \times D^\sigma K(t, \tau, z_{n,\tau} + x_\tau, \hat{\sigma}) d\tau, K(t, \tau, z + x_\tau, \hat{\sigma}) d\hat{\tau} ds.$$

Since $f$ is continuous, we obtain $D_0(\rho z_n, \rho z) \to 0$ as $n \to \infty$. Thus $\rho$ is continuous.

Step 2. $\rho$ maps bounded sets into bounded sets in $C_0$.

Indeed, it is enough to show that for any $\eta > 0$, there exists a positive constant $L$ such that for each $z \in B_\eta = \{z \in C_0 | D_0(z, \hat{0}) \leq \eta\}$ one has $D_0(\rho z, \hat{0}) \leq L$. Let $z \in B_\eta$. From $(H1) - (H3)$, we have for each $t \in [0, \hat{t}]$,

$$D[\rho z(t), \hat{0}] \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t, s) D[f(s, z_t + x_s), \hat{0}] ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t, s) D^\sigma K(t, \tau, z_\tau + x_\tau, \hat{\sigma}) d\tau, \hat{0} d\hat{\sigma} ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t, s) D[\rho_1(s) + q_1(s) D(\rho_0(s) + q_0(s))] ds,$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t, s) D[\rho_2(s) + q_2(s) D(\rho_0(s) + q_0(s))] ds,$$

$$\leq \frac{M_{\rho_1} ||q_1||_\infty}{\Gamma(\alpha + 1)} + \frac{M_{\rho_2} ||q_2||_\infty}{\Gamma(\alpha + 1)} \eta.$$
where

\[
D_\infty(\tilde{z}_s + x_s, \tilde{\theta}) \leq D_\infty(\tilde{z}_s, \tilde{\theta}) + D_\infty(x_s, \tilde{\theta}) \\
\leq K_0 \eta + M_0 D_\infty(\phi, \tilde{\theta}) = \eta^*
\]

\[
D_\infty(\tilde{z}_t + x_t, \tilde{\theta}) \leq D_\infty(\tilde{z}_t, \tilde{\theta}) + D_\infty(x_t, \tilde{\theta}) \\
\leq K_0 \eta + M_0 D_\infty(\phi, \tilde{\theta}) = \eta^*
\]

and \( M_0 = \sup \{ \|M(t)\| / t \in [0, \bar{t}] \} \). Hence \( D_0(pz, \tilde{\theta}) \leq L \).

Step 3. \( \rho \) maps bounded sets into equicontinuous sets of \( C_0 \). Let \( t_1, t_2 \in [0, \bar{t}], t_1 < t_2 \) and let \( B_{\bar{t}} \) be a bounded set of \( C_0 \) as in Step 2. Then we have

\[
D[pz(t_2), pz(t_1)] \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha - 1} U(t_2 - s) - (t_1 - s)^{\alpha - 1} U(t_1 - s)] ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} [(t_2 - s)^{\alpha - 1} U(t_2 - s) - (t_1 - s)^{\alpha - 1} U(t_1 - s)] ds \\
+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \big[ ((t_2 - s)^{\alpha - 1} U(t_2 - s) - (t_1 - s)^{\alpha - 1} U(t_1 - s)) \big] ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \big[ ((t_2 - s)^{\alpha - 1} U(t_2 - s) - (t_1 - s)^{\alpha - 1} U(t_1 - s)) \big] ds \\
\leq \frac{\|p_1\|_{\infty} + \|q_1\|_{\infty} \eta^*}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha - 1} U(t_2 - s) - (t_1 - s)^{\alpha - 1} U(t_1 - s)] ds \\
+ \frac{\|p_1\|_{\infty} + \|q_1\|_{\infty} \eta^*}{\Gamma(\alpha)} \int_{t_1}^{t_2} [(t_2 - s)^{\alpha - 1} U(t_2 - s) - (t_1 - s)^{\alpha - 1} U(t_1 - s)] ds \\
+ \frac{\|p_2\|_{\infty} + \|q_2\|_{\infty} \eta^*}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha - 1} U(t_2 - s) - (t_1 - s)^{\alpha - 1} U(t_1 - s)] ds \\
+ \frac{\|p_2\|_{\infty} + \|q_2\|_{\infty} \eta^*}{\Gamma(\alpha)} \int_{t_1}^{t_2} [(t_2 - s)^{\alpha - 1} U(t_2 - s)] ds
\]

The right hand side as tends to zero \( t_1 \to t_2 \). The equicontinuity for case \( t_1 < t_2 \leq 0 \) and \( t_1 \leq 0 \leq t_2 \) is obvious. As a consequence of Step 1–3, together with Arzela-Ascoli theorem, we can conclude that \( \rho : C_0 \to C_0 \) is continuous and
completely continuous.

Step 4. A priori bounds. We now show there exists an open set $U \subseteq C_0$ with $z = \lambda \rho(z)$ for $\lambda \in (0, 1)$ and $z \in \partial U$. Let $z \in C_0$ and $z = \lambda \rho(z)$ for some $0 < \lambda < 1$. Then for each $t \in [0, \hat{b}]$ we see that

$$z(t) = \lambda \int_0^t (t-s)^{\alpha-1} U(t, s) q_1(s) D_\infty(\tilde{z}_s + y_s, \hat{0}) ds + \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} U(t, s) \int_0^s K(t, r, \tilde{z}_r + y_r) dr ds.$$

This implies by $(H1)-(H3)$.

$$D[z(t), \hat{0}] \leq \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} U(t, s) q_1(s) D_\infty(\tilde{z}_s + y_s, \hat{0}) ds + \frac{M t^\beta \|p_1\|_{\infty}}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} U(t, s) q_2(s) D_\infty(\tilde{z}_s + y_s, \hat{0}) ds + \frac{M t^\beta \|p_2\|_{\infty}}{\Gamma(\alpha + 1)}.$$

But

$$D_\infty(\tilde{z}_s + y_s, \hat{0}) \leq D_\infty(\tilde{z}_0, \hat{0}) + D_\infty(y_0, \hat{0})$$

$$\leq K(t) \sup \{D[z(s), \hat{0}] / 0 \leq s \leq t\} + M(t) D_\infty(z_0, \hat{0}) + K(t) \sup \{D[y(s), \hat{0}] / 0 \leq s \leq t\} + M(t) D_\infty(y_0, \hat{0})$$

$$\leq K_b \sup \{D[z(s), \hat{0}] / 0 \leq s \leq t\} + M_b D_\infty(\phi, \hat{0}). \quad (6.3.1)$$

If we take $\nu(t)$ the right hand of (3.2), then we get

$$D_\infty(\tilde{z}_s + y_s, \hat{0}) \leq \nu(t)$$
and therefore,

\[
D[\varphi(t), \hat{\varphi}] \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) \varphi_1(s) w(s) \, ds \\
+ \frac{M \beta^p \|p_1\|_{\infty}}{\Gamma(\alpha+1)} \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} U(t,s) \varphi_2(s) w(s) \, ds \\
+ \frac{M \beta^p \|p_2\|_{\infty}}{\Gamma(\alpha+1)},
\]

\(t \in [0, \bar{t}].\)

Using the above inequality and the definition of \(w\) we see that

\[
w(t) \leq M_\varphi D_{\infty} (\phi, \hat{\varphi}) + \frac{M K_b b^p \|p_1\|_{\infty}}{\Gamma(\alpha+1)} + \frac{M K_b \|q_1\|_{\infty}}{\Gamma(\alpha)} \\
\int_0^t (t-s)^{\alpha-1} U(t,s) w(s) \, ds + \\
M_\varphi D_{\infty} (\phi, \hat{\varphi}) + \frac{M K_b b^p \|p_2\|_{\infty}}{\Gamma(\alpha+1)} + \frac{M K_b \|q_2\|_{\infty}}{\Gamma(\alpha)} \\
\int_0^t (t-s)^{\alpha-1} U(t,s) w(s) \, ds, \quad t \in [0, \bar{t}].
\]

Then there exists \(K = K(\alpha)\) such that

\[
|w(t)| \leq M_\varphi D_{\infty} (\phi, \hat{\varphi}) + \frac{M K_b b^p \|p_1\|_{\infty}}{\Gamma(\alpha+1)} + \frac{K(\alpha) M K_b \|q_1\|_{\infty}}{\Gamma(\alpha)} \\
\int_0^t (t-s)^{\alpha-1} U(t,s) R ds \\
+ M_\varphi D_{\infty} (\phi, \hat{\varphi}) + \frac{M K_b b^p \|p_2\|_{\infty}}{\Gamma(\alpha+1)} + \frac{M K(\alpha) K_b \|q_2\|_{\infty}}{\Gamma(\alpha)} \\
\int_0^t (t-s)^{\alpha-1} U(t,s) w(s) \, ds
\]

where

\[
R = 2 M_\varphi D_{\infty} (\phi, \hat{\varphi}) + \frac{M K_b b^p \|p_1\|_{\infty}}{\Gamma(\alpha+1)}.
\]

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Hence,
\[
\|w\|_\infty \leq R + \frac{M R K(\alpha) b^2 K_b}{\Gamma(\alpha + 1)} + \frac{M K_b b^\alpha p_{2\alpha} \|p_2\|_\infty}{\Gamma(\alpha + 1)} + \frac{M K(\alpha) K_b b^\alpha q_{2\alpha} \|q_2\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varpi(s) ds
\]
\[
= M_1.
\]

Then,
\[
D(z, \hat{0}) \leq M_1^* \|I^a q_1\| + \|I^a q_2\| + \frac{M b^\alpha \|p_1\|_\infty}{\Gamma(\alpha + 1)} + \frac{M b^\alpha \|p_2\|_\infty}{\Gamma(\alpha + 1)} = M^*.
\]

Set
\[
U = \{ z \in C_0 / D_0(z, \hat{0}) \leq M^* + 1 \}
\]

\(\rho : \bar{U} \rightarrow C_0\) is continuous and completely continuous. From the choice of \(U\), there is no \(z \in \partial U\) such that \(z = \lambda \rho(z)\) for \(\lambda \in (0, 1)\). As a consequence of the nonlinear alternative of Leray-Schauder type [36], we deduce that \(\rho\) has a fixed point \(z\) in \(U\).

### 6.4 Example

In this section we give an example to illustrate the usefulness of our main results.

Let us consider the following integrofractional differential equation
\[
D^a u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) + \int_0^t K(t, s - t, \xi, u(s, \xi)) ds
\]
\[
+ f(t, t - s, \xi, u(s, \xi)), \quad 0 < \alpha < 1,
\]
\[
(t, x) \in [0, \frac{b}{2}] \times (0, \pi), \quad t = \frac{b}{2}
\]
\[
u(t, 0) = u(t, \pi), \quad t \in [0, \frac{b}{2}]
\]
\[
u(t, 0) = u(t, \pi), \quad t \in [0, \frac{b}{2}]
\]
(6.4.1)

Let \(X = L^2([0, \pi])\). Define \(D = H^a([0, \pi])^\top H^a([0, \pi]), \) and \(A(t) = \frac{\partial^2}{\partial y^2} u\) for
$u \in D$ which can determined a strongly continuous evolutionary process $\{U(\cdot, \cdot)\}$
in $L^2([0, \phi])$ and it is also compact and there exists a $M > 0$ such that $\|U(\cdot, \cdot)\| \leq M$. Define

$$\mathcal{S}^{t} \mathcal{K}(t, s, x_{0}) = \int_{0}^{t} \mathcal{K}(t, s, t - s, \xi, u(s, \xi)) ds$$

and

$$f(t, x_{t}) = f(t, t - s, \xi, u(s, \xi))$$

Clearly Now let $u_{0} \in X$, $\mathcal{S}^{t} \mathcal{K}(t, s, x_{s})$ and $f(t, \cdot)$ is continuous function. All conditions of theorem 6.3.1 are now fulfilled so we deduce that 6.4.1 has an integral solution.