Chapter 5

Conclusion and future work

The aim of this work is to study how associated primes behave in various types of ring extensions; in particular, the skew polynomial extensions and this thesis concerns the same question. We try to connect some earlier results and generalize some. In this thesis we investigate the nature of associated prime ideals of skew polynomial rings over a Noetherian ring $R$ and their relation with those of the coefficient ring $R$.

The study of prime ideals in skew polynomial rings $O(R) = R[x; \sigma, \delta]$ has proved to be a challenging task to many non commutative ring theorists, see [1, 2, 41, 55]. The difficulties that arise are that, for a prime ideal $P \subseteq R$, $a \in P \subseteq P[x] \subseteq O(R), ax = x\sigma(a) + \delta(a)$ need not to lie in $P[x]$, since $\sigma(a)$ or $\delta(a)$ need not lie in $P$. Thus, $P[x]$ need not even to be an ideal in $O(R)$.

In 2000, Carl Faith has proved in [21] that for any commutative ring $R$, the associated prime of $R[x]$ are all extended from the associated prime of $R$ in natural way. He has proved that if $R$ is a commutative ring, then the associated prime ideals of the usual polynomial ring $R[x]$ (viewed as module over itself) are precisely the ideals of the form $P[x]$, where $P$ is an associated prime ideal of $R$. 
In 2002, S. Annin generalized this result simultaneously in three different directions in his Ph. D Thesis (Chapter 2.2 of [3]) to general polynomial modules, to noncommutative rings and to Ore extensions. In order to extend Faith’s results Annin imposed \((\sigma, \delta)-\)compatibility condition on the module \(M_R\). and proved in Theorem (2.3) of [2].

In our work in this thesis we prove some results for a Noetherian ring \(R\), which is also an algebra over \(\mathbb{Q}\) such that \(\sigma(\delta(a)) = \delta(\sigma(a))\), for all \(a \in R\), where \(\sigma\) an automorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\).

The main results proved in this thesis are given below.

In Proposition (2.1.6) of Chapter 2 and Proposition (5.1.5) of Chapter 5 we generalize the results of Seidenberg and Gabriel for \(\sigma\)-derivation \(\delta\) of \(R\) regarding invariance of minimal prime ideals and associated prime ideals of a Noetherian \(\mathbb{Q}\)-algebra under \(\sigma\)-derivation \(\delta\), see (Proposition (10) of [11]).

In Theorem (2.3.4) give a necessary and sufficient condition for a Noetherian ring \(R\) to be a \(\sigma(*)\)-ring, in the sense that a Noetherian ring \(R\) is a \(\sigma(*)\)-ring if and only if for each minimal prime \(U\) of \(R\), \(\sigma(U) = U\) and \(U\) is completely prime ideal of \(R\), see (Theorem (1) of [12]). We also find a relation between a \(\delta\)-ring and a 2-primal ring in Theorem (2.3.6), see (Theorem (3) of [12]).

In Theorem (3.4.1)of Chapter 3 we find a relation between the minimal prime ideals of \(R\) and those of the differential operator ring \(R[x; \delta]\), where \(R\) is a Noetherian \(\mathbb{Q}\)-algebra and \(\delta\) is a derivation of \(R\), see (Theorem (2) of [14]). In Proposition (3.4.2) we prove that the extension of a minimal prime ideal of a Noetherian \(\sigma\)-ring \(R\) is a completely prime ideal of \(S(R)\), see (Proposition (2) of [12]).

In Chapter 4 we prove in Theorem (4.1.1) that for a Noetherian \(\sigma(*)\)-ring \(R\),
$P(R[x;\sigma]) = P(R)[x;\sigma]$, see (Theorem (2) of [12]). In Proposition (4.2.1) we find a relation between the completely prime ideals of a ring $R$ with the completely prime ideals of $O(R)$, see (Proposition (4) of [12]) and in Theorem (4.3.1) we find a relation between the prime radical of a 2-primal ring $R$ and that of $O(R)$, see (Theorem (4) of [12]). We generalize this result for a Noetherian $\mathbb{Q}$-algebra $R$.

Structure of associated prime ideals of $O(R)$ is given in Chapter 5 and in Theorem (5.2.1) the following result is proved:

Let $R$ be a Noetherian $\mathbb{Q}$-algebra, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. If $P \in \text{Ass}(O(R)_{O(R)})$ is such that $\sigma(P \cap R) = P \cap R$, then $P \cap R \in \text{Ass}(R_R)$ and if $P_1 \in \text{Ass}(R_R)$ is such that $\sigma(P_1) = P_1$, then $O(P_1) \in \text{Ass}(O(R)_{O(R)})$. A similar result is proved for minimal prime ideals, see (Theorem (1) of [12]). These results are useful to examine the primary decomposition and existence of artinian quotient ring of $O(R)$.

There are many interesting questions left for future work. For example, if we don’t impose the condition $\sigma(\delta(a)) = \delta(\sigma(a))$, $a \in R$, where $R$ is a Noetherian ring (or even an algebra over $\mathbb{Q}$), $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation, then:

1. Find the possibility of invariance of minimal prime ideals and associated prime ideals of a Noetherian $\mathbb{Q}$-algebra under $\sigma$-derivation $\delta$.

2. Find a relation between $\text{MinSpec}(R)$ and $\text{MinSpec}(O(R))$.

3. Find a relation between $\text{Ass}(R_R)$ and $\text{Ass}(O(R)_{O(R)})$. 