Chapter 4

Associated Prime Ideals of Skew Polynomial Rings

Associated prime ideals have long played a central role in commutative ring theory with applications to algebraic geometry, localization theory, and elsewhere. More recently, a noncommutative theory of associated prime ideals has been developed. A great many interesting results, particularly for skew polynomial rings, have arisen from this work.

In this Chapter we give a complete description and structure of associated prime ideals of the skew polynomial ring $O(R) = R[x; \sigma, \delta]$, where $\sigma$ is an automorphism and $\delta$ a $\sigma$-derivation of a Noetherian $\mathbb{Q}$-algebra $R$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$.

This chapter is divided into two sections. In the first Section (5.1) we give some basic results about associated prime ideals of $O(R)$. In the next Section (5.2), we prove that $P \in \text{Ass}(O(R)_{O(R)})$ implies that $P = O(P_1)$, for some $P_1 \in \text{Ass}(R_R)$ with $\sigma(P_1) = P_1$ and conversely $P_1 \in \text{Ass}(R_R)$ with $\sigma(P_1) = P_1$ implies that $O(P_1) \in \text{Ass}(O(R)_{O(R)})$.

We also prove a similar result for minimal prime ideals, see [11]. These results are useful to examine the primary decomposition and existence of artinian quotient ring
of $O(R)$.

### 4.1 Elements of $\text{Ass}(O(R)_{O(R)})$

**Proposition 4.1.1.** Let $R$ be a Noetherian $\mathbb{Q}$-algebra. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Then $e^{t\delta}$ is an automorphism of the skew power series ring $T = R[[t, \sigma]]$.

**Proof.** The proof is same as in Seidenberg [61] and a sketch in noncommutative case is provided by Blair and Small in [5].

**Remark 4.1.1.** Let $R$ be a Noetherian ring and $I$ an ideal of $R$. Then $I.R[[t, \sigma]] = \{b_0 + tb_1 + t^2b_2 + \ldots | b_i \in I\}$. We denote it by $I[[t, \sigma]]$.

**Lemma 4.1.2.** Let $R$ be a Noetherian $\mathbb{Q}$-algebra and $\delta$ a derivation of $R$. An ideal $I$ of $R$ is $\delta$-invariant if and only if $TI$ is $e^{t\delta}$-invariant.

**Proof.** Let $TI$ be $e^{t\delta}$-invariant.

Now, for any $a \in I, a \in TI$.

Therefore, $e^{t\delta}(a) \in TI$.

$\Rightarrow a + t\delta(a) + \left(\frac{t^2}{2!}\right)\delta^2(a) + \ldots \in TI$.

Thus we have, $\delta(a) \in I$.

Conversely, let $\delta(I) \subseteq I$ and $f = \sum t^j a_j \in TI, j = 0, 1, 2, 3, \ldots$

Then, $e^{t\delta}(f) = f + t\delta(f) + \left(\frac{t^2\delta^2}{2!}\right)(f) + \ldots$

$= \sum t^j a_j + t \left(\sum t^j \delta(a_j)\right) + \ldots \in TI$ as $\delta(a_i) \in I$.

Therefore, $e^{t\delta}(TI) \subseteq TI$.

Replacing $e^{t\delta}$ by $e^{-t\delta}$ we get, $e^{t\delta}(TI) = TI$. 


Note that if $I$ is an ideal of a ring $R$ such that $\sigma(I) = I$, then $IT = TI$ is an ideal of $T$.

The following two Propositions in which a structure of associated prime ideals and minimal prime ideals of the formal power series ring is given, are crucial in proving the main result, namely Theorem (5.2.1):

**Proposition 4.1.3.** Let $R$ be a ring and $T = R[[t, \sigma]]$. Then

1. $Q \in \text{Ass}(R_R)$ such that $\sigma(Q) = Q$ implies that $QT \in \text{Ass}(T_T)$

2. $P \in \text{Ass}(T_T)$ such that $\sigma(P \cap R) = P \cap R$ implies that $P \cap R \in \text{Ass}(R_R)$ and $P = (P \cap R)T$.

**Proof.** 1. Let $Q = \text{Ann}(I) = \text{Ass}(I_I), I \subseteq R$ an ideal of $R$.

We will show that $QT \in \text{Spec}(T)$.

Let $f = a_0 + ta_1 + t^2a_2 + \ldots, g = b_0 + tb_1 + t^2b_2 + \ldots$

be two elements of $T$ with $fTg \subseteq QT$ but $f \notin QT$.

Now for all $h = c_0 + tc_1 + t^2c_2 + \ldots, fhg \in QT$.

$\Rightarrow (a_0 + ta_1 + \ldots)(c_0 + tc_1 + \ldots)(b_0 + tb_1 + \ldots) \in QT$.

$\Rightarrow (a_0c_0 + t(\sigma(a_0)c_1 + a_1c_0) + \ldots)(b_0 + tb_1 + \ldots) \in QT$.

$\Rightarrow (a_0c_0b_0 + t(\sigma(a_0)c_0b_1 + \sigma(a_0)c_1b_0 + a_1c_0b_0) + \ldots) \in QT$.

Without loss of generality suppose $a_0 \notin Q$.

Now $a_0c_0b_0 \in Q$ for all $c_0 \in R$, and therefore $b_0 \in Q$.

Also $\sigma(a_0)c_0b_1 + \sigma(a_0)c_1b_0 + a_1c_0b_0 \in Q$, for all $\sigma(c_0), c_0, c_1 \in R$.

$\Rightarrow \sigma(a_0)c_0b_1 \in Q$ for all $\sigma(c_0) \in R$.

$\Rightarrow \sigma(a_0)Rb_1 \in Q$. Now $\sigma(a_0) \notin Q \Rightarrow b_1 \in Q$.

With the same process we get $b_i \in Q$ for all $i \geq 2$. 

Therefore, \( g \in QT \), which implies that \( QT \in \text{Spec}(T) \).

Now we claim that \( QT = \text{Ann}(IT) \). We have \((QT)(IT) = 0\) as \( QI = 0 \), and we have \( QT \subseteq \text{Ann}(IT) \).

If \( J = \text{Ann}(IT) \) and \( QT \subseteq J \), then there exists \( l = d_0 + td_1 + t^2d_2 + \ldots \), \( l \notin QT \).

Without loss of generality let \( d_0 \notin Q \).

Now, \((IT)l = 0\) implies that \( Id_0 = 0 \) and which in turn implies that \( d_0 \in Q \), which is a contradiction.

Therefore, \( QT = \text{Ann}(IT) \). Now let \( K \subseteq IT \) be such that \( \text{Ann}(K) = L \supset QT \).

Then there exists \( p = u_0 + tu_1 + t^2u_2 + \ldots \in L \) and \( p \notin QT \).

With out loss of generality suppose \( u_0 \notin Q \), then as above we get a contradiction.

Hence \( QT = \text{Ann}(K) \) and \( QT \in \text{Ass}(T_T) \).

2. Suppose that \( P \in \text{Ass}(T_T) \).

Let \( f = a_0 + ta_1 + t^2a^2 + \ldots \in T \) be such that \( P = \text{Ann}(fT) = \text{Ass}(fT_{fT}) \).

Now \( a_iR(P \cap R) = 0 \) for all \( i \). Choose \( a_n \neq 0 \) from coefficients of \( f \).

Let \( Q = \text{Ass}(a_nR) \) so that \( P \cap R \subseteq Q \).

Now \( Q = \text{Ann}(a_nR) = \text{Ass}(a_nR_{a_nR}), a_nr \neq 0 \).

Now \( QT \in \text{Ass}(T_T) \) by above paragraph.

In fact \( QT = \text{Ass}(a_nR_{RT_{a_nR}}) = \text{Ann}(a_nrRT) \) by above paragraph.

Now \( a_nRQ = 0 \) implies that for any \( h = d_0 + td_1 + t^2d_2 + \ldots \in T \), \( a_nR_hQ = 0 \). Which implies \( a_nrRTQ = 0 \) and therefore \( Q \subseteq P \cap R \).

Thus \( Q = P \cap R \in \text{Ass}(R_R) \).

In fact \( P \cap R \in \text{Ass}(a_iR)_{\text{Ass}(a_iR)} = \text{Ann}(a_iR) \) for all \( i \).

Now \( (P \cap R)T = \text{Ann}(a_iT) = \text{Ass}(a_iT_{a_iT}), \) for all \( i, a_i \neq 0 \).
Hence $(P \cap R)T = \text{Ass}(fTfT) = P$.

\[ \text{Proposition 4.1.4.} \] Let $R$ be a ring and $T = R[[t, \sigma]]$. Then

1. $P \in \text{MinSpec}(T)$ implies that $P \cap R \in \text{MinSpec}(R)$ and $P = (P \cap R)T$.

2. $Q \in \text{MinSpec}(R)$ implies that $QT \in \text{MinSpec}(T)$.

\[ \text{Proof.} \] 1. $P \in \text{MinSpec}(T)$ implies that $P \cap R \in \text{Spec}(R)$.

If possible, suppose $P \cap R \notin \text{MinSpec}(R)$.

Let $P_1 \subset P \cap R$ and $P_1 \in \text{MinSpec}(R)$.

Then $P_1T \subset (P \cap R)T \subseteq P$, which is not possible as $P_1T \in \text{Spec}(R)$.

Therefore, $P \cap R \in \text{MinSpec}(R)$.

Now, it is easy to see that $(P \cap R)T = P$.

2. Let $Q \in \text{MinSpec}(R)$. Then, $QT \in \text{Spec}(T)$.

Suppose $QT \notin \text{MinSpec}(T)$ and $Q_1 \subset QT$ be a minimal prime ideal of $T$.

Then $Q_1 \cap R \subset QT \cap R = Q$ which is a contradiction as $Q_1 \cap R \in \text{Spec}(R)$.

Hence $QT \in \text{MinSpec}(T)$.

\[ \text{Proposition 4.1.5.} \] Let $R$ be a Noetherian $\mathbb{Q}$-algebra. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. If $P \in \text{Ass}(R_R)$ is such that $\sigma(P) = P$, then $\delta(P) \subseteq P$.

\[ \text{Proof.} \] Let $T = R[[t, \sigma]]$.

Now by Proposition (5.1.1) $e^{t\delta}$ is an automorphism of $T$.

Let $P \in \text{Ass}(R_R)$. 

\[ \Box \]
Then by Proposition (5.1.3) $PT \in \text{Ass}(T)$. 

So, there exists an integer $n \geq 1$ such that $(e^{t\delta})^n(PT) = PT$; i.e., $e^{n t\delta}(PT) = PT$.

But $R$ is a $Q$-algebra, therefore $e^{t\delta}(PT) = PT$.

Hence by Lemma (5.1.2) $\delta(P) \subseteq P$. \hfill \Box

Note that the above Proposition (5.1.5) also holds if $P \in \text{MinSpec}(R)$. [See, Proposition (2.1.6)]

### 4.2 Associated Primes

Now we are in a position to give a description of $\text{Ass}(O(R)_{O(R)})$ and $\text{MinSpec}(O(R))$ in the form of the following:

**Theorem 4.2.1.** Let $R$ be a Noetherian $Q$-algebra. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. Then:

1. If $P \in \text{Ass}(O(R)_{O(R)})$ is such that $\sigma(P \cap R) = P \cap R$, then $P \cap R \in \text{Ass}(R_R)$ and if $P_1 \in \text{Ass}(R_R)$ is such that $\sigma(P_1) = P_1$, then $O(P_1) \in \text{Ass}(O(R)_{O(R)})$.

2. If $P \in \text{MinSpec}(O(R))$ is such that $\sigma(P \cap R) = P \cap R$, then $P \cap R \in \text{MinSpec}(R)$ and $P_1 \in \text{MinSpec}(R)$ such that $\sigma(P_1) = P_1$, then $O(P_1) \in \text{MinSpec}(O(R))$.

**Proof.**

1. Let $P_1 \in \text{Ass}(R_R)$ with $\sigma(P_1) = P_1$.

Then $\delta(P_1) \subseteq P_1$ by Proposition (5.1.5).

Let $P_1 = \text{Ann}(cR) = \text{Assas}(cR), c \in R$.

Since, $cP_1 = 0$, we have $\delta^k(c)P_1 = 0$ for all $k \geq 0$. 


Now, let $h = \sum x^jb_j \in O(R), 0 \leq j \leq u$.

Then, for all $r \in R, crhP_1 = 0$; i.e. $cRhP_1 = 0$.

Therefore, $cRhO(R) = 0$.

Now it can be seen that $O(P_1) \in Spec(O(R))$.

Suppose $O(P_1) \neq Ann(chO(R))$.

Then there exists an ideal $K$ of $O(R)$ such that $O(P1) \subseteq K$ and $K = Ann(chO(R)), ch \neq 0$.

Now, by Proposition (3.1.2), there exists $g = \sum x^jd_j \in K, 0 \leq j \leq u$ such that $d_t \in C(P_1)$.

Now $chO(R)K = 0$. So $chRg = 0$ which implies that for all $s \in Rchsg = 0$;

i.e., $c(x^ub_u+ \ldots +b_0)s(x^td_t+ \ldots +d_0) = 0$

or $(x^{u+i}c+ \ldots +d^i(c))b_usd_t+ \ldots +cb_0sd_0 = 0$.

Therefore, $cb_usd_t = 0$ for all $s \in R$;

i.e., $cb_uRd_t = 0$ and we have $d_t \in Ann(cb_uR) = P_1$, a contradiction as $d_t \in C(P_1)$.

Thus $O(P_1) = Ann(chO(R))$ for all $h \in O(R)$; i.e., $O(P_1) = Assas(chO(R))$.

Conversely suppose that $P \in Ass(O(R)_{O(R)})$.

Choose $f = \sum x^ja_j \in O(R), 0 \leq j \leq n, an \neq 0$, $f$ of least degree such that $P = Ann(fO(R)) = Assas(fO(R))$.

Now $a_iR(P \cap R) = 0$ for all $i, 1 \leq i \leq n$.

Let $P_1 \in Ass(anR)$ with $\sigma(P_1) = P_1$.

Then there exists $s \in R$ such that $a_ns \neq 0$ and $P_1 = Ann(a_nsR) = Assas(a_nsR)$.

Now $\delta(P_1) \subseteq P_1$ by Proposition (5.1.5) and by above paragraph $O(P_1) \in Ass(O(R)_{O(R)})$. 
Now $a_n s R P_1 = 0$ implies that $\delta^t(a_n s) R P_1 = 0$ for all integers $t \geq 1$ and for any $h = \sum x^j b_j \in O(R), 0 \leq j \leq u, a_n s R h P_1 = 0$.

Therefore, $a_n s R O(R) P_1 = 0$ which implies that $P_1 \subseteq P \cap R$.

Also $P \cap R \subseteq P_1$ as $a_n s R (P \cap R) = 0$.

Thus $P_1 = P \cap R$ and $O(P_1) = O(P \cap R) \in Ass(O(R)_{O(R)}$).

2. Let $P_1 \in MinSpec(R)$ with $\sigma(P_1) = P_1$.

Then by Proposition (2.1.6), $\delta(P_1) \subseteq P_1$.

Now it can be easily seen that $O(P_1) \in Spec(O(R))$.

If possible suppose that $O(P_1) \notin MinSpec(O(R))$, and $P_2 \subset O(P_1)$ be a minimal prime ideal of $O(R)$.

Then we have $P_2 = O(P_2 \cap R) \subset O(P_1) \in MinSpec(O(R))$.

Therefore, $P_2 \cap R \subset P_1$, which is a contradiction as $P_2 \cap R \in Spec(R)$.

Hence, $O(P_1) \in MinSpec(O(R))$.

Conversely let $P \in MinSpec(O(R))$ with $\sigma(P \cap R) = P \cap R$.

Then it can be easily seen that $P \cap R \in Spec(R)$ and $O(P \cap R) \in Spec(O(R))$.

Therefore, $O(P \cap R) = P$. We now show that $P \cap R \in MinSpec(R)$.

Suppose that $P_3 \subset P \cap R$, and $P_3 \in MinSpec(R)$.

Then $O(P_3) \subset O(P \cap R) = P$.

But $O(P_3) \in Spec(O(R))$ and, $O(P_3) \subset P$, which is not possible.

Thus, we have $P \cap R \in MinSpec(R)$.

We now have the following two Corollaries:

**Corollary 4.2.2.** If in Theorem (5.2.1) $\sigma$ is the identity map and $D(R) = R[x; \delta]$, then...
then:

1. $P \in \text{Ass}(D(R)_{D(R)})$ if and only if $P \cap R \in \text{Ass}(R)$ and $D(P \cap R) = P$.

2. $P \in \text{MinSpec}(D(R))$ if and only if $P \cap R \in \text{MinSpec}(R)$ and $D(P \cap R) = P$.

In the following Corollary we use the fact that for any $P \in \text{Ass}(R) \cup \text{MinSpec}(R)$, $\sigma(P) \in \text{Ass}(R) \cup \text{MinSpec}(R)$.

**Corollary 4.2.3.** If in Theorem (5.2.1) $\delta = 0$ and $S(R) = R[x, \sigma]$, then:

1. $P \in \text{Ass}(S(R)_{S(R)})$ if and only if there exists $Q \in \text{Ass}(R)$ and an integer $m \geq 1$ such that $S(P \cap R) = P$ and $P \cap R = \cap_{j=0}^{m}\sigma^j(Q)$.

2. $P \in \text{MinSpec}(S(R))$ if and only if there exists $Q \in \text{MinSpec}(R)$ and an integer $m \geq 1$ such that $S(P \cap R) = P$ and $P \cap R = \cap_{j=0}^{m}\sigma^j(Q)$.

**Problem:** We are unable to verify whether the above result is true for any $\sigma$ and $\delta$ with out the condition that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. This is now an open problem.

**Theorem 4.2.4.** Let $R$ be a Noetherian $\sigma(*)$-ring which is also an algebra over $\mathbb{Q}$. Then $P \in \text{MinSpec}(R)$ implies $O(P) \in \text{MinSpec}(O(R))$ and $P$ is completely prime ideal of $R$.

**Proof.** By Theorem (2.3.4); $\sigma(P) = P$ and $P$ is completely prime ideal of $R$.

Now, $\sigma(P) = P$ and $P \in \text{MinSpec}(R)$, so by Proposition (5.1.5); $\delta(P) \subseteq P$.

Hence by Lemma (3.3.3), $O(P)$ is a prime ideal of $O(R)$.

Now suppose that $K \subset O(P)$ be a minimal prime ideal of $O(R)$.

Then $(K \cap R) \subset (O(P) \cap R) = P$ (Lemma (3.3.3)) and $K \cap R$ is a prime ideal of $R$, a contradiction. Therefore, $O(P)$ is a minimal prime ideal of $O(R)$. \qed