Chapter 3

Reflection and refraction of \( SH \)-waves at a corrugated interface between two monoclinic elastic half-spaces\(^1\)

3.1 Introduction

The problems concerning the wave propagation in anisotropic elastic media play a major role in the field of geophysics, earthquake engineering and also in signal processing. The studies relating to reflection and transmission of plane harmonic elastic waves from boundaries within the anisotropic media are, therefore, of considerable interest for determining the presence or absence of valuable anisotropic materials beneath the earth surface. Such studies are also helpful in obtaining the knowledge about the rock structure (i.e. the solid part of the interior of the earth) and their elastic properties. They provide the information regarding the minerals present inside the earth. The crystals, which are elastic solids and bounded by natural plane surfaces, are generally anisotropic and occur in different forms. Monoclinic form is one of them. Monoclinic materials possess one plane symmetry. It is proved in the literature that there are 13 independent elastic constants needed to characterize such type of materials. A number of researchers have attempted the problems of reflection and refraction of elastic waves from the plane boundary between two different monoclinic media \( e.g. \) Deresiewicz and Mindlin (1957), Chattopadhyay and Bandyopadhyay (1986), Chattopadhyya \textit{et al.}\(^1\)


Chattopadhyay et al. (1997) studied the reflection and transmission of shear waves at the plane interface between two monoclinic type media. In this chapter, we have considered a problem of reflection and transmission of an incident plane shear wave at a corrugated interface between two dissimilar monoclinic elastic half-spaces. Expressions of the reflection and transmission coefficients have been derived in closed form for the first order approximation of the corrugation. The equations determining the reflection and transmission coefficients for the second order approximation of the corrugation are also presented. Numerical computations are performed for a specific model and depicted graphically showing the effect of anisotropy, frequency of the incident wave, the angle of incidence and the undulation of the boundary surface on these coefficients have been given. The problems considered by Asano (1960) and Chattopadhyay et al. (1997) have been shown as particular cases of the present problem. Some special cases and some particular cases have also been presented.

3.2 Formulation of the problem and its solution

We consider two monoclinic elastic solid half spaces, namely $H_1$ and $H_2$ with different elastic properties and separated by a corrugated interface, whose equation is given by $y = \zeta(z)$. Let the upper half space and the lower half space be denoted by $H_1$ and $H_2$ respectively. The $x$-axis and the $z$-axis are taken on the horizontal plane, while the $y$-axis is taken vertically downwards. The geometry of the problem is shown in Fig 3.1. Here $\zeta$ is a periodic function of $z$, independent of $x$, whose mean value is zero and may be expanded by means of Fourier series as

$$\zeta = \sum_{n=1}^{\infty} [\zeta_n e^{i\lambda_n z} + \zeta_{-n} e^{-i\lambda_n z}], \quad (3.1)$$

where $\zeta_n$ and $\zeta_{-n}$ are defined earlier in (2.2) of Chapter 2.

In a special case, when the interface is periodic and is of the form $\zeta = a \cos k^* z$, the wavelength of corrugation is $2\pi/k^*$. Here, $a$ is the amplitude of the corrugation. Let us use the following notations

$$u_x \equiv u, \quad u_y \equiv v, \quad u_z \equiv w$$

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\[ \tau_{xx} \equiv T_1, \quad \tau_{yy} \equiv T_2, \quad \tau_{zz} \equiv T_3, \quad \tau_{yz} \equiv T_4, \quad \tau_{zx} \equiv T_5, \quad \tau_{xy} \equiv T_6. \]

The stress-strain relations for a rotated \( y \)-cut quartz crystal, which exhibit monoclinic symmetry with \( x \)-axis being the diagonal, are given by

\[
\begin{bmatrix}
T_1 & T_2 & T_3 & T_4 & T_5 & T_6
\end{bmatrix} =
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\
c_{21} & c_{22} & c_{23} & c_{24} & 0 & 0 \\
c_{31} & c_{32} & c_{33} & c_{34} & 0 & 0 \\
c_{41} & c_{42} & c_{43} & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{55} & c_{56} \\
0 & 0 & 0 & 0 & c_{65} & c_{66}
\end{bmatrix}
\begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
S_5 \\
S_6
\end{bmatrix}, \tag{3.2}
\]

where

\[
S_i = \frac{\partial u}{\partial x}, \quad S_2 = \frac{\partial v}{\partial y}, \quad S_3 = \frac{\partial w}{\partial z}, \quad S_4 = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad S_5 = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad S_6 = \frac{\partial v}{\partial x}, \tag{3.3}
\]

and \( c_{ij} = c_{ji} \) \((i, j = 1, 2, 3, \ldots, 6)\) are the elastic constants.

For a plane \( SH \)-wave having vibrations in \( y-z \) plane and causing displacement in the \( x \)-direction only, we shall take \( u = u(y, z, t) \) and \( v = w = 0 \).

Thus, in this case, the expressions of the quantities \( S_i \) and \( T_i \), \((i = 1, 2, 3, \ldots, 6)\) become

\[
S_1 = 0 = S_2 = S_3 = S_4, \quad S_5 = \frac{\partial u}{\partial z}, \quad S_6 = \frac{\partial u}{\partial y},
\]

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\[ T_1 = 0 = T_2 = T_3 = T_4, \]
\[ T_5 = c_{55} S_5 + c_{56} S_6 = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}, \]
\[ T_6 = c_{65} S_5 + c_{66} S_6 = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}, \]
\[ \tau_{xy,y} + \tau_{zz,z} = \rho \ddot{u}, \]
\[ \text{or} \quad \frac{\partial}{\partial y} \left( c_{55} \frac{\partial u}{\partial z} + c_{56} \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( c_{55} \frac{\partial u}{\partial z} + c_{56} \frac{\partial u}{\partial y} \right) = \rho \ddot{u} \]
\[ \text{or} \quad c_{66} \frac{\partial^2 u}{\partial y^2} + 2c_{56} \frac{\partial^2 u}{\partial y \partial z} + c_{55} \frac{\partial^2 u}{\partial z^2} = \rho \frac{\partial^2 u}{\partial \eta^2}. \] (3.5)

Therefore, the equation of motion in a monoclinic elastic medium, in the absence of body forces, when \( SH \)-wave is propagating in \( y - z \) plane and causing displacements in the \( x \)-direction only, reduces to

\[ \tau_{xy,y} + \tau_{zz,z} = \rho \ddot{u}, \]
\[ \text{or} \quad \frac{\partial}{\partial y} \left( c_{55} \frac{\partial u}{\partial z} + c_{56} \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( c_{55} \frac{\partial u}{\partial z} + c_{56} \frac{\partial u}{\partial y} \right) = \rho \ddot{u} \]
\[ \text{or} \quad c_{66} \frac{\partial^2 u}{\partial y^2} + 2c_{56} \frac{\partial^2 u}{\partial y \partial z} + c_{55} \frac{\partial^2 u}{\partial z^2} = \rho \frac{\partial^2 u}{\partial \eta^2}. \] (3.5)

For a plane harmonic \( SH \)-wave propagating in the positive direction of \( z \)-axis, we can assume the solution of equation (3.5) in the form

\[ u(y, z, t) = U(y) e^{i(\omega t - k z)} \] (3.6)

where \( k \) is the wavenumber, \( \omega \) is the circular frequency and \( \omega = kc, c \) being the phase velocity. Substituting (3.6) into (3.5) and solving the resulting equation for \( U(y) \), we finally get the following solution of equation (3.5) as

\[ u(y, z, t) = e^{-\frac{\alpha}{2} y} \left( A \cos \eta y + B \sin \eta y \right) e^{i(\omega t - k z)}, \] (3.7)

where \( A \) and \( B \) are constants and

\[ \eta = \sqrt{\frac{\rho \omega^2 - k^2 c_{55}}{c_{66}}} + \frac{k^2 c_{55}^2}{c_{66}^2} = \frac{1}{2 \sqrt{4\beta - \alpha^2}}, \] (3.8)

\[ \alpha = -2ik(c_{56}/c_{66}), \quad \beta = (\rho \omega^2 - c_{55}k^2)/c_{66}. \]

The equation of \( SH \)-wave motion, in the absence of body forces, for the upper medium \( H_1 \) and for the lower medium \( H_2 \) may be written respectively as

\[ c_{66} \frac{\partial^2 u_1}{\partial y^2} + 2c_{56} \frac{\partial^2 u_1}{\partial y \partial z} + c_{55} \frac{\partial^2 u_1}{\partial z^2} = \rho_1 \frac{\partial^2 u_1}{\partial \eta^2}, \] (3.9)
and

\[ d_{66} \frac{\partial^2 u_2}{\partial y^2} + 2d_{56} \frac{\partial^2 u_2}{\partial y \partial z} + d_{55} \frac{\partial^2 u_2}{\partial z^2} = \rho_2 \frac{\partial^2 u_2}{\partial t^2}, \]  

(3.10)

where \( u_1 \) and \( u_2 \) are the displacement components in the \( z \)-direction. The quantities \( c_{55}, c_{66}, d_{55} \) and \( d_{56} \) and \( d_{66} \) are the elastic constants in the medium \( H_1 \) and \( H_2 \) respectively, \( \rho_1 \) is the density in the medium \( H_1 \).

Let a unit amplitude plane \( SH \)-wave propagating through the upper medium \( H_1 \) be incident at the corrugation interface making an angle \( \gamma \) with the \( y \)-axis. This incident wave would give rise to a regularly reflected and a regularly refracted waves along with irregularly reflected and refracted waves. The displacements in the upper medium \( H_1 \) caused due to incident and regularly reflected \( SH \)-waves respectively, are given by

\[ u^{inc} = e^{ik(\alpha_1 - p)} e^{i\omega(t - \frac{\sin \gamma}{\beta_{h1}})} , \quad u^{refle} = Be^{ik(\alpha_1 + p)} e^{i\omega(t - \frac{\sin \gamma}{\beta_{h1}})}, \]  

(3.11)

where \( B \) is the amplitude of the regularly reflected \( SH \)-wave and

\[ \alpha_1 = \frac{c_{66}}{c_{56}} , \quad p = \frac{1}{c_{66}} \sqrt{c_{66}(\frac{\rho_1 \beta_{h1}^2}{\sin^2 \gamma} - c_{55}) + c_{56}^2} , \quad \beta_{h1} = \sqrt{c_{66}/\rho_1} , \]  

(3.12)

\( \beta_{h1} \) is the velocity of \( SH \)-wave in the upper medium. Thus, we can write

\[ u^{inc+refle} = e^{ik\alpha_1 y} (e^{-ik\nu} + Be^{ik\nu}) e^{i\omega(t - \frac{\sin \gamma}{\beta_{h1}})}. \]  

(3.13)

Similarly, the displacement in the lower medium \( H_2 \) due to regularly refracted wave is given by

\[ u^{refr} = De^{ik(\alpha_2 - q)} e^{i\omega(t - \frac{\sin \delta}{\beta_{h2}})}, \]  

(3.14)

where \( D \) is the amplitude of the regularly refracted wave and

\[ \alpha_2 = \frac{d_{56}}{d_{66}} , \quad q = \frac{1}{d_{56}} \sqrt{d_{66}(\frac{\rho_2 \beta_{h2}^2}{\sin^2 \delta} - d_{55}) + d_{56}^2} , \quad \beta_{h2} = \sqrt{d_{66}/\rho_2} , \]  

(3.15)

\( \beta_{h2} \) is the velocity of \( SH \)-wave in the lower medium, \( \delta \) is the angle made by the refracted wave with the \( y \)-axis and is connected with the angle of incidence \( \gamma \) through the Snell’s law as already explained in equation (2.17) of Chapter 2.

Due to corrugated nature of the interface, the \( n^{th} \) order spectrum for the scattered reflected waves (called irregularly reflected waves) is given by

\[ u^{n-refl} = e^{ik\alpha_1 y} [B_n e^{ik\nu} e^{i\omega(t - \frac{\sin \gamma}{\beta_{h1}})} + B_n e^{ik\nu} e^{i\omega(t - \frac{\sin \delta}{\beta_{h2}})}], \]  

(3.16)
where $B_n$ and $B'_n$ are constants and

$$p_n = \frac{1}{c_{66}} \left( \frac{\rho_1 \beta_{b1}^2}{\sin^2 \gamma_n} - c_{55} \right) + c_{26}, \quad p'_n = \frac{1}{c_{66}} \left( \frac{\rho_1 \beta_{b1}^2}{\sin^2 \gamma'_n} - c_{55} \right) + c_{26}.$$  

(3.17)

$\gamma_n$ are the angles made with the y—axis by the $n^{th}$ order spectrum in the left side of the regularly reflected wave and $\gamma'_n$ are the angles made with the y—axis by the $n^{th}$ order spectrum in the right side of the regularly reflected wave.

Similarly, the $n^{th}$ order spectrum for scattered refracted waves (called irregularly refracted waves) is given by

$$u_{ir-refr} = e^{ik_2y}[D_n e^{-ik_2ny} e^{i\omega(t - \frac{c_{26}}{\beta_{b2}})} + D'_n e^{-ik_2ny} e^{i\omega(t - \frac{c_{26}}{\beta_{b2}'})}],$$  

(3.18)

where $D_n$ and $D'_n$ are constants and

$$q_n = \frac{1}{d_{66}} \sqrt{d_{66} \left( \frac{\rho_2 \beta_{b2}^2}{\sin^2 \delta_n} - d_{55} \right) + d_{26}^2}, \quad q'_n = \frac{1}{d_{66}} \sqrt{d_{66} \left( \frac{\rho_2 \beta_{b2}^2}{\sin^2 \delta'_n} - d_{55} \right) + d_{26}^2}. \quad (3.19)$$

$\delta_n$ and $\delta'_n$ are the angles made with the y—axis by the $n^{th}$ order spectrum on the left side and right side respectively of the regularly refracted waves. The angles $\gamma_n$, $\gamma'_n$, $\delta_n$ and $\delta'_n$ are connected with the angles $\gamma$ and $\delta$ through the Spectrum theorem given by equation (2.22).

The total displacement $u_1$ in the upper medium $H_1$ is, thus, the sum of the displacements due to regularly reflected wave, the irregularly reflected waves and the incident wave

$$u_1 = e^{ik_1y}[D e^{-ik_1y} + B e^{ik_1y} + \sum B_n e^{ik_1ny} e^{-ink_1z} + \sum B'_n e^{ik_1ny} e^{ink_1z}] e^{i\omega(t - \frac{\omega_{b1}}{\beta_{b1}})}.$$  

(3.20)

Similarly, the total displacement $u_2$ in the lower medium $H_2$ is the sum of the displacements due to the regularly and the irregularly the refracted waves

$$u_2 = e^{ik_2y}[D e^{-ik_2y} + \sum D_n e^{-ik_2ny} e^{-ink_2z} + \sum D'_n e^{-ik_2ny} e^{ink_2z}] e^{i\omega(t - \frac{\omega_{b2}}{\beta_{b2}})}, \quad (3.21)$$

where the use of Spectrum theorem given in (2.22) has been made in writing equations (3.20) and (3.21). The constants $B$, $D$, $B_n$, $D_n$, $B'_n$ and $D'_n$ can be determined by satisfying the following boundary conditions at the interface.
3.3 Boundary conditions

The boundary conditions to be satisfied at the corrugated interface \( y = \zeta(z) \) are the continuity of \( x \)-component of the displacement and the traction.

First, we deduce the expression of relevant traction on the corrugated interface. The direction cosine of the normal \( \nu \), to the corrugated interface \( y = \zeta(z) \) are given by

\[
< 0, \frac{1}{\sqrt{1 + \zeta'^2}}, \frac{-\zeta'}{\sqrt{1 + \zeta'^2}} >
\]

The normal traction to the corrugated boundary surface, thus, becomes

\[
\begin{bmatrix}
T_1 & T_5 & T_3 \\
T_6 & T_2 & T_4 \\
T_7 & T_4 & T_3
\end{bmatrix}
\begin{bmatrix}
0 \\
\frac{1}{\sqrt{1 + \zeta'^2}} \\
\frac{-\zeta'}{\sqrt{1 + \zeta'^2}}
\end{bmatrix}
= [T_6 - T_5\zeta']\frac{1}{\sqrt{1 + \zeta'^2}}
\]

The mathematical form of the boundary conditions at the corrugated interface \( y = \zeta(z) \) can be written as

\[(I) \quad u_1 = u_2, \quad \text{and} \quad (II) \quad [T_6^{(1)} - T_5^{(1)}\zeta']\frac{1}{\sqrt{1 + \zeta'^2}} = [T_6^{(2)} - T_5^{(2)}\zeta']\frac{1}{\sqrt{1 + \zeta'^2}},\]

where \( \zeta' \) is the derivative of \( \zeta \) with respect to \( z \), \( T_6^{(l)} \) and \( T_5^{(l)} \) \((l = 5,6)\) are the stress components in the medium \( H_1 \) and \( H_2 \) respectively.

Inserting equations (3.20) and (3.21) into the above boundary conditions and making use of Snell’s law (2.17), we obtain

\[
e^{ik_1\zeta}[e^{-ik_1\zeta} + Be^{ik_1\zeta} + \sum B_n e^{ik_1\zeta} e^{-nk_1^*z} + \sum B_n e^{ik_1\zeta} e^{nk_1^*z}]
\]

\[
= [D e^{-ik_1\zeta} + \sum D_n e^{-ik_1\zeta} e^{-nk_1^*z} + \sum D_n e^{-ik_1\zeta} e^{nk_1^*z}]e^{ik_2\zeta}, \quad (3.22)
\]

and

\[
c_{66}[k(-p + \alpha_1)e^{-ik_1\zeta} + ik(p + \alpha_1)Be^{ik_1\zeta} + \sum B_n tk(p_n + \alpha_1)e^{ik_1\zeta} e^{-nk_1^*z} + \sum B_n tk(p_n + \alpha_1)e^{ik_1\zeta} e^{nk_1^*z}]
\]

\[
+ \sum B_n [k(p_n + \alpha_1)e^{ik_1\zeta} e^{nk_1^*z}]e^{ik_1\zeta} - c_{66}[\frac{\omega \sin \gamma}{\beta_1}(e^{-ik_1\zeta} + Be^{ik_1\zeta}) + \sum B_n e^{ik_1\zeta} (nk_1^* + \frac{\omega \sin \gamma}{\beta_1})e^{-nk_1^*z} - \sum B_n e^{ik_1\zeta} (nk_1^* - \frac{\omega \sin \gamma}{\beta_1})e^{nk_1^*z}]
\]

\[
- \zeta'[c_{66}[k(-p + \alpha_1)e^{-ik_1\zeta} - c_{66}[\frac{\omega \sin \gamma}{\beta_1}(e^{-ik_1\zeta} + Be^{ik_1\zeta}) + \sum B_n e^{ik_1\zeta} (nk_1^* + \frac{\omega \sin \gamma}{\beta_1})e^{-nk_1^*z}]
\]

\[
- \sum B_n e^{ik_1\zeta} (nk_1^* - \frac{\omega \sin \gamma}{\beta_1})e^{nk_1^*z} - \zeta'[c_{66}[k(-p + \alpha_1)e^{-ik_1\zeta}]
\]

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These equations enable us to determine the amplitude ratios for the \(n\)th order approximation of the corrugation. Here, we shall discuss the case of first and second order approximation of the corrugation.

### 3.4 Solution of the first order approximation

We assume that the corrugation of the interface \(y = \zeta(z)\) is so small that higher powers of \(\zeta\) can be neglected. Using the approximation as already explained in (2.30) into equations (3.22) and (3.23), the first order approximation for \(B\) and \(D\) can be obtained by collecting the terms independent of \(z\) and \(\zeta\) as follows

\[
1 + B = D, \tag{3.24}
\]

\[
(c_{66}\alpha_1 - c_{56} - pc_{66}) + (c_{66}\alpha_1 - c_{56} + pc_{66})B = [d_{56}(-q + \alpha_2) - d_{56}] D. \tag{3.25}
\]

These formulae give the values of the coefficients \(B\) and \(D\) at the plane interface. The coefficient \(B\) corresponds to the reflection coefficient at the plane interface, while the
coefficient $D$ corresponds to the transmission coefficient at the plane interface. Solving equations (3.24) and (3.25), we obtain

$$B = \frac{1 - M}{1 + M}, \quad D = \frac{2}{1 + M},$$

(3.26)

where

$$M = \frac{d_{66}}{c_{66}} \sqrt{\frac{\frac{d_{56}}{d_{66}} - \left(\frac{d_{56}}{d_{66}}\right)^2 - \frac{c^2}{\beta_{h_2}^2}}{\left(\frac{c_{56}}{c_{66}} - \left(\frac{c_{56}}{c_{66}}\right)^2 - \frac{c^2}{\beta_{h_2}^2}\right)^2}}.$$  

(3.27)

We see that the expressions of the coefficients $B$ and $D$ in (3.26) are the same as obtained by Chattopadhyay et al. (1997) for the relevant problem. Further, to find the solution of the first order approximation for the coefficients $B_n$ and $D_n$, we collect the coefficients of $e^{\imath n k^* z}$, obtaining

$$B_n - D_n = \imath k\left[(-q + \alpha_2)D + p(1 - B) - \alpha_1(1 + B)\right]\zeta_{-n},$$

(3.28)

$$\ell[c_{66}k(p_n + \alpha_1) - c_{56}(nk^* + \frac{\omega \sin \gamma}{\beta_{h_1}})]B_n + \ell[d_{66}k(q_n - \alpha_2) + d_{56}(nk^* + \frac{\omega \sin \gamma}{\beta_{h_1}})]D_n$$

$$= [(1 + B)[c_{56}k^2(p^2 + \alpha_1^2) - k\alpha_1 c_{56}(\frac{\omega \sin \gamma}{\beta_{h_1}} - nk^*) - c_{55}(nk^* \omega \sin \gamma)]$$

$$+ (1 - B)[(\frac{\omega \sin \gamma}{\beta_{h_1}} - nk^*)k p c_{56} - 2k^2 p \alpha_1 c_{66}] + D[d_{66}k^2(-q^2 - \alpha_2^2 + 2q \alpha_2)$$

$$+ d_{56}k(-q + \alpha_2)(\frac{\omega \sin \delta}{\beta_{h_2}} - nk^*) + d_{55}(\frac{nk^* \omega \sin \delta}{\beta_{h_2}})]\zeta_{-n}.$$  

(3.29)

Similarly, equating the coefficients of $e^{\imath n k^* z}$, we obtain the first order approximation for the coefficients $B'_n$ and $D'_n$ as follows

$$B'_n - D'_n = \imath k[(-q + \alpha_2)D + p(1 - B) - \alpha_1(1 + B)]\zeta_n,$$

(3.30)

$$\ell[c_{66}k((p'_n + \alpha_1) + c_{66}(nk^* - \frac{\omega \sin \gamma}{\beta_{h_1}})]B'_n + \ell[d_{66}k(q'_n - \alpha_2) - d_{56}(nk^* - \frac{\omega \sin \gamma}{\beta_{h_1}})]D'_n$$

$$= [(1 + B)[c_{56}k^2(p^2 + \alpha_1^2) - c_{56}(\frac{\omega \sin \gamma}{\beta_{h_1}} - nk^*)k \alpha_1] + c_{55}(\frac{nk^* \omega \sin \gamma}{\beta_{h_1}}]$$

$$+ (1 - B)[c_{56}k p(\frac{\omega \sin \gamma}{\beta_{h_1}} + nk^*) - 2k^2 p \alpha_1 c_{66}] + D[d_{66}k^2(-q^2 - \alpha_2^2 + 2q \alpha_2)$$

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The equations (3.28) - (3.31) give the values of $B_n$, $D_n$, $B'_n$ and $D'_n$ as follows

\begin{align*}
B_n &= \frac{\Delta b_n}{\Delta n}, \quad D_n = \frac{\Delta d_n}{\Delta n}, \quad B'_n = \frac{\Delta b'_n}{\Delta n}, \quad D'_n = \frac{\Delta d'_n}{\Delta n}, \quad \text{(3.32)}
\end{align*}

where

\begin{align*}
\Delta b_n &= [(1+B)[c_{66}k^2(p^2+\alpha_1^2) - k\alpha_1c_{66}(\frac{\omega\sin\gamma}{\beta_{h1}} - nk\epsilon_{n}) - c_{55}(\frac{nk^*\omega\sin\gamma}{\beta_{h1}}) - d_{56}k\alpha_1(nk^* + \frac{\omega\sin\delta}{\beta_{h2}})]
+k^2d_{66}\alpha_1(q_n - \alpha_2)] + (1-B)[(\frac{\sin\gamma_4}{\beta_{h1}} - nk\epsilon_{n})kpc_{56} - 2k^2p\alpha_1c_{66} - d_{56}kp(nk^* + \frac{\omega\sin\delta}{\beta_{h2}})]
-k^2d_{66}(q_n - \alpha_2)] + D(d_{66}k^2(q_n - q) + 2nk^*d_{56}k)(q - d_{55}(\frac{nk^*\omega\sin\delta}{\beta_{h2}}))]\zeta_n, \quad \text{(3.33)}
\end{align*}

\begin{align*}
\Delta d_n &= [(1+B)[c_{66}k^2(p^2 - p_n\alpha_1) + 2k\alpha_1n^*c_{56} - c_{55}(\frac{nk^*\omega\sin\gamma}{\beta_{h1}})] + (1-B)[-2nk^*kp c_{56}
+k^2p (p_n - \alpha_1)c_{66}] + D[d_{66}k^2(-q^2 - \alpha_2^2 + 2q\alpha_2) + [d_{56}k(-q + \alpha_2)(\frac{\omega\sin\delta}{\beta_{h2}} - nk^*)
+d_{56}(\frac{nk^*\omega\sin\delta}{\beta_{h2}}) + [k^2c_{66}(p_n + \alpha_1) + c_{56}k(nk^* + \frac{\omega\sin\delta}{\beta_{h2}})](-q + \alpha_2)]\zeta_n, \quad \text{(3.34)}
\end{align*}

\begin{align*}
\Delta b'_n &= [(1+B)[c_{66}k^2(p^2 + \alpha_1^2) - k\alpha_1c_{66}(\frac{\omega\sin\gamma}{\beta_{h1}} + nk^*) + c_{55}(\frac{nk^*\omega\sin\gamma}{\beta_{h1}}) + k^2\alpha_1d_{66}(q'_n - \alpha_2)]
-d_{56}\alpha_1k(nk^* - \frac{\omega\sin\delta}{\beta_{h2}})] + (1-B)[(\frac{\omega\sin\gamma}{\beta_{h1}} + nk^*)kpc_{56} - 2k^2p\alpha_1c_{66} - k^2d_{66}(q'_n - \alpha_2)
+d_{56}kp(nk^* - \frac{\omega\sin\delta}{\beta_{h2}})] - D[(d_{66}k^2(q - q'_n) + 2nk^*d_{56}k)(q - \alpha_2) + d_{55}(\frac{nk^*\omega\sin\delta}{\beta_{h2}}))]\zeta_n, \quad \text{(3.35)}
\end{align*}

\begin{align*}
\Delta d'_n &= [(1+B)[c_{66}k^2(p^2 - p'_n\alpha_1) - 2k\alpha_1c_{66} nk^* + c_{55}(\frac{nk^*\omega\sin\gamma}{\beta_{h1}})]
+(1-B)[2nk^*kp c_{56} + k^2p(p'_n - \alpha_1)c_{66}] + D[d_{66}k^2(-q^2 - \alpha_2^2 + 2q\alpha_2) + [d_{56}k(\frac{\omega\sin\delta}{\beta_{h2}} + nk^*)
+c_{56}k(nk^* - \frac{\omega\sin\delta}{\beta_{h2}}) + k^2\alpha_6(p'_n + \alpha_1)](-q + \alpha_2) - d_{55}(\frac{nk^*\omega\sin\delta}{\beta_{h2}})]\zeta_n. \quad \text{(3.36)}
\end{align*}

\begin{align*}
\Delta n &= \epsilon[d_{66}k(q_n - \alpha_2) + c_{66}k(p_n + \alpha_1) + (nk^* + \frac{\omega\sin\gamma}{\beta_{h1}})(d_{56} - c_{56})] \quad \text{(3.37)}
\end{align*}
\[ \Delta'_n = \nu[\cos k' \theta' - \cos k \theta - \nu k' \cos k' \theta' \cos (k + k') \theta + \nu k \cos k \theta \cos (k + k') \theta + \nu k \sin \gamma \sin \phi]. \quad (3.38) \]

The values of the coefficients \( B \) and \( D \) appearing in the above expressions are given by equation (3.26). Here \( B_n \) and \( B'_n \) are the reflection coefficients for the first order approximation of the corrugation; \( D_n \) and \( D'_n \) are the refraction coefficients for the first order approximation of the corrugation.

If the terms of higher order than \( \zeta^2 \) are neglected, we have

\[ e^{-iQc} \approx 1 - Q\zeta - \frac{Q^2\zeta^2}{2}, \quad (3.39) \]

To find the solution of the second order approximation for the coefficients \( B, D, B_n, D_n \), \( B'_n \) and \( D'_n \), we collect the terms independent of \( z \), equating the coefficients of \( e^{-i\kappa z} \) and coefficients of \( e^{i\kappa z} \) from equations (3.22) and (3.23) after inserting (3.39), obtaining

\[
(1 + B) - k^2[(-p + \alpha_1)^2 + B(p + \alpha_1)^2] \zeta_n \zeta_{-n} + ikB_n(p_n + \alpha_1)\zeta_n + ikB'_n(p'_n + \alpha_1)\zeta_{-n} = \]

\[ D[1 - k^2(\alpha_2 - q)^2\zeta_n - \nu kD_n(\alpha_2 - q_n)\zeta_n + \nu kD'_n(\alpha_2 - q'_n)\zeta_{-n}, \quad (3.40) \]

\[
[1 - k^2(-p + \alpha_1)^2\zeta_n][\cos \beta_n + \cos \beta_6 (-p + \alpha_1)] + B(\cos \beta_6 + \cos \beta_6 (p + \alpha_1)) [1 - k^2(p + \alpha_1)^2\zeta_n] \]

\[ + B_n \nu \zeta_n(p_n + \alpha_1)] \cos \beta_6 k(p_n + \alpha_1) - \cos \beta_6 (n\kappa^* + \sin \gamma \sin \phi)] + B'_n \nu \zeta_{-n}(p'_n + \alpha_1) \]

\[
[\cos \beta_6 k(p'_n + \alpha_1) - \cos \beta_5 (n\kappa^* - \sin \gamma \sin \phi)] = D[-d_{56} + d_{66}(\alpha_2 - q) (1 - k^2(\alpha_2 - q)^2\zeta_n \zeta_{-n})] \]

\[ + D_n \nu \zeta_n(\alpha_2 - q_n)]d_{56} k(\alpha_2 - q_n) - d_{56}(n\kappa^* + \sin \gamma \sin \phi)] \]

\[ + D'_n \nu \zeta_{-n}(\alpha_2 - q'_n)]d_{56} k(\alpha_2 - q'_n) - d_{56}(n\kappa^* - \sin \gamma \sin \phi)]], \quad (3.41) \]

\[ \nu \zeta_{-n}[(-p + \alpha_1) + B(p + \alpha_1) + D_n[1 - k^2(p_n + \alpha_1)\zeta_n] - D'_n k^2(p'_n + \alpha_1)^2\zeta_{-n} = \]

\[ = Dk\zeta_{-n}(\alpha_2 - q) + D_n[1 - k^2(\alpha_2 - q_n)^2\zeta_n] - D'_n k^2(\alpha_2 - q'_n)^2\zeta_{-n}, \quad (3.42) \]

\[ -k\zeta_{-n}[(-p + \alpha_1) \cos \beta_5 + \cos \beta_6 (-p + \alpha_1) + n\kappa^* \cos \beta_5 + \cos \beta_6 (-p + \alpha_1)] \]

\[ -Bk\zeta_{-n}(p + \alpha_1) \cos \beta_5 + \cos \beta_6 (p + \alpha_1) k + n\kappa^* \cos \beta_5] \]

\[ + B_n \nu[1 - k^2(p_n + \alpha_1)^2\zeta_n]\cos \beta_6 k(p_n + \alpha_1) - \cos \beta_6 (n\kappa^* + \sin \gamma \sin \phi)] + B'_n \nu \zeta^2 \]

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\[ \times [(nk^* - \frac{\omega \sin \gamma}{\beta_{h_1}})(p_n + \alpha_1)(nk^* c_{55} - \frac{k(p_n' + \alpha_1)}{2} c_{56}) - (p_n' + \alpha_1)'(\frac{k^2(p_n' + \alpha_1)}{2} c_{66} - knk^* c_{56})] \]

\[ = D k \zeta_n [(a_2 - q)(kd_{66} - kd_{66}(a_2 - q) + nk^* d_{56}) + nk^* d_{55}] - D_n l \left[(nk^* + \frac{\omega \sin \delta}{\beta_{h_2}})d_{56} \right] \]

\[ - d_{66} k(a_2 - q_n)(1 - k^2(a_2 - q_n)^2 \zeta_n \zeta_{-n}) - D_n' \zeta_n^2 [k^2(a_2 - q_n)^2(\frac{k}{2}(a_2 - q_n)d_{66} - nk^* d_{56})] \]

\[ + (nk^* - \frac{\omega \sin \delta}{\beta_{h_2}})k(a_2 - q_n)(\frac{k}{2}(a_2 - q_n)d_{66} - nk^* d_{55})]. \tag{3.43} \]

\[ \frac{\alpha_2 - q + \alpha_1}{B(p + \alpha_1)} \zeta_n - B_n k^2(p_n + \alpha_1)^2 \zeta_n^2 + B_n'[1 - k^2(p_n' + \alpha_1)^2 \zeta_n \zeta_{-n}] \]

\[ = D(\alpha_2 - q) \zeta_n - D_n' k \frac{(\alpha_2 - q_n)^2}{2} \zeta_n^2 + D_n'[1 - k^2(a_2 - q_n)^2 \zeta_n \zeta_{-n}]. \tag{3.44} \]

\[ - k \zeta_n[(p - p + \alpha_1)(-c_{66} + c_{66}(-p + \alpha_1)) + nk^*(c_{55} - c_{56}(-p + \alpha_1))] - B k^2 \zeta_n [(p + \alpha_1) \times (k c_{56} + c_{66} k(p + \alpha_1) - nk^* c_{56}) + nk^* c_{55}] + B_n k^2 c_n^2 [(nk^* + \frac{\omega \sin \delta}{\beta_{h_2}})(p_n + \alpha_1)] \times (k c_{56} + c_{66} k(p + \alpha_1) - nk^* c_{56}) + nk^* c_{55}] + B_n'[1 - k^2(p_n' + \alpha_1)^2 \zeta_n \zeta_{-n}] \times \zeta_n (c_{56} (nk^* - \frac{\omega \sin \gamma}{\beta_{h_1}}) + c_{66} k(p_n + \alpha_1))] = D_n \zeta_n k[(\alpha_2 - q)(kd_{66} - kd_{66}(a_2 - q)) \times (nk^* d_{56}) - nk^* d_{55}] - D_n c_n^2 (a_2 - q_n)^2(k \frac{(\alpha_2 - q_n)}{2} d_{66} + nk^* d_{56}) \]

\[ + nk^* d_{55}] - D_n c_n^2 (a_2 - q_n)^2(k \frac{(\alpha_2 - q_n)}{2} d_{66} + nk^* d_{55}) \]

\[ - k(nk^* + \frac{\omega \sin \delta}{\beta_{h_2}})(a_2 - q_n)(k \frac{(\alpha_2 - q_n)}{2} - nk^* d_{55}) \]

\[ + D_n'[1 - k^2(a_2 - q_n)^2 \zeta_n \zeta_{-n}]. \tag{3.45} \]

These equations enable us to give the reflection and transmission coefficients for the second order approximation of the corrugation.

### 3.5 Energy relation

To derive the expression of the energy flux for the incident, reflected and refracted waves, first of all, we shall prove that for a plane time harmonic wave, the total energy per unit volume is twice the mean of the kinetic energy per unit volume.

Let us consider the displacement for a wave propagating in the \( x \)- direction as

\[ u(x, t) = A e^{ik(x-ct)}, \]

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where $A$ is a constant and defines the amplitude of the wave. The argument $k(x - ct)$ is called the phase of the wave, $c$ is the phase velocity.

At any instant $t$, $u(x, t)$ is a periodic function of $x$ with wavelength $\Lambda$, where $\Lambda = 2\pi/k$. The quantity $k = 2\pi/\Lambda$, which counts the number of wavelengths over $2\pi$, is termed as wavenumber. At any position, the displacement $u(x, t)$ is time-harmonic with time period $T$, where $T = 2\pi/\omega$ and $\omega$ is the circular frequency, which is given by $\omega = kc$.

Let $\ddot{u}(x, t)$ and $\tau_z(x, t)$ be the particle velocity and stress component respectively, then

$$\ddot{u} = \frac{\partial u}{\partial t} = -ik\ c\ A\ e^{i[k(x - ct)]}, \quad u_x = \frac{\partial u}{\partial x} = ik\ A\ e^{i[k(x - ct)]},$$

Clearly,

$$\ddot{u} = -cu_x.$$ 

The stress component $\tau_z$ is given by

$$\tau_z = \rho\ c^2\ \frac{\partial u}{\partial x} = \rho\ c^2\ u_x = -\rho\ \dot{u}.$$ 

The instantaneous rate of work of the traction acting on the element is the product of the stress components $\tau_z$ and the particle velocity $\ddot{u}(x, t)$. This instantaneous rate of work is called the power per unit area and is denoted by $\mathcal{P}$. Therefore, we have

$$\mathcal{P} = -\tau_z\ddot{u} = \rho\ c\ \ddot{u}^2, \quad (: \dot{\tau}_z = -\rho\ \dot{u}).$$

As the power defines the rate at which energy is communicated per unit time across a unit area, clearly, $\mathcal{P}$ represents the energy flux across the area element and it must, therefore, be related to the total energy density $\mathcal{H}$.

The total energy density per unit volume is equal to the sum of the kinetic energy density $\mathcal{K}$ and the strain energy density $\mathcal{U}$, thus

$$\mathcal{H} = \mathcal{K} + \mathcal{U} = \frac{1}{2}\ \rho\ \ddot{u}^2 + \frac{1}{2}\ \rho\ c^2\ u_x^2 = \frac{1}{2}\ \rho\ \ddot{u}^2 + \frac{1}{2}\ \rho\ c^2\ \left(\frac{-\ddot{u}}{c}\right)^2 \quad (: \ddot{u} = -c\ u_x) = 2\ \left[\frac{1}{2}\ \rho\ \ddot{u}^2\right].$$

Thus, for the plane time harmonic waves, the total energy per unit volume is twice the mean of the kinetic energy per unit volume.
The expression for the energy flux for the incident, reflected and refracted waves is obtained by multiplying the total energy per unit volume (which is twice the mean of kinetic energy density) by the velocity of propagation and the area of the wave front involved. The area of the wave front is proportional to the cosine of the angle between the wave normal and the vertical.

Thus, the expression of energy for the incident SH− wave in the medium $H_1$, is given by

$$E^{inc} = \frac{1}{2} \rho_1 k^2 c^2 \beta_{h_1} \cos \gamma \exp \{2i k (\alpha_1 - p)y + c(t - z \frac{\sin \gamma}{\beta_{h_1}})\}.$$ 

Similarly, the expressions of energy for regularly reflected, irregularly reflected, regularly refracted and irregularly refracted $SH$- waves for the unit area of wave front can be written respectively as

$$E^{reg-refl} = \frac{1}{2} \rho_1 k^2 c^2 \beta_{h_1} [B_n^2 \cos \gamma_n \exp \{2i k (\alpha_1 + p_n)y + c(t - z \frac{\sin \gamma_n}{\beta_{h_1}})\} + B'_{n}^2 \cos \gamma'_n \exp \{2i k (\alpha_1 + p'_n)y + c(t - z \frac{\sin \gamma'_n}{\beta_{h_1}})\}],$$

$$E^{reg-refr} = \frac{1}{2} \rho_2 k^2 c^2 \beta_{h_2} [D_n^2 \cos \delta_n \exp \{2i k (\alpha_2 - q_n)y + c(t - z \frac{\sin \delta_n}{\beta_{h_2}})\} + D'_{n}^2 \cos \delta'_n \exp \{2i k (\alpha_2 - q'_n)y + c(t - z \frac{\sin \delta'_n}{\beta_{h_2}})\}],$$

Using Snell’s law, the Spectrum theorem as given in (2.17) and (2.22) and dividing by the energy flux of the incident wave, the energy equation can be written as

$$1 = B^2 + D^2 \frac{\beta_{h_2} \rho_2 \cos \delta}{\beta_{h_1} \rho_1 \cos \gamma} + \sum B_n^2 \cos \gamma_n \cos \gamma + \sum B'_{n}^2 \cos \gamma'_n \cos \gamma + \sum D_n^2 \frac{\beta_{h_2} \rho_2 \cos \delta_n}{\beta_{h_1} \rho_1 \cos \gamma} + \sum D'_{n}^2 \frac{\beta_{h_2} \rho_2 \cos \delta'_n}{\beta_{h_1} \rho_1 \cos \gamma}.$$ 

(3.46)

The partition of energy at a plane interface between two different monoclinic half-spaces can be readily reduced from equation (3.46) by putting the values of the coefficients $B_n$, $D_n$, $B'_n$ and $D'_n$ equal to zero, as they are proportional to the amplitude of corrugated interface. We obtain

$$1 = B^2 + D^2 \frac{\beta_{h_2} \rho_2 \cos \delta}{\beta_{h_1} \rho_1 \cos \gamma}.$$ 

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This relation is similar to and in fact, reduces to the relation of partitioning of energy due to incident $SH$-wave at a plane interface between two isotropic elastic media given in (Udias, 1999).

### 3.6 Special case

Now we consider a special case, when the boundary surface is given by $y = a \cos k^*z$, where $a$ is the amplitude of the corrugation. From equations (3.33)-(3.38), we obtain the following formulae of the coefficients $B_1$, $D_1$, $B'_1$ and $D'_1$ for the first order approximation of the corrugation by putting $n = 1$ and $\zeta = \zeta_{-1} = \frac{3}{2}$ as

\[
B_1 = \frac{\Delta B_1}{\Delta_1}, \quad D_1 = \frac{\Delta D_1}{\Delta_1}, \quad B'_1 = \frac{\Delta B'_1}{\Delta'_1}, \quad D'_1 = \frac{\Delta D'_1}{\Delta'_1}
\]

(3.47)

where the values of quantities $\Delta B_1$, $\Delta D_1$, $\Delta B'_1$ and $\Delta D'_1$ and $\Delta_1$, $\Delta'_1$ are given by

\[
\Delta B_1 = \frac{a}{2}[(1+B)[c_{66}k^2(p^2 + \alpha_1^2) - k\alpha_1c_{66}(\frac{\omega \sin \gamma}{\beta_{h_1}} - k^*) - c_{55}k^*\omega \sin \gamma \beta_{h_1}] + d_{56}k\alpha_1(k^* + \frac{\omega \sin \delta}{\beta_{h_2}}) + k^2d_{66}(q_1 - \alpha_2)] + (1 - B)[(\frac{\omega \sin \gamma}{\beta_{h_1}} - k^*)kpc_{56} - 2k^2p_1c_{66} - d_{56}k p(k^* + \frac{\omega \sin \delta}{\beta_{h_2}}) + k^2d_{66}p_1(q_1 - q) - kk^*(2d_{66}(-q + \alpha_2) + d_{55}\frac{\omega \sin \delta}{\beta_{h_2}})],
\]

\[
\Delta D_1 = \frac{a}{2}[(1 + B)[c_{66}k^2(p^2 - p_1\alpha_1) + 2k\alpha_1c_{66}k^* - c_{55}\frac{k^*\omega \sin \gamma}{\beta_{h_1}}] + (1 - B)[-2k^*kpc_{56} + k^2p(p_1 + \alpha_1)c_{66}] + D[d_{66}k^2(-q^2 - \alpha_2^2 + 2q_2\alpha_2) + d_{56}k(-q + \alpha_2)(\frac{\omega \sin \delta}{\beta_{h_2}} - k^*) + \frac{k^*\omega \sin \delta}{\beta_{h_2}} - k^2c_{66}(p_1 + \alpha_1) - c_{66}k(nk^* + \frac{\omega \sin \delta}{\beta_{h_2}})(-q + \alpha_2)],
\]

\[
\Delta B'_1 = \frac{a}{2}[(1 + B)[c_{66}k^2(p^2 + \alpha_1^2) - k\alpha_1c_{66}(\frac{\omega \sin \gamma}{\beta_{h_1}} + k^*) + c_{55}\frac{k^*\omega \sin \gamma}{\beta_{h_1}} + k^2\alpha_1d_{66}(q_1' - \alpha_2)
\]

\[
- d_{56}\alpha_1k(k^* - \frac{\omega \sin \delta}{\beta_{h_2}})(-q + \alpha_2)], + (1 - B)[(\frac{\omega \sin \gamma}{\beta_{h_1}} + k^*)kpc_{56} - 2k^2p_1c_{66} - k^2d_{66}p(q_1' - \alpha_2)
\]

\[
+ d_{66}k p(k^* - \frac{\omega \sin \delta}{\beta_{h_2}})] + D[(d_{66}k^2(q - q_1') + 2k^*d_{66}k)(-q + \alpha_2) - d_{55}\frac{k^*\omega \sin \delta}{\beta_{h_2}}],
\]

\[
\Delta D'_1 = \frac{a}{2}[(1 + B)[c_{66}k^2(p^2 - p_1\alpha_1) - 2k\alpha_1c_{66}k^* + c_{55}\frac{k^*\omega \sin \gamma}{\beta_{h_1}}] + (1 - B)[2k^*kpc_{56} + k^2p(p_1 - \alpha_1)c_{66}] + D[d_{66}k^2(-q^2 - \alpha_2^2 + 2q_2\alpha_2) + d_{56}k(-q + \alpha_2)(\frac{\omega \sin \delta}{\beta_{h_2}} + k^*)]
\]

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\[
-d_{55} \frac{k^* \omega \sin \delta}{\beta_h} + [k^2 c_{66} (p'_1 + \alpha_1) + c_{56} k (k^* - \frac{\omega \sin \delta}{\beta_h})](-q + \alpha_2)],
\]

\[
\Delta_1 = i [d_{66} k(q_1 - \alpha_2) + c_{56} k(p_1 + \alpha_1) + (k^* + \frac{\omega \sin \gamma}{\beta_{h1}})(d_{66} - c_{56})],
\]

\[
\Delta'_1 = i [d_{66} k(q'_1 - \alpha_2) + c_{66} k(p'_1 + \alpha_1) + (\frac{\omega \sin \gamma}{\beta_{h1}}) - k^*](d_{66} - c_{56})].
\]

In case of normal incidence, that is, when \( \gamma = 0^\circ \), these reflection and transmission coefficients reduce to

\[
B_1 = \frac{a}{2 \Delta_1}[(1 + B)[c_{66} k^2 (p^2 + \alpha_1^2) + k \alpha_1 c_{66} k^* + d_{56} k \alpha_1 k^* + k^2 d_{66}( q_1 - \alpha_2)] - (1 - B)\times \left[k^* k p c_{56} + 2 k^2 p \alpha_1 c_{66} + d_{56} k p k^* + k^2 d_{66} p \alpha_1 (q_1 - \alpha_2) + D[d_{66} k^2 (q_1 - q) + 2 k k^* d_{56} (q_1 - \alpha_2)]\right],
\]

\[
D_1 = \frac{a}{2 \Delta_1}[(1 + B)[c_{66} k^2 (p^2 - p_1 \alpha_1) + 2 k \alpha_1 c_{66} k^*] + (1 - B)\left[-2 k^* k p c_{56} + k^2 p (p_1 + \alpha_1) c_{66}\right]
\]

\[
+ D[d_{66} k^2 (2 q_1 q_2 - q^2 - \alpha_2^2) + d_{56} k (q - \alpha_2) k^* + (k^2 c_{66} (p_1 + \alpha_1) - c_{56} k k^*) (-q + \alpha_2)],
\]

\[
B'_1 = \frac{a}{2 \Delta_1}[(1 + B)[c_{66} k^2 (p^2 + \alpha_1^2) - k \alpha_1 c_{66} k^* + k^2 \alpha_1 d_{66}(q'_1 - \alpha_2) - d_{56} \alpha_1 k k^*] + (1 - B)\left[k^* k p c_{56} - 2 k^2 p \alpha_1 c_{66} - k^2 p d_{66}(q'_1 - \alpha_2) + d_{56} k p k^*\right] + D[d_{66} k^2 (q - q_1) + 2 k^* d_{66} k (-q + \alpha_2)],
\]

\[
D'_1 = \frac{a}{2 \Delta_1}[(1 + B)[c_{66} k^2 (p^2 - p'_1 \alpha_1) - 2 k \alpha_1 c_{66} k^* + (1 - B)\left[2 k^* k p c_{56} + k^2 p (p'_1 - \alpha_1) c_{66}\right]
\]

\[
+ D[d_{66} k^2 (2 q_1 q_2 - q^2 - \alpha_2^2) + (d_{56} k k^* + k^2 c_{66} (p'_1 + \alpha_1) + c_{56} k (k^* - \frac{\omega \sin \delta}{\beta_h})) (-q + \alpha_2)],
\]

where now

\[
\Delta_1 = i [d_{66} k(q_1 - \alpha_2) + c_{66} k(p_1 + \alpha_1) + k^*(d_{66} - c_{56})]
\]

\[
\Delta'_1 = i [d_{66} k(q'_1 - \alpha_2) + c_{66} k(p'_1 + \alpha_1) - k^*(d_{66} - c_{56})].
\]

Further, it can be seen that if the corrugation is absent i.e. when \( a = 0 \), the coefficients \( B_1, D_1, B'_1 \) and \( D'_1 \) vanish.

### 3.6.1 Particular cases

(a) If both the half-spaces are orthotropic elastic, we shall take \( c_{56} = d_{56} = 0 \). With the help of these values, the reflection and transmission coefficients given in (3.26) at the plane interface between two dissimilar orthotropic elastic half-spaces reduce to

\[
B = \frac{c_{66} \sqrt{c_{55} - \frac{c^2}{\beta_{h1}^2}} - d_{66} \sqrt{d_{55} - \frac{c^2}{\beta_{h2}^2}}}{c_{66} \sqrt{c_{55} - \frac{c^2}{\beta_{h1}^2}} + d_{66} \sqrt{d_{55} - \frac{c^2}{\beta_{h2}^2}}},
\]

\[ (3.48) \]

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The expressions of the coefficients $B_1$, $D_1$, $B'_1$ and $D'_1$ given by (3.47), in this case become

\[ B_1 = \frac{a}{2\Delta_1} \left[ ((1+B)k^2c_{66}p^2-c_{55}k^\omega \sin \gamma) - (1-B)k^2pd_{66}q_1 + D(d_{66}k^2q(q_1-q) + d_{55}k^\omega \sin \delta) \right], \]

\[ D_1 = \frac{a}{2\Delta_1} \left[ ((1+B)k^2c_{66}p^2-c_{55}k^\omega \sin \gamma) + (1-B)k^2pc_{66}p_1 - D[k^2q(d_{66}q+c_{66}p_1) - d_{55}k^\omega \sin \delta] \right], \]

\[ B'_1 = \frac{a}{2\Delta'_1} \left[ ((1+B)k^2c_{66}p^2+c_{55}k^\omega \sin \gamma) - (1-B)k^2pd_{66}q_1 + D[d_{66}k^2q(q_1-q) - d_{55}k^\omega \sin \delta] \right], \]

\[ D'_1 = \frac{a}{2\Delta'_1} \left[ ((1+B)k^2c_{66}p^2+c_{55}k^\omega \sin \gamma) + (1-B)k^2pc_{66}p_1 - D[k^2q(d_{66}q+c_{66}p_1) + d_{55}k^\omega \sin \delta] \right], \]

where now

\[ \Delta_1 = \pm k[d_{66}q_1 + c_{66}p_1], \quad \Delta'_1 = \pm k[d_{66}q_1' + c_{66}p_1']. \]

(b) If both the half-spaces are transversely isotropic, we shall take $c_{55} = N_1$, $c_{66} = M_1$, $c_{56} = d_{56} = 0$, $d_{55} = N_2$, $d_{66} = M_2$. With the help of these values, the expressions for the quantities $p$, $q$, $p_n$, $q_n$, $p'_n$ and $q'_n$ given in equations (3.12), (3.15), (3.17) and (3.19) reduce to

\[ p = \frac{\omega}{\beta_1} \left[ \frac{N_1}{M_1} \cos \gamma \right], \quad q = \frac{\omega}{\beta_1} \left[ \frac{N_2}{M_2} \left( \frac{\beta_1^2}{\beta_2^2} - \sin^2 \gamma \right) \right], \]

\[ p_n = \frac{\omega}{\beta_1} \left[ \frac{N_1}{M_1} \cos \gamma_n \right], \quad q_n = \frac{\omega}{\beta_1} \left[ \frac{N_2}{M_2} \left( \frac{\beta_1^2}{\beta_2^2} - \sin^2 \gamma_n \right) \right], \]

\[ p'_n = \frac{\omega}{\beta_1} \left[ \frac{N_1}{M_1} \cos \gamma'_n \right], \quad q'_n = \frac{\omega}{\beta_1} \left[ \frac{N_2}{M_2} \left( \frac{\beta_1^2}{\beta_2^2} - \sin^2 \gamma'_n \right) \right], \]

With these modified values, we shall have the reflection and transmission coefficients for the transversely isotropic media for the first order approximation of the corrugation.

(c) For the isotropic elastic media, we shall have $c_{55} = c_{66} = \mu_1$, $c_{56} = d_{56} = 0$, $d_{55} = d_{66} = \mu_2$. With the help of these values, the reflection and transmission coefficients given in (3.27) at the plane interface between two elastic half-spaces become

\[ B = \frac{\mu_1}{\mu_1} \sqrt{1 - \frac{c^2}{\beta_1^2}} - \mu_2 \sqrt{1 - \frac{c^2}{\beta_2^2}}, \quad D = \frac{\mu_1}{\mu_1} \sqrt{1 - \frac{c^2}{\beta_1^2}} + \mu_2 \sqrt{1 - \frac{c^2}{\beta_2^2}}. \]
These results match with the well known results given in (Udias, 1999) for the relevant problem. The expressions of the coefficients $B_i$, $D_i$, $B'_i$ and $D'_i$, in this case, reduce to

\[ B_1 = \frac{a}{2\Delta_1} \mu_1 [(1 + B)(k^2 p^2 - k^* \omega \frac{\sin \gamma}{\beta_h}) - (1 - B)k^2 p q_1 \frac{\mu_2}{\mu_1} + D \frac{\mu_2}{\mu_1} (k^2 (q q_1 - q^2) + k^* \omega \frac{\sin \delta}{\beta_h})], \]

\[ D_1 = \frac{c}{2\Delta_1} \mu_1 [(1 + B)(k^2 p^2 - k^* \omega \frac{\sin \gamma}{\beta_h}) + (1 - B)k^2 p q_1 \frac{\mu_2}{\mu_1} - D \frac{\mu_2}{\mu_1} (k^2 (q q_1 + q) + k^* \omega \frac{\sin \delta}{\beta_h})], \]

\[ B'_1 = \frac{c}{2\Delta'_1} \mu_1 [(1 + B)(k^2 p^2 - k^* \omega \frac{\sin \gamma}{\beta_h}) + (1 - B)k p q_1 \frac{\mu_2}{\mu_1} - D \frac{\mu_2}{\mu_1} (k^2 q (q q_1 + q) + k^* \omega \frac{\sin \delta}{\beta_h})], \]

\[ D'_1 = \frac{c}{2\Delta'_1} \mu_1 [(1 + B)(k^2 p^2 - k^* \omega \frac{\sin \gamma}{\beta_h}) - (1 - B)k^2 p q_1 \frac{\mu_2}{\mu_1} + D \frac{\mu_2}{\mu_1} (k^2 (q q_1 - q^2) + k^* \omega \frac{\sin \delta}{\beta_h})], \]

\[ \Delta_1 = \eta k (p_1 + \frac{\mu_2}{\mu_1} q_1) \mu_1, \quad \Delta'_1 = \eta k (p'_1 + \frac{\mu_2}{\mu_1} q'_1) \mu_1. \]

These formulae give the refraction and reflection coefficients for the first order approximation of the corrugated interface between two uniform elastic half spaces. In this case, the expressions for $p, q, p_n, q_n, p'_n$ and $q'_n$ given in equations (3.12), (3.15), (3.17) and (3.19) using the result \( \frac{\omega}{\beta_h} = \sqrt{\sigma_1 h_1} \) reduce to

\[ p = \frac{\omega}{\beta_h} \cos \gamma = \sqrt{\sigma_1 h_1} \cos \gamma, \quad q = \frac{\omega}{\beta_h} \sqrt{\frac{\beta_{h_1}^2}{\beta_{h_2}^2} - \sin^2 \gamma} = \frac{\omega}{\beta_h} \cos \delta = \sqrt{\sigma_2 h_2} \cos \delta, \]

\[ p_n = \frac{\omega}{\beta_h} \cos \gamma_n = \sqrt{\sigma_1 h_1} \cos \gamma_n, \quad q_n = \frac{\omega}{\beta_h} \sqrt{\frac{\beta_{h_1}^2}{\beta_{h_2}^2} - \sin^2 \gamma_n} = \frac{\omega}{\beta_h} \cos \delta_n = \sqrt{\sigma_2 h_2} \cos \delta_n, \]

\[ p'_n = \frac{\omega}{\beta_h} \cos \gamma'_n = \sqrt{\sigma_1 h_1} \cos \gamma'_n, \quad q'_n = \frac{\omega}{\beta_h} \sqrt{\frac{\beta_{h_1}^2}{\beta_{h_2}^2} - \sin^2 \gamma'_n} = \sqrt{\sigma_2 h_2} \cos \delta'_n. \]

With the help of these substitutions, the boundary conditions (3.24), (3.25), (3.28) and (3.29) match with those of Asano (1960) for the relevant problem.

### 3.7 Numerical results and discussion

In order to study the effect of corrugation of the interface, frequency of the incident wave and the monoclinic behavior of the media on the reflection and the transmission coefficients numerically, we take the values of relevant parameters as

\[ c_{55} = 0.60 \times 10^{10} \, N/m^2, \quad c_{66} = 0.75 \times 10^{10} \, N/m^2, \quad c_{56} = 0.09 \times 10^{10} \, N/m^2, \]

\[ \rho_1 = 4700 \, kg/m^3, \quad d_{55} = 0.94 \times 10^{10} \, N/m^2, \quad d_{66} = 0.93 \times 10^{10} \, N/m^2, \]

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\( \rho_2 = 7450 \text{ kg/m}^3, \quad d_{56} = -0.11 \times 10^{10} \text{ N/m}^2, \quad \omega a / \beta_{h_1} = 0.5, \) and \( k^* a = 0.00125, \) wherever not mentioned.

To study the effect of frequency on the reflection and transmission coefficients for both plane and corrugated interfaces, we have taken different values of the frequency parameter \( \omega a / \beta_{h_1} = 0.1, 0.5, 1.0 \) and 1.5. Figure 3.2 shows the variation of the reflection and transmission coefficients with respect to the angle of incidence \( \gamma \) at the plane interface between the half-spaces \( H_1 \) and \( H_2. \) We observe from Figure 3.2 that the reflection and transmission coefficients are not affected at all angles of incidence for different values of frequency parameter. However, the transmission coefficient is found to be greater than the reflection coefficient. Figures 3.3-3.6 show that the coefficients \( B_1, D_1, B'_1 \) and \( D'_1 \) are influenced by the frequency parameter \( \omega a / \beta_{h_1}. \) Each of these coefficients increases with the increase of the value of \( \omega a / \beta_{h_1}. \) We notice from these

![Amplitude Ratios (B, D)](image)

Fig. 3.2 Variation of Coefficients B and D with angle of incidence (\( \gamma \)), \( F = \frac{\omega a}{\beta_{h_1}}. \)
figures that the effect of frequency on the reflection coefficients $B_1$ and $B_1'$ is maximum near $\gamma = 0^\circ$ and minimum at $\gamma = 90^\circ$, whereas the reverse behavior is observed in the case of transmission coefficients $D_1$ and $D_1'$. However, these coefficients are found to be continuous functions of $\gamma$ at every value of frequency chosen. Figures 3.7 and 3.8 show the variation of reflection and transmission coefficients with respect to the frequency at fixed angle of incidence $\gamma = 45^\circ$ for the plane and for the corrugated interface respectively. It is very clear from Figure 3.7 that the reflection and trans-
mission coefficients are not affected by the frequency of the incident wave at the plane interface. However, the values of the coefficient $D$ are greater than that of $B$ for all values of the frequency. Figure 3.8 shows that these coefficients increase rapidly with the increase of the frequency parameter.

As the coefficients $B$ and $D$ represent the reflection and transmission coefficients at the plane interface, it is obvious that they should not depend on the corrugation of the
interface. This is also clear from our analytical results given in (3.26). To study the effect of corrugation parameter $k^*a$ on the coefficients $B_1$, $D_1$, $B'_1$ and $D'_1$, we have computed them for very small values of $k^*a$. Figures 3.9-3.12 show the variation of the coefficients $B_1$, $D_1$, $B'_1$ and $D'_1$ versus $k^*a$ when $\omega a/\beta_{h_1}$ takes the values as 0.5, 1.0 and 1.5; and $\gamma = 45^0$. Here, we notice that these coefficients are strongly affected by the corrugation parameter $k^*a$. The values of the coefficients $B_1$ and $D_1$ increase monotonically with increase of $k^*a$, whereas the values of the coefficients $B'_1$ and $D'_1$ decrease monotonically with increase of $k^*a$.

To discuss the effect of monoclinic and isotropic properties simultaneously on the reflection and transmission coefficients, we consider the upper medium $H_1$ as monoclinic elastic and the lower medium $H_2$ as isotropic elastic. For this purpose, we take the values of the relevant elastic parameters as

\[
\begin{align*}
c_{55} &= 0.60 \times 10^{10} \text{ N/m}^2, \\
c_{66} &= 0.75 \times 10^{10} \text{ N/m}^2, \\
c_{56} &= 0.09 \times 10^{10} \text{ N/m}^2, \\
\rho_1 &= 4700 \text{ kg/m}^3, \\
d_{55} &= d_{66} = 0.211 \times 10^{10} \text{ N/m}^2, \\
\rho_2 &= 7400 \text{ kg/m}^3.
\end{align*}
\]
Figures 3.13-3.15 depict the variations of the reflection and transmission coefficients with respect to the angle of incidence at plane and at corrugated interface between the medium $H_1$ and $H_2$. Here, we notice that the values of the coefficients $B$, $B_1$ and $B_1'$ decrease in the range $0^\circ < \gamma \leq 62^\circ$ and in the rest of the range $62^\circ < \gamma \leq 90^\circ$, the values of these coefficients increase. The refraction coefficient $D$ is most affected as
compared to $D_1$ and $D'_1$ at $\gamma = 0^0$. In this case, the values of the refraction coefficients are found to be greater than that of the reflection coefficients at all angles of incidence. Figures 3.16 and 3.17 depict the variations of the reflection and transmission coefficients $B$, $D$, $B_1$ and $D_1$ with respect to the corrugation parameter $k^*a$ at $\gamma = 45^0$. We notice from Figure 3.16 that the reflection coefficient $B_1$ decreases in the range of $0^0$.
corrugation parameter $0.0 < k^*a < 0.05$ and thereafter, it increases, whereas the values of the coefficient $D_i$ increases slowly with corrugation parameter. Figure 3.17 shows the variation of $B'_i$ and $D'_i$ versus corrugation parameter $k^*a$ with the same values of parameter as taken for Figure 3.16. Here, we notice that the reflection coefficient $B'_i$ increases significantly while the transmission coefficient $D'_i$ decreases slowly with the

\[
\frac{\omega}{\beta} = \frac{\beta}{\beta_n} \text{ at } \gamma = 45^\circ \text{ when the upper medium is monoclinic and the lower medium is isotropic.}
\]

\[
\text{Fig. 3.19 Variation of the Reflection and the Refraction Coefficient s } D_i, B_i, D'_i \text{ with frequency } F = \frac{\omega}{\beta} \text{ at } \gamma = 45^\circ \text{ and } K_{Ph Pi},
\]

\[
\text{Fig. 3.20 Variation of the Reflection Coefficient } B \text{ with frequency } F = \frac{\omega}{\beta} \text{ at } \gamma = 45^\circ \text{ and } R = \frac{\beta_n}{\beta}.
\]
corrugation parameter. We also observe from these figures that the value of \( D_1 \) is more than the value of \( D'_1 \) as \( k*a \) takes the values from 0.0 to 0.10.

Figures 3.18 and 3.19 show the variations of reflection and transmission coefficients \( B, D, B_1, D_1, B'_1 \) and \( D'_1 \) with respect to the frequency parameter when the wave is incident at \( \gamma = 45^\circ \). From Figure 3.18, it is clearly noticed that the reflection and
transmission coefficients at the plane interface are not affected by the frequency parameter. Also, the values of the coefficients $B_1$, $D_1$, $B'_1$ and $D'_1$ increase rapidly with the frequency parameter.

To study the effect of velocity contrast on the reflection and transmission coefficients for both plane and corrugated interfaces, we have taken different values of the velocity ratio $\beta_{h_1}/\beta_{h_2}$ namely 1.0, 1.5, 2.0 and 2.5. Figures 3.20-3.25 show the variations of

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.23}
\caption{Variation of the Refraction Coefficient $D_i$ with frequency $F = \frac{\omega a_h}{\beta_h}$ at $\gamma = 45^\circ$ and $R = \frac{\beta_{h_i}}{\beta_{h_i}}$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.24}
\caption{Variation of the Refraction Coefficient $B'_i$ with frequency $F = \frac{\omega a_h}{\beta_h}$ at $\gamma = 45^\circ$ and $R = \frac{\beta_{h_i}}{\beta_{h_i}}$.}
\end{figure}
the reflection and transmission coefficients $B, D, B_1, D_1, B'_1$ and $D'_1$ with respect to the frequency parameter $F(= \omega a/\beta_{h_1})$ at $\gamma = 45^\circ$. We observe from the Figures 3.20 and 3.21 that the value of the reflection coefficient $B$ increases, while the value of transmission coefficient $D$ decreases with velocity ratio $(\beta_{h_1}/\beta_{h_2})$, however, they are constant for a given value of velocity ratio.

From Figures 3.22-3.25, we note that the variations of the reflection and transmission coefficients $B_1, D_1, B'_1$ and $D'_1$ increase with the increase of the frequency parameter and velocity ratio. This, in general, as the velocity ratio takes higher and higher values, the values of these coefficients increase with increase of the frequency parameter.

![Diagram of Amplitude Ratio vs Frequency](image)

**Fig. 3.25** Variation of the Refraction Coefficient $D'_1$ with frequency $F(= \omega a/\beta_{h_1})$ at $\gamma = 45^\circ$ and $R = \frac{\beta_{h_1}}{\beta_{h_2}}$. 

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