Chapter 1

Introduction

There are mainly two types of bodies in nature: rigid bodies and deformable bodies. A body is said to be rigid if the relative position of its particles remains unchanged when an external force is applied on it, otherwise, it is said to be deformable. In nature, all the bodies get deformed in shape or in size or in both. So, there is no natural body which is absolutely rigid. Further, deformable bodies are classified into two categories - elastic bodies and non-elastic bodies. If a body is enable to recover its original shape and size after the removal of applied external forces, the body is said to be elastic, otherwise, it is said to be non-elastic. The property of a body to be elastic depends on the magnitude of the applied external force. If the applied forces are sufficiently large, even elastic bodies do not recover instantaneously to its original shape and size when the applied forces are removed. The changes in the shape and size of a body are described by strain. The complete state of strain at a point can be described by the components of the strain tensor ($e_{ij}$). When a body is strained, internal forces get develop in it to resist the strained state. These internal forces measured per unit area, are called stresses. The complete state of stress at a point can be described by the components of the stress tensor ($\tau_{ij}$).

The study of mathematical theory of deformation of a continuum is in the province of the subject of mechanics. A body is said to be continuum, if it obeys the continuum hypothesis. In the continuum hypothesis, the term-particle is used to mean a geometrical point without dimensions. Density and mechanical properties are considered as continuous functions of the spatial co-ordinates and time, whereas the atomic and molecular nature of the particles are not considered. The study of the mechanical behavior of a continuum can be traced back to the discovery of Hooke in 1678, known
as Hooke’s law. The generalized form of Hooke’s law is stated as: within elastic limits, stress is proportional to strain.

The bodies are further categorized into homogeneous and inhomogeneous, isotropic and anisotropic. A body is said to be homogeneous, if its elastic properties are independent of geometric positions of the points, otherwise, it is said to be inhomogeneous (non-homogeneous). A body, in which the elastic properties at a point are independent of directions, it is said to be isotropic body, otherwise, it is called an anisotropic body. Directions for which the elastic properties are the same are said to be elastically equivalent. For an isotropic body, all the directions drawn through a given point are elastically equivalent, while for an anisotropic body, not all, but only some are elastically equivalent. Depending on the structure, a body may be isotropic or anisotropic and at the same time homogeneous and non-homogeneous.

The mathematical formulations of various problems related to elastic and non-elastic deformations are based on fundamental equations and constitutive equations. The fundamental equations are based upon the universal laws of physics such as law of conservation of mass, linear momentum, moment of momentum and energy. The constitutive equations are those which characterize the behavior of specific idealized materials based upon their internal constitutions. Mathematically, the usefulness of these constitutive equations is to describe the relationship between the kinematic, mechanical and thermal field equations and to permit the formulations of well-posed problems in continuum mechanics.

In brief, the general forms of the fundamental equations and constitutive equations are as:

(i) The momentum equation or equation of motion for an elastic continuum

\[ \partial_j \tau_{ij} + f_i = \rho \partial_t u_i, \quad (i, j = x, y, z) \]  

(1.1)

where

- \( \tau_{ij} \) - the components of stress tensor,
- \( f_i \) - the components of body force (i.e. the force which is proportional to the volume of the body, e.g. gravitational force),
- \( u_i \) - the components of displacement vector \( \mathbf{u} \),
- \( \partial_j \) - partial derivative with respect to the spatial co-ordinates \( x_j \),
- \( \partial_t \) - double time derivative,
(ii) The stress-strain relation for a linear elastic solid, (called generalized Hooke's law), assuming infinitesimal strains, is given by (Sokolnikoff, 1956)

\[ \tau_{ij} = C_{ijkl}e_{kl}, \tag{1.2} \]

where \( C_{ijkl} \) is a fourth order tensor representing the elastic properties of the body and \( e_{kl} = \frac{1}{2} (\partial_k u_l + \partial_l u_k) \) represents the components of strain tensor.

For an isotropic elastic medium, all the components of tensor \( C_{ijkl} \) can be expressed only by two constants \( \lambda \) and \( \mu \), called Lame's constants, as

\[ C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{lj} + \delta_{lj} \delta_{ik}), \tag{1.3} \]

where the symbol \( \delta_{mn} \) is a well known Kronecker delta.

Inserting (1.3) into (1.2), the generalized Hooke's law for an isotropic elastic medium can be written in the form

\[ \tau_{ij} = \lambda \delta_{ij} \Theta + \mu (\partial_i u_j + \partial_j u_i), \tag{1.4} \]

where \( \Theta = \partial_k u_k \) denotes the change in volume per unit volume and is known as cubical dilatation.

The subject of elasticity is an important branch of solid mechanics and has wide applications in various fields such as geophysics (interpretation of seismic data using elastic wave analysis), material sciences (modelling the mechanical properties of solids such as rubber), non-destructive evaluation of the integrated materials using elastic waves, engineering structural mechanics, biomechanics etc.

1.1 Elastic waves in an isotropic elastic medium

Wave is a disturbance that travels through a medium without giving the medium as a whole, any permanent displacement. When a wave propagates, the energy is transferred progressively from one place to another in the medium. The disturbance, which causes the elastic deformation in the medium, takes the form of a wave known as elastic wave. Inserting (1.4) into (1.1) and neglecting the body forces, the vector form of the equation of small motion in a uniform elastic body can be written as (Love, 1944);

\[ \rho \ddot{u} = (\lambda + 2\mu)\nabla (\nabla \cdot u) - \mu \nabla \times (\nabla \times u), \tag{1.5} \]
(i) For equivoluminal displacement, i.e., when \( \nabla \cdot \mathbf{u} = \Theta = 0 \) and using the vector identity,
\[
\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u},
\]
equation (1.5) reduces to
\[
\nabla^2 \mathbf{u} = \frac{1}{\beta^2} \mathbf{u}, \quad \beta = \sqrt{\frac{\mu}{\rho}}.
\]
This is a standard wave equation in which \( \beta \) defines the velocity with which the wave propagates. These waves are called *distortional waves*. Here, the shear modulus \( \mu \) characterizes the distortion and rotation of the volume element.

(ii) In the case, when motion is irrotational, i.e., when \( \nabla \times \mathbf{u} = 0 \) and using the above vector identity, the general equation (1.5) becomes
\[
\nabla^2 \mathbf{u} = \frac{1}{\alpha^2} \mathbf{u}, \quad \alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}.
\]
This is again a standard wave equation in which \( \alpha \) is the velocity with which the wave propagates in the medium. These are called *dilatational or irrotational waves*.

The solutions of the equations (1.6) and (1.7) represent two types of waves that propagate in the medium with velocities \( \alpha \) and \( \beta \). These velocities are functions of the elastic coefficients \( \lambda, \mu \) and density \( \rho \). In the derivation of equation (1.6), we have considered \( \nabla \times \mathbf{u} \neq 0 \) and \( \nabla \cdot \mathbf{u} = 0 \). This means that the medium changes in shape but not in volume. The particles do not move in the direction of propagation, but they simply oscillate up and down about their individual equilibrium position as the wave advances. These waves are known as a *shear or equivoluminal or distortional or transverse or rotational wave*. Further, in the derivation of equation (1.7), we have considered \( \nabla \cdot \mathbf{u} \neq 0 \) and \( \nabla \times \mathbf{u} = 0 \). This means that there is a change in the volume element. As these waves propagate, the elastic material expands and contracts keeping the same form i.e. the particle displacement is parallel to the direction of wave propagation. Therefore, solution of equation (1.7) corresponds to the waves known as *compressional or irrotational or dilatational or longitudinal waves*. Sound waves are classical example of longitudinal waves. In an infinite homogeneous isotropic elastic medium, there exists only these two types of waves. These waves are called body waves (Bullen and Bolt, 1985). The existence of these waves was first shown theoretically by Poisson (1829). When earth is regarded as a perfectly elastic body, the elastic
waves travelling through the earth medium are called Seismic waves. The longitudinal wave is called $P$-wave and the transverse wave is called $S$-wave in seismology. The $S$-waves, being transverse in nature, can be polarized in two planes: horizontal plane and vertical plane. If polarization takes place in horizontal plane, these waves are called $SH$-waves and if polarization takes place in vertical plane, these are called $SV$-waves. When earth is regarded as a layered half-space, it is possible for the waves to propagate parallel to the plane boundary. These types of waves are called surface waves. These waves are restricted in the neighborhood of the free surface or in exceptional circumstances at an internal interface. The surface waves are further categorized into two types: (i) Rayleigh type surface waves, which propagate along the plane free surface of a semi-infinite solid with velocity less than that of both $P$ and $SV$-waves, involving both longitudinal and transverse motions. (ii) Love type surface waves, which are horizontally polarized shear waves propagating in a superficial layer of a material overlying a half-space and can exist only when shear wave velocity in the layer is less than that of in the half-space.

Stoneley (1924) investigated the other type of surface waves propagating along the plane interface between two distinct uniform elastic solid half-spaces in perfect contact, known as Stoneley waves. These waves are harmonic waves, produce continuous traction and displacement across the interface and attenuate exponentially with distance normal to the interface in both the half-spaces, provided the range of the elastic constants of the two solids lie within some suitable limits (Scholte, 1947).

1.2 Importance of elastic waves

Elastic wave propagation is an elegant and fascinating subject that deals with numerous problems in various fields such as engineering, telecommunication (signal processing), medicines (echography) and metallurgy (non-destructive testing). These waves are useful in detection of notches and faults in different types of materials such as detection of faults in railway lines, detection of buried land-mines etc. An additional application of elastic waves lies in the field of geophysics; particularly in seismology. The propagation of elastic waves and their reflection and refraction from various types of surfaces/interfaces are of great importance in the field of seismology, earthquake engineering and also in signal processing. The techniques of seismic wave propagation is a tool for
investigating the internal structure of the earth as well as for exploration of valuable materials such as oils, water, chemical, etc. beneath the earth surface. The nature of the earth materials plays an important role in changing the characteristics of the seismic signals recorded on the surface of the earth. Therefore, these seismic signals carry a lot of information with them about the internal composition of the earth’s interior and its dynamic characteristics can be carried out up to a large extent. The elastic surface waves are very helpful for the study of crustal and upper mantle structures. The knowledge about the upper structure of the earth helps the seismologists in studying about the nature of the possible sources of the earthquakes and which, in turn, further help in studying the problems of the prediction of earthquakes.

In the theoretical study of seismic waves, the earth is approximated by various models. These models are mathematical frameworks within which observed seismograms are related to the earth’s interior via model parameters. To achieve a better study of Earth’s materials, especially for deep interior, the wave propagation studies in realistic Earth’s models become more important. Most of the literature on the studies of elastic waves that deals with an idealized earth model and boundary conditions is widely available including the books by Ewing et al. (1957), Brekhovskikh (1960), Achenbach (1973), Ben Menahem and Singh (1981), Bullen and Bolt (1985), Aki and Richards (1988), Sheriff and Geldart (1995), Lay and Wallace (1995), Shearer(1999), Udias (1999), Pujol (2003), Chapman (2004).

1.2.1 Waves in an anisotropic medium

When the elastic properties at a point of an elastic medium, vary with the direction, the medium is said to be anisotropic medium. Such types of media exist in nature in the form of rocks and crystals. Elastically, they are fully described by the fourth order tensor $C_{ijkl}$, which has 81 components. Owing to the symmetry of stress tensor $\tau_{ij}$ and strain tensor $e_{kl}$, these components reduce to 36 in number. These elastic constants are further reduced to 21, to ascertain the existence of strain-energy density function (Sokolnikoff, 1956). If a medium has one plane of elastic symmetry, there are only 13 elastic constants. A medium in which the elastic system is represented by these 13 constants is known as anisotropic medium with monoclinic symmetry or monoclinic medium. When there are two orthogonal planes of symmetry, then the number of
non-zero independent elastic constants reduces to 9. Such a medium is said to be orthotropic medium. If a medium possesses symmetry about an axis, the medium is called transversely isotropic medium. In this type of medium, there are only five independent elastic constants. The body, in which the elastic behavior is independent of direction, is called isotropic body. For isotropic bodies, the independent elastic constants reduces to only 2. A detailed derivation and the reduction of number of elastic constants in elastic medium possessing various symmetries are given in the book by Love (1944).

The interior of the earth is made up of solids, liquids and gases. The solid part is in the form of rocks, minerals and crystals with definite geometrical outlines. The stratifications or horizontal alignments of layers structure and the preferential alignments of crystals or rocks generates anisotropy within the earth. Different types of anisotropic forms are possible. Monoclinic form is one of them. It is found that almost one-third of all the materials possesses monoclinic symmetry (Mason, 1964). The monoclinic system corresponds to those crystals which can be referred to three unequal axes, two of which intersect at an oblique angle, while the third is perpendicular to the other two. Rotated $y-$ cut quartz, which exhibits monoclinic symmetry, is formed by rotation of thickness direction around the $x-$ axis. These $y-$ cut crystals are very useful to generate shear vibration in the solids. Seismologists recognized that the seismic waves propagated in sedimentary or crystalline rocks do not always behave as if the medium were isotropic. Seismic wave velocities are usually greater in crystalline rocks than is sedimentary rocks. This property of velocity anisotropy is characterized by the variations of velocity with the direction of propagation. Thus, the study of wave propagation in anisotropic media helps the seismologists to obtain detailed knowledge about the rock structure as well as the elastic properties and at the same time the information regarding minerals present inside the earth.

Stoneley (1949) was one of the earliest investigators to discuss the seismological implications of anisotropy in the Earth. Observational and theoretical study of the propagation of the seismic waves have revealed the existence of anisotropy in the medium of Earth’s interior by many investigators including Uhrig and Van-Melle (1955), Synge (1957), Harkrider and Anderson (1962), Anderson and Harkrider (1962) and Vanderstoep (1966). Stoneley (1955) studied the propagation of surface waves in a cubic crystalline horizontally stratified medium and discussed the possibility of propagation of elastic waves, analogous to Rayleigh waves and Love waves, over a surface of cubical
crystal. In his model, he considered the symmetrical cases in which the direction of propagation is parallel to the \( x \)-axis or makes an angle of 45° with it and showed that the waves with exponentially falling amplitude with distance from the free face, \textit{i.e.}, Rayleigh type surface waves, do not exist for every set of values of the three elastic constants describing anisotropic medium chosen at random. For crystals of aluminium and copper, these Rayleigh-type waves, do not exist, but for salt rocks, their existence is demonstrated both for the symmetrical cases and for an asymmetrical cases. Stoneley (1961) treated the propagation of Rayleigh surface waves and Love waves in two types of crystalline media, namely, transversely isotropic media and crystals of cubic symmetry. He discussed that how the method of treatment can be extended to other crystal classes and to the general problem of wave propagation on the surface of a general anisotropic solid.

Musgrave (1961) studied the problem of reflection and refraction of plane waves propagating from a point disturbance at a plane boundary. He discussed the characterization of these waves in terms of slowness surfaces and wave surfaces, and established the connections between the two types of surfaces. Anderson (1961) studied the problem of surface wave propagation in layered anisotropic media. He discussed the effect of anisotropy on Rayleigh and Stoneley waves. Buchwald (1961) studied Rayleigh waves in anisotropic media and has shown that, in general, Rayleigh waves in semi-infinite anisotropic media are either (i) dissipative, or (ii) not propagated at all. It is possible that non-dissipative Rayleigh waves can travel in certain discrete directions. Dutta (1965) studied Rayleigh wave propagation in a two-layered anisotropic media and discussed the possibility of propagation of Rayleigh waves in an incompressible crust when the density remains constant and rigidity varies exponentially with depth, lying on a semi-infinite transversely isotropic medium. Kolodner (1966) studied the problem of existence of longitudinal wave in a linear, homogeneous anisotropic media and established that there exist at least three distinct directions along which longitudinal waves can propagate. Crampin and Taylor (1971) investigated the problem of surface wave propagation in an anisotropic media by permitting any combination of plane layers possessing any anisotropic symmetry. A problem of reflection and refraction of plane waves at a plane boundary between anisotropic media was studied by Henneke (1972). He discussed the effect of anisotropy on the reflection and refraction coefficients.

The propagation of Love type \( SH \)-waves in a monoclinic crystalline internal stratum
of finite thickness lying between two semi-infinite non-isotropic elastic layers, has been 
investigated by De (1975). In the case of crystalline media, he derived the dispersion 
relation and compared the obtained results with an isotropic and transversely isotropic 
cases. Daley and Hron (1977) obtained the reflection and refraction coefficients when 
estatic waves impinging at an interface between two different transversely isotropic half­ 
spaces and at an interface between a transversely isotropic layer and vacuum. Crampin 
(1977) reviewed various studies related to the observation of wave motion in anisotropic 
and cracked elastic media. Fryer and Frazer (1987) studied seismic waves in a stratified 
anisotropic media and derived analytical expressions of eigenvalues and eigenvectors of 
estatic systems with a horizontal plane of symmetry. Neyfeh (1991) derived the ana­ 
lytical expressions for the reflection and transmission coefficients when a acoustic wave 
is incident at the interface of liquid and solid-anisotropic half-spaces possessing up to 
as low as monoclinic symmetry and the expressions for the distributions of stresses 
and displacements throughout the fluid-solid system have been derived. He has also 
deduced the variations of phase velocity and beam shifting parameters with azimuthal 
angles.

Chattopadhyay and Choudhury (1995) studied a problem of reflection of $P$ and $SV$­ 
waves from a plane free boundary of a monoclinic type solid half-space and computed 
Rayleigh wave velocity and reflection coefficients. They observed that Rayleigh wave 
velocity is significantly affected due to the anisotropy of the medium and the reflection 
coefficients due to an incident $P$ and $SV$-waves is affected significantly when the angle 
of incidence is 60° and 88°. Singh (1999) commented that in a monoclinic solid medium, 
$P$-waves are purely longitudinal and $SV$-waves are purely transverse only in a specific 
direction and, in general, the angle of incidence is not equal to the angle of reflection. 
Due to this difference in the nature of $SV$-waves in monoclinic medium, $SV$-waves 
in monoclinic medium are said to be quasi $SV$-waves or $qSV$-waves. Similarly, in a 
monoclinic medium, the $P$-type of waves are said to be quasi $P$-waves or $qP$-waves. 
Chattopadhyaya etal. (1996) discussed the reflection and transmission of shear waves 
in monoclinic media and obtained dispersion equations for a monoclinic layer lying 
over a monoclinic half-space. The $SH$- type of surface waves in layered anisotropic 
half-space is also discussed by Romeo (1997) using wave-splitting approach. He de­ 
derived the dispersion equation in terms of the Laplace transform of a function, which 
characterizes the inhomogeneity.
Chattopadhyay and Saha (1999) attempted the problem of reflection and refraction of quasi-SV or qSV - waves at an interface between two different monoclinic half-spaces. Singh and Khurana (2002) investigated the problem of reflection of P- and SV-waves in an anisotropic elastic medium possessing monoclinic symmetry. Sengupta and Nath (2001) studied the propagation of surface waves in anisotropic fibre-reinforced elastic media and found that the Rayleigh type of wave velocity in anisotropic fibre-reinforced elastic medium increases to a considerable amount in comparison with the Rayleigh wave velocity in isotropic materials. Singh (2002) commented their paper and pointed out an error in the basic calculations made by Sengupta and Nath (2001) for the surface waves in the fibre-reinforced anisotropic elastic media. Rychlewski (2001) discussed a problem of elastic wave propagation in a general anisotropic medium whose material properties are characterized by the fourth order tensor $C_{ijkl}$. He considered the propagation of homogeneous time-harmonic plane waves. It is shown that for such materials the longitudinal wave speed does not depend on the arbitrary propagation direction. Hence, he deduced that in general anisotropic materials, longitudinal wave propagates with the same manner as in isotropic materials. Caviglia and Morro (2002) analyzed the interaction between a plane wave and a planarly stratified, inhomogeneous, dissipative and anisotropic medium. The excitation occurs at oblique incidence and a Riccati equation is established as the law that connects the reflection characteristics to the thickness of the layer. Caviglia and Morro (2003) studied a problem of reflection and transmission of transient waves in anisotropic elastic multilayers sandwiched between two homogeneous half-spaces. The half-spaces and the multilayer consist of anisotropic elastic materials. The uniqueness of the solution in the whole space domain is proved by means of an energy method and a closed-form solution is determined, for a homogeneous layer, where the reflected and transmitted displacement is given in terms of the incident displacement. Kuznetsov (2004) investigated the existence of Love type surface waves in stratified monoclinic media. They developed a modified transfer matrix method and obtained closed form dispersion relations for media consisting of one or two orthotropic layers lying on an orthotropic substrate. Sharma (2005) studied the wave propagation in a general anisotropic poroelastic solid saturated with a viscous fluid flowing through its pores of anisotropic permeability. He discussed the effects of anisotropy, frequency, viscosity and variations of polarizations of quasi-waves with their phase direction. He concluded that the presence of anisotropy changes the relative
motion between fluid and solid particles of porous aggregate. The particles of the two constituents will no longer be moving parallel to each other, for any of the quasi-waves. Chattopadhyay and Rajneesh (2006) studied the reflection and refraction of plane waves at a plane interface between highly anisotropic triclinic crystalline half-space and homogeneous isotropic half-space. They discussed the phase velocities of all the three types of waves, namely, \( qP \), \( qSV \) and \( qSH \). They also presented a relation between the direction of motion and the direction of propagation. Udias (1999) in his book presented the basic literature of wave propagation in anisotropic media.

1.2.2 Waves in heterogeneous medium

When the material properties of a medium vary with the position, the medium is said to be heterogeneous or inhomogeneous. When, in particular, the material properties of a medium vary along horizontal direction only, it is said to be laterally heterogeneous and if these properties vary along vertical direction only, the medium is said to be vertically heterogeneous. There are evidences in the literature that certain regions of the earth’s crust behave like inhomogeneous material. Karal and Keller (1959) developed a general method for the progressive type of wave solution of the linearized equations of elasticity for both homogeneous and inhomogeneous media. This method is based upon an expansion of the solution which is useful for pulses and for rapidly changing wave forms. The method is not restricted by the usual considerations, which depend upon separation of variables. The solution consists of a series of terms, the first of which describes the wave motion predicted by geometrical optics. Subsequent terms account for certain types of diffraction effects. The series is not necessarily convergent but is presumably asymptotic to the solution. Hooke (1961) discussed various cases of the separability of vector wave equation for isotropic media when the elastic parameters \( \lambda \) and \( \mu \), and the density \( \rho \) are functions of one or more coordinates. He represented the displacement in terms of potentials and studied the problem of separation of vector wave equation of elasticity. He investigated the cases of certain type of inhomogeneity in the medium, in which separation takes place. He obtained three independent solutions of the vector wave equation in terms of potential functions and showed that one of these potential functions represents generalized \( SH \)-waves. The
other two potential functions satisfy independent second order wave equation only if the parameters of the medium satisfy certain ordinary differential equations. Hooke (1962a, b) discussed the problems of generalization of the method of potentials for the vector wave equation of elasticity for inhomogeneous media and contributed to a theory of separability of vector wave equation of elasticity for inhomogeneous media. Alverson et al. (1963) extended the work of Hooke to a media with variable Poisson ratio and concluded that the three constitutive parameters of the medium must satisfy a pair of simultaneous non-linear differential equations. Karlsson and Hooke (1963) used the method of separation of vector wave equation to study a two-dimensional problem in an inhomogeneous, isotropic, half-space whose elastic parameters and density vary as the square of the depth with constant velocities of propagation of $P$ and $S$ waves. The Rayleigh wave dispersion curve for this medium is examined. They found that the displacements vary inversely with depth and the relative effects of the inhomogeneity as compared to the homogeneous medium increase with distance from the source. On the surface, the amplitude of the Rayleigh wave varies sinusoidally with distance from the source. The displacements at times before and behind the Rayleigh wave are similar, i.e., approximately sinusoidal, variation with distance from the source.

Lock (1963) studied the wave propagation in an inhomogeneous transversely isotropic medium and discussed the various effects of inhomogeneity of the medium by taking the variation of the parameters in the form $\{\mu, \rho\} = \{\mu', \rho'\} \exp(pz)$, where $\mu'$ and $\rho'$ are constants and $p$ is the heterogeneity factor. Avtar (1967) discussed a problem of Love wave propagation in two homogeneous layers overlying an inhomogeneous half-space. He found the exact solution of the differential equation in particular cases, but for the general case, he obtained the solution in two forms; one is an asymptotic series solution and the second is obtained by WKBJ (Wentzel-Kramer-Brillioun-Jeffreys) approximation. Gupta (1967) studied the propagation of plane $SH$-waves in an inhomogeneous elastic half-space. He showed that the equation of motion and boundary conditions for $SH$-waves in solids and for sound waves in fluids are equivalent for inhomogeneous media. Sinha (1969) investigated the possibility of existence of Love waves in a heterogeneous crust over a homogeneous half-space.

Acharya (1970) determined the effect of inhomogeneity of the medium on the reflection of $P$-waves from a free surface. He concluded that if the properties of the medium are varying slowly then the reflection coefficients at a free surface are complex.
and the phase change upon the vertical reflection is not 180° as in the case of homogeneous media. Bhattacharya (1970) considered the problem of Love-wave dispersion in a layer in which shear wave velocity and shear modulus vary as linear functions of the horizontal distance. He obtained the Love-wave dispersion curves for the first two modes. Chatterjee (1972) and Negi and Singh (1973) have investigated the Love-wave dispersion in a layer with laterally inhomogeneity varying exponentially. The principle of constructive interference have been applied to derive the frequency equation for Love waves in a laterally inhomogeneous layer lying over a homogeneous half-space.

Kennett (1972) considered the problem of seismic waves scattering by lateral inhomogeneities. He made use of the perturbation technique to calculate the scattering effect of lateral inhomogeneities in multilayered media. Singh (1974) discussed Love-wave propagation in a transversely isotropic inhomogeneous layer lying over a homogeneous isotropic, half-space. He considered the horizontal variations in the rigidity and density of the layer and presented the dispersion curves. Singh et al. (1976) derived the frequency equation for Love waves in a laterally and vertically inhomogeneous layered half-space using Thomson-Haskell matrix method. They have assumed the elastic parameters to be the function of both x and z co-ordinates but shear wave velocity to be independent of x co-ordinate. Singh et al. (1978) discussed the problem of reflection and refraction of $SH$-waves at a plane interface between two laterally and vertically heterogeneous elastic media and showed that these coefficients are strongly influenced by the heterogeneity of the half-spaces. Malhotra et al. (1982) derived the expressions for reflection coefficients and the surface amplitude of $SH$-waves when the crust is both laterally and vertically heterogeneous and transversely isotropic.

Kumari et al. (1983) investigated a problem of Love wave propagation, using the parabolic equation and other approximations, for two cases: (i) when the density and rigidity are functions of the depth only, and (ii) when these are functions of both, the vertical and the horizontal coordinates respectively of the point. The propagation of waves on a multilayered interface is investigated by Rokhlin et al. (1986) using matrix method. They obtained the solution as a series expansion with respect to the thickness of the layers. In the first approximation, the solution is determined by a simple characteristic equation for the velocity of the interface wave, where the interface layer is described by effective elastic properties, depending mainly on shear moduli. The propagation of waves in a medium with time-dependent spatial inhomogeneities in its
velocity profile is studied by Abraham (1988). The rigorous formulation for source and incident field is described in the form of integral equation. This was in the spirit of Rayleigh, whose aim was to bypass complications arising from boundary conditions. A perturbation expansion, correct to all orders, is then obtained with the aid of a simple identity, for the two cases: (i) weak scatterers and (ii) strong scatterers. The expressions are derived for the wavefunctions when the inhomogeneity in the medium is due to distributions of moving particles. John (1988) discussed the effects of random heterogeneity on wave fields. His aim was to highlight a richness in the observations as a result of random heterogeneity, and to provide a discussion of the possible physical bases for these observations. Rossikhin and Shitikova (1992) attempted a problem of slightly inhomogeneous surface wave in an isotropic half-space in the presence of weak anisotropy by its surface. They considered two types of weakly anisotropic layers: the layer in which the shear wave velocity is the same as the half-space ("neutral" layer) and the layer in which the velocity of the shear wave is smaller than the velocity of the shear wave in the half-space ("decelerating" layer). The new slightly inhomogeneous surface waves with the predominating transverse displacement components parallel to a free surface (the quasi-transverse waves of \( SH \) type) were shown to exist in the described two layered media. These waves penetrate deeply into a backing. The wave corresponding to the "neutral" layer propagates with the bigger velocity and penetrates deeper into the backing than the wave conforming to the "decelerating" layer.

Robins (1994) attempted a problem of generation of shear and compression waves in an inhomogeneous elastic medium. He considered an inhomogeneous solid sediment lying between two homogeneous media: an upper fluid layer representing the ocean and a semi-infinite homogeneous solid substrate. He determined the reflection coefficient of a plane wave incident on the sediment from above. It is assumed that the shear modulus in the sediment is small compared with the bulk modulus. With a suitably chosen density variation in the sediment, the equations are solved analytically. Exact solutions are derived and are used to investigate the effect of a continuous density variation within the sediment, on the reflection of an incoming plane wave from the upper layer. Dey et al. (1996) considered a problem of torsional surface waves in a nonhomogeneous and anisotropic medium. They discussed the possibility of propagation of torsional surface waves in (i) a nonhomogeneous elastic medium with polynomial variation of rigidity and density, (ii) exponential variation of rigidity with constant
density, and (iii) exponential variation of both rigidity and density. It is shown that homogeneous isotropic and anisotropic media will not allow torsional surface waves to propagate but some types of inhomogeneity in the rigidity and density in the medium allow such propagation. It is also observed that in certain types of nonhomogeneity, there may be two torsional wavefronts. Zhang et al. (1997) studied the propagation of seismic waves in a laterally inhomogeneous medium.

The scattering of elastic waves in heterogeneous media is discussed by Joseph and Phanidhar (2001). They derived explicit expressions for the attenuation of longitudinal and transverse elastic waves in terms of the statistics of the density and Lame parameter fluctuations. Hasanyan et al. (2003), considered two dynamical problems for an isotropic elastic medium with spatially varying functional inhomogeneity, the propagation of surface anti-plane shear $SH$ waves and the stress deformation state of an anti-plane vibrating medium with a semi-infinite crack. These problems are considered for five different types of inhomogeneities. It is shown that the propagation of surface anti-plane shear waves is possible in all these cases. The existence conditions and the speed of propagation of surface waves have been obtained. Fourier transforms along with the Wiener-Hopf technique are employed to the investigation of the stress deformation state of a vibrating medium with a semi-infinite crack. They analyzed that the inhomogeneity can have both the quantitative and the qualitative impact on the character of the stress distribution near the crack. Saravanan and Rajagopal (2003) studied the role of inhomogeneities in the deformation of elastic bodies and discussed that it is quite common to approximate ‘mildly’ inhomogeneous bodies as homogeneous bodies belonging to a certain constitutive class in view of the simplification that such an approximation accords. They investigated the consequences of such an assumption and showed that it is clearly inappropriate for many classes of inhomogeneous bodies. They also investigated an important class of deformations, which in view of the paucity of boundary value problems that have been solved for non-linear inhomogeneous solids.

1.2.3 Waves in viscoelastic medium

In reality, no body is perfectly elastic. When a force is applied on any body, it takes some time to respond. There is definitely time delay of the elastic deformation with respect to the applied forces. Due to this behavior of the bodies, every real body is not
perfectly elastic. The time delay in such deformations for the system to recover the initial state, is called relaxation time. If in a body, the applied stress is proportional to the time rate of strain, then such bodies are called *viscous bodies*. Imperfectly elastic bodies that possess properties of elastic and viscous bodies both are called *viscoelastic bodies*. In viscoelastic bodies, if stress is held constant, the strain increases with time (creep), and if strain is held constant, the stress decreases with time (relaxation). Their behavior can be understood in terms of one dimensional mechanical models consisting in combinations of springs and dashpots (Aki and Richards, 1988). A body which consists in an elastic (spring) and a viscous (dashpot) element in series is called Maxwellian. Thus, the materials which exhibit complete recovery after sufficient time following creep or relaxation are called viscoelastic materials.

The wave theory in anelastic medium has proved its worth in solid mechanics by explaining the real phenomenon of energy loss. Other fields of application are solid earth geophysics for improving estimates of the earth’s inner composition, electrical engineering for analysis of delay lines, and applied mechanics and physics for dynamical analysis of materials. It is well established in the literature that the earth can not behave as a perfectly elastic bodies. The lack of perfect elasticity decrease the wave amplitude due to anelastic attenuation. In order to delineate the fine structure of the earth accurately, the imperfection of the elasticity of the earth materials like metals, concrete, soil, mud can not be disregarded. The unconsolidated materials generally exhibit the highest degree of anelastic response towards high frequency waves. The waves of high frequency are widely used in seismic prospecting. The linear theory of visco-elasticity describes the linear behavior of both elastic and viscous materials and provides a basis for describing the attenuation of seismic waves due to anelasticity. The attenuation of seismic waves with distance and time ensures the anelastic behavior of Earth. The anelastic behavior of the earth’s material plays an important role in changing the characteristics of seismic waves in defining seismic source functions and in determining the internal structure of the earth (Brune, 1970). Therefore, in order to study the behavior of the earth’s material more precisely, the effects of anelasticity must be incorporated into the earth models. We now briefly review the earlier work done in this direction.

Before 1960, most of the work on linear viscoelastic wave propagation for which explicit solutions were obtained, was essentially one dimensional and for specific ma-
terials. However, a foundation of three dimensional linear visco-elasticity theory was laid down by Bland (1960). He concluded that similar to a perfectly elastic isotropic medium, under the assumption of small displacements, two types of waves can propagate in an isotropic viscoelastic medium when body forces are absent. The first type of wave, known as the dilatational or longitudinal or $P$-type wave, propagates through a linear viscoelastic isotropic material with the complex velocity $\alpha^* = \sqrt{\frac{\lambda^* + 2\mu^*}{\rho^*}}$, where $\lambda^*$ and $\mu^*$ are complex valued modified Lame’s parameters. The second type of wave, known as shear or equivoluminal or distortional or $S$-wave, propagates with the complex velocity $\beta^* = \sqrt{\frac{\mu^*}{\rho^*}}$. However, Hunter (1960) explored the application of the general theory to study the nature of plane waves in anelastic media. Lockett (1962) studied the problem of reflection and refraction of waves at an interface between viscoelastic materials. He found that the body waves in visco-elastic media are damped exponentially with distance travelled. Also, the waves reflected from or refracted through a plane interface can have an associated attenuation vector which is a linear combination with positive coefficients of the forward normal to the wave surface and of the inward normal to the interface. These waves are more complex than those of elasticity and arise when sinusoidal dilatational or rotational waves are incident on an interface in a viscoelastic medium.

Cooper and Reiss (1966) solved three dimensional wave propagation problems for a wide class of homogeneous, isotropic and linear viscoelastic media. They applied a method for specific viscoelastic materials to high frequency time harmonic waves. Their method consists in representation of the solution in the form of asymptotic series in inverse integer powers of the frequency. The first term in the expansion is called the geometrical theory of viscoelasticity and the subsequent terms provide corrections to the theory. They employed this method to the problem of reflection of time harmonic dilatational waves from a smooth rigid plane and derived the asymptotic reflection laws. The approximate solution up to the first two terms of the reflected wave was obtained analytically. Again, in 1966, they studied the reflection of time harmonic plane dilatational and shear waves from a rigid or stress free boundary of an arbitrary linear viscoelastic half space. Using the properties of general plane wave, i.e., the plane wave whose amplitude varies across the wave front, they determined the exact solution of the reflection problem for a wider class of linear viscoelastic media and noticed that,
in general, the reflected wave is inhomogeneous, i.e. attenuates in a direction different from the direction of propagation. Also, when certain specific conditions prevail, the reflected wave is homogeneous, i.e., the wave attenuates in the direction of propagation. The necessary and sufficient conditions on the material parameters for the existence of the critical angle were derived. They noticed that, unlike elastic materials in which the reflected wave propagates along the interface for a certain range of super critical incident angle, in viscoelastic medium the wave travels along the interface only for discrete angles. However, for a special class of viscoelastic materials, there may be a range of incident angle that produces a reflected wave propagated along the interface. Such types of materials, they called, elastic like materials. In general, the phase and reflected angles are functions of the angle of incidence, the frequency and the materials properties. Cooper (1967) studied the problem of transmission and reflection of time harmonic plane dilatational and shear waves at a plane interface between two linearly viscoelastic half-spaces. He concluded that except for some special cases, the reflected and transmitted waves are general plane waves. Because the speeds of the reflected and transmitted waves are, in general, functions of the angle of incidence, he concluded that the angles of reflection and transmission depend on the incident angle in a more complicated way than in the limiting elastic case. The results were found entirely consistent with those he and Reiss (1966) while dealing with the problem of reflection and transmission from rigid and free boundaries. The results obtained by Cooper (1967) agree with Lockett’s (1962) result except at one point where an inconsistency occurs. Lockett (1962) indicates that if the interface waves exist, they occur for all angles of incidence greater than some critical angle, while Cooper (1967) showed that interface wave occurs for discrete angles of incidence.

Shaw and Bugl (1969) studied the problem of reflection and transmission of plane harmonic waves in a multilayered viscoelastic medium using matrix method and showed that nature of the $P$ and $S$ waves in an anelastic medium is different from that in an elastic medium. They also showed that the inhomogeneous reflected and transmitted waves will be formed even if the incident waves are not of this form that (i) the actual speed of propagation of wave fronts differs from the usual wave speed, (ii) the interface wave can not exist in the visco-elastic layers under the elastic half-space unless one of the wave speeds in the layers is real (iii) the interface wave cannot exist in the visco-elastic layers under the visco-elastic half-space only under specific conditions on
the material properties and incident angle. Schoenberg (1971) studied the reflection and transmission of $P$ and $SV$ waves at an elastic/viscoelastic interface. He used the complex notation and the solution in the viscoelastic medium is obtained by using the correspondence principle. He, for the first time, presented the particle motion in the general visco-elastic wave as a function of the angle of incidence and frequency and gave the solution for a particular viscoelastic material. The remark of Hunter (1960) that only the study of waves in an elastic medium has been fully exploited and not in viscoelastic media, inspired Buchen (1971a) to study a class of plane inhomogeneous waves in linear viscoelastic media. He presented the detailed theoretical description of the physical properties and the energy associated with plane inhomogeneous $P$ and $SV$ waves in a linear viscoelastic medium. He defined a non-dimensional parameter $Q^{-1}(=2Im(\frac{\omega}{\beta})/Re(\frac{\omega}{\beta}))$, called the loss factor, often used in expressing the attenuation of seismic body waves and derived its expression. The mathematical theory suggested that the nature of plane waves in anelastic media is distinctly different from the nature of plane waves in elastic media. Buchen also found that the particle motion for $P$- and $SV$- waves in a viscoelastic medium is elliptical while for perfectly elastic medium it is rectilinear. Buchen (1971b) studied the reflection and transmission of $SH$-waves emitted from an axisymmetric line source in one of the two half-spaces and situated parallel to the interface in one of the half-space. He presented the asymptotic expressions for the direct, reflected, transmitted and diffracted waves. He also mentioned that these expressions are not valid in the neighborhood of the critical angle. He showed that distinct differences in the amplitude distributions for the head wave and transmitted wave occur, if the quality factor $Q$, of one medium is greater or less than that for the other. When $Q$ is same for both the medium, the wave display properties similar to the case of perfect elasticity.

Borcherdt (1973a), keeping his approach different from Buchen (1971a), has also studied in detail the nature and the physical properties of plane waves in elastic and linear viscoelastic media by using an explicit energy conservation relation, valid for an arbitrary steady state viscoelastic radiation field. He remarked that the definition for $2\pi Q^{-1}$, as accepted by Buchen (1971a), is incorrect for homogeneous plane waves except for solids with a small amount of absorption. He defined $2\pi Q^{-1}$, as the ratio of the loss in energy density per cycle of forced oscillations to the peak energy density stored during that cycle. Thus, the general expression of $Q^{-1}$ (the loss factor)
derived by Borcherdt differs from that derived by Buchen. He also showed that in an elastic medium, the only type of inhomogeneous wave that can propagate is one for which planes of constant phase are perpendicular to the planes of constant amplitude while in an anelastic medium, this is the only type of inhomogeneous wave that cannot propagate. Silva (1976) studied the problem of incident $P$- and $SV$- waves in anelastic half-space and studied the effects of vertical variations in attenuation as well as velocity and density on body waves. He presented an extension of the Thomson-Haskell matrix method to viscoelastic problems. In support of his anelastic earth model, he gave the examples of soil, crust and core-mantle boundary. Taking into account the parameters of the regional earth structure, he obtained synthetic seismograms to study the attenuation effect. Borcherdt (1977) studied analytically the problem of reflection and transmission of general plane $SH$-waves at an elastic and anelastic media. He showed that unlike general $P$ and $SV$ waves in an anelastic medium, which exhibits elliptical particle motion, the general plane $SH$-waves exhibit same particle motion as the elastic plane $SH$-waves, i.e., linear motion perpendicular to the direction of incidence. He concluded that the characteristics of the general plane $SH$-waves reflected and refracted at plane anelastic boundaries are: (i) velocity and maximum attenuation are functions of the angle of incidence and frequency, (ii) maximum energy flows at a different velocity and in a different direction than phase propagation and (iii) energy flow across the boundary due to interaction of the incident and reflected waves. He also showed that none of these characteristics are predicted for the plane $SH$-waves described by elastic theory. Krebes and Hron (1980a, b) incorporated the effects of viscoelasticity into their models to study the seismic wave problems. They interpreted that certain regions of the earth behave like a linear visco-elastic media.

Kaushik and Chopra (1980) studied the effect of inhomogeneity on the reflection and transmission coefficients when a inhomogeneous plane $SH$-wave is incident at an interface between homogeneous and inhomogeneous visco-elastic half spaces. The results were compared with the corresponding perfectly elastic case and with the case when both the media are homogeneous visco-elastic. Kaushik and Chopra (1981) also studied the reflection and transmission of plane $SH$-waves at an anisotropic elastic/viscoelastic interface. They obtained the expressions for the reflection and transmission coefficients. Kaushik and Chopra (1983, 1984) studied the problem of reflection and transmission of $SH$-waves at an interface by incorporating heterogeneity in linear viscoelastic half
space and discussed the behavior of the amplitude ratios with angle of incidence. They have also discussed the existence of critical angle of the incident wave for some cases. Gogna and Chander (1985) discussed the reflection and transmission of $SH$-waves at an interface between anisotropic inhomogeneous elastic and viscoelastic half-spaces. They used the correspondence principle to find the expressions for the reflection and transmission coefficients for viscoelastic medium, which states that the solution for a viscoelastic problem is given by the known elastic solution with the complex viscoelastic moduli substituted in place of the corresponding elastic moduli. They concluded that the moduli and the phases of reflection and transmission coefficients are significantly affected by the angle of incidence, frequencies of the medium, horizontal shear velocity and vertical shear velocity. John (1988) in their note on reflection and transmission of waves at a boundary between two viscoelastic media pointed out the errors in the equations that are presented by Cooper (1967). They noted that the errors do not arise from the physical treatment of the problem but well due to printing or a similar cause. They listed the corrected equations to give some demonstration of their validity. Le et al. (1992) presented a complete description of $SH$-wave propagation in homogeneous transversely isotropic linear viscoelastic medium. They showed that the energy propagates along a direction dictated by the real phase vector, real attenuation vector and the real and imaginary parts of the rigidities. Nchtschein and Hron (1997) described plane wave approach and asymptotic ray theory to obtain the reflection and transmission coefficients between two anelastic media.

Paulino and Jin (2001) presented an extension of the correspondence principle (as applied to homogeneous viscoelastic solids) to nonhomogeneous viscoelastic solids under the assumption that the relaxation (or creep) moduli be separable functions in space and time. The revisited correspondence principle extends to specific instances of thermo viscoelasticity and fracture of functionally graded materials and shown that the viscoelastic correspondence principle remains valid for a linearly isotropic viscoelastic functionally graded material with separable relaxation (or creep) functions in space and time. Mukherjee and Paulino (2003) revisited the problem of Paulino and Jin (2001) and examined the reasons behind the success or failure of the correspondence principle for viscoelastic functionally graded materials by addressing some subtle points regarding the results. For the inseparable class of nonhomogeneous materials, the correspondence principle fails because of an inconsistency between the replacements of the
moduli and of their derivatives. Cerveny (2004) discussed three basic approaches to the determination of the slowness vector of an inhomogeneous plane wave propagating in a homogeneous viscoelastic anisotropic medium. These three approaches differ in the specification of the mathematical form of the slowness vector: directional specification, componental specification and mixed specification of the slowness vector. He concluded that the simplest, most straightforward and transparent algorithms to determine the phase velocities and slowness vector of inhomogeneous plane wave propagating in a viscoelastic anisotropic media can be obtained, if the mixed specification of the slowness vector is used. These algorithms are based on the solution of an algebraic equation of the sixth degree. Contrary to the mixed specification, the directional specification can hardly be used to determine the slowness vector of inhomogeneous plane wave propagating in a viscoelastic anisotropic media. Morro (2005) considered the reflection and transmission of mechanical waves for a uniaxially-inhomogeneous viscoelastic layer, which is sandwiched between two homogeneous elastic half-spaces. The problem is framed within the time domain as originated by a normally-incident wave. Uniqueness results for reflection and transmission in a viscoelastic layer are established for an initial value problem through an energy functional.

1.2.4 Mathematical description of linear viscoelastic materials

For a small motion, the stress-strain relation in an isotropic linear viscoelastic medium is given by Buchen (1971, b)

\[
\tau_{ij} = 2g \ast \varepsilon_{ij}(t) + \left(f(t) - \frac{2}{3}g(t)\right) \ast \delta_{kk}(t)\delta_{ij}
\]  

(1.8)

where \(f(t)\) and \(g(t)\) are the relaxation functions characterizing the shear and bulk behaviors of the material, \(\delta_{ij}\) is a Kronecker delta and (*) denotes the convolution operation defined as

\[
f(t) \ast g(t) = \int_{-\infty}^{t} f(t - \tau)g(\tau)\,d\tau
\]

The equation of small motion (1.1), in the absence of body forces, is re-written as

\[
\tau_{ij,j} = \rho \dddot{u}_i \quad (i, j = x, y, z),
\]  

(1.9)

where "comma" in the subscript denotes the partial derivative with respect to spatial coordinate and superposed dotes on the right hand side denote the double time
derivative. Putting \( i = x \) into (1.9), we have
\[
\tau_{xx, x} + \tau_{xy, y} + \tau_{xz, z} = \rho \ddot{u}_x.
\]
For homogeneous linear viscoelastic solid, \( \rho, g \) and \( f \) are taken independent of spatial co-ordinates. Inserting equation (1.8) into the above equation, we obtain
\[
\rho \ddot{u}_x = 2g \ast d(e_{xx})_x + (f - \frac{2}{3}g) \ast d(e_{kk})_x + 2g \ast d(e_{xy})_y + 2g \ast d(e_{zz})_z,
\]
\[
= 2g \ast d(u_{xx}, x) + (f - \frac{2}{3}g) \ast d(u_{xx} + u_{yy} + u_{zz}),
\]
\[
+ 2g \ast d(\frac{1}{2}(u_{xx} + u_{yy})), y + 2g \ast d(\frac{1}{2}(u_{xx} + u_{zz})), z,
\]
\[
= 2g \ast d(\frac{\partial^2 u_x}{\partial x^2}) + (f - \frac{2}{3}g) \ast d(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_x}{\partial x \partial z}) + g \ast d(\frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_y}{\partial x \partial z}) + g \ast d(\frac{\partial^2 u_z}{\partial z^2} + \frac{\partial^2 u_z}{\partial x \partial z} + \frac{\partial^2 u_z}{\partial x \partial y}),
\]
\[
= (f - \frac{2}{3}g) \ast d(\frac{\partial}{\partial x}(e_{kk}))+ g \ast d(\nabla^2 u_x) + g \ast d(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}),
\]
\[
\]
Similarly, by putting \( i = y \) and \( i = z \) into equation (1.9), we obtain
\[
\rho \ddot{u}_y = (f + \frac{1}{3}g) \ast d(\frac{\partial}{\partial y}(e_{kk})) + g \ast d(\nabla^2 u_y),
\]
\[
\rho \ddot{u}_z = (f + \frac{1}{3}g) \ast d(\frac{\partial}{\partial z}(e_{kk})) + g \ast d(\nabla^2 u_z).
\]
Combining all these equations, the vector form of equation of motion for homogeneous linear viscoelastic solids is given by
\[
\rho \ddot{\mathbf{u}} = (f + \frac{1}{3}g) \ast d(\nabla(\nabla \cdot \mathbf{u})) + g \ast d(\nabla^2 \mathbf{u}).
\]
(1.10)
For a time harmonic progressive waves, we take
\[
\mathbf{u} = U \exp(\omega t)
\]
(1.11)
where \( U \) denotes the complex function of the spatial co-ordinates and \( \omega \) is the angular frequency. On substituting (1.11) into (1.10), we obtain
\[
-\omega^2 \rho U = (f + \frac{1}{3}g) \ast d(\nabla(\nabla \cdot U)) + g \ast d(\nabla^2 U).
\]
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Taking the Fourier transform to both sides of the above equation and using the property that Fourier transform of convolution of two functions is equal to the product of Fourier transform of the functions, we obtain

\[-\omega^2 \rho F\{U\} = F\{(f + \frac{1}{3}g)\} F\{d(\nabla (\nabla \cdot U))\} + F\{g\} F\{d(\nabla^2 U)\},\]

where \(F\{\cdot\}\) denotes the Fourier transform of \(\cdot\).

Using the following definition and property of the Fourier transform, given by respectively

\[F\{h\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(\tau)e^{-i\omega \tau} d\tau \]

and

\[F\{dh\} = -i\omega F\{h\},\]

we obtain

\[(K + \frac{M}{3})\nabla (\nabla \cdot \widetilde{U}) + M\nabla^2 \widetilde{U} + \rho \omega^2 \widetilde{U} = 0, \quad (1.12)\]

where

\[\widetilde{U} = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\tau)e^{-i\omega \tau} d\tau,\]

\[K = \frac{-i\omega}{2\pi} \int_{-\infty}^{\infty} f(\tau)e^{-i\omega \tau} d\tau,\]

\[M = \frac{-i\omega}{2\pi} \int_{-\infty}^{\infty} g(\tau)e^{-i\omega \tau} d\tau.\]

Here \(K\) and \(M\) are the complex bulk modulus and complex shear modulus respectively. The quantity \(\widetilde{U}\) is the Fourier transform of the complex displacement vector \(U\), i.e. \(F\{U\} = \widetilde{U}\). For a plane \(SH\)-wave, we have

\[\nabla \cdot \widetilde{U} = 0.\]

Hence, the equation of harmonic motions (1.12) governing \(SH\)-wave, reduces to

\[\nabla^2 \widetilde{U} + k^2_{\beta} \widetilde{U} = 0, \quad k^2_{\beta} = \rho \omega^2 / M. \quad (1.13)\]

Since the expression representing a plane wave in a linearly visco-elastic medium is slightly different from the elastic case because of the amplitude decay due to the presence of visco-elasticity in the medium, therefore, we introduce an attenuation vector \(A\).
along with a propagation vector $\mathbf{P}$ as follows. For a harmonic plane wave propagating in this type of medium, the displacement vector is expressed as:

$$u = U_0 \exp(-\mathbf{A} \cdot \mathbf{r}) \exp(-i(\mathbf{P} \cdot \mathbf{r}) \exp(i\omega t) = U_0 \exp(-i(\mathbf{k} \cdot \mathbf{r} - \omega t))$$

where $\mathbf{A}$ is in a direction perpendicular to the planes of constant amplitude defined by $\mathbf{A} \cdot \mathbf{r} = \text{constant}$. The vector $\mathbf{P}$ is called the propagation vector, which is perpendicular to the plane of constant phase defined by $\mathbf{P} \cdot \mathbf{r} = \text{constant}$. The wavevector $\mathbf{k}$ is a complex wavenumber given as $\mathbf{k} = \mathbf{P} - i\mathbf{A}$. Here, the $\mathbf{P}$ and $\mathbf{A}$ are the two vectors generally not parallel because the amplitude of the plane wave propagating in a viscoelastic medium can vary along the wavefront (Borcherdt, 1977). If the vectors $\mathbf{P}$ and $\mathbf{A}$ are parallel, the wave is called homogeneous plane wave. If the vectors $\mathbf{P}$ and $\mathbf{A}$ are not parallel, the plane wave is called inhomogeneous. The modules of the vectors $\mathbf{P}$ and $\mathbf{A}$ are given by (Borcherdt, 1973)

$$|\mathbf{P}| = \left[ \frac{1}{2} ((Re k^2 + ((Re k^2)^2 + \frac{(Im k^2)^2}{cos^2 \gamma})^{\frac{1}{2}}) \right]^{\frac{1}{2}}$$

$$|\mathbf{A}| = \left[ \frac{1}{2} ((-Re k^2 + ((Re k^2)^2 + \frac{(Im k^2)^2}{cos^2 \gamma})^{\frac{1}{2}}) \right]^{\frac{1}{2}} = \frac{-Im(k^2)}{2|\mathbf{P}| \cos \gamma}$$

and satisfy the relations

$$\mathbf{P} \cdot \mathbf{P} - \mathbf{A} \cdot \mathbf{A} = Re(k^2), \quad \mathbf{P} \cdot \mathbf{A} = |\mathbf{P}||\mathbf{A}| \cos \gamma = (\frac{-1}{2})Im(k^2); \quad (1.14)$$

where $\gamma$ is the angle between attenuation and propagation vectors. The physical requirement that the amplitude of the wave not increase in the direction of propagation is satisfied by requiring that the angle between attenuation and propagation vectors lies in the range $0 \leq \gamma \leq \frac{\pi}{2}$. For an elastic material (i.e. when $Im(k^2) = 0$), equation (1.14) implies that either $\mathbf{A} = 0$ (which means that the homogeneous plane wave is not attenuated) or $\gamma = \frac{\pi}{2}$ (which means that the direction of maximum attenuation is perpendicular to the direction of propagation). Conversely, equation (1.14) shows that, if $\mathbf{A} = 0$ or $\gamma = \frac{\pi}{2}$, the material is non-dissipative. We can also say that a material is dissipative if and only if $\mathbf{A} \neq 0$ and $\gamma \neq \frac{\pi}{2}$. In brief, these results show that the only type of inhomogeneous plane wave that propagates in elastic media does not propagate in dissipative media and vice-versa. This difference in the nature of inhomogeneous waves in the two types of materials is a basic result. It implies that the results of several problems of linear anelastic media are substantially different from those in elastic media.
1.2.5 Waves in initially stressed medium

Since earth is a large elastic body under gravitational field, therefore, it becomes gravitationally prestressed or initially stressed. There are some other factors like slow process of creep inside the earth, variations of temperature etc. which make the earth initially stressed medium. These stresses may be approximated as homogeneous near the surface (Biot, 1940). An elegant and elaborate exposition of the theory, explaining the elastodynamics of a body under initial stress, is found in the book entitled "Mechanics of Incremental Deformations" by Biot (1965). If we regard the earth as an initially stressed deformable body, the mechanical changes in the characteristics of the earth can further be studied. Keeping it in mind, many researchers in the field of wave propagation studied wave propagation in an initially stressed medium. Hayes (1963) studied a problem of wave propagation and uniqueness of the displacement boundary-value problem for infinitesimal deformation in prestressed elastic solids. He obtained the necessary and sufficient conditions for the waves to travel with a real speed of propagation in a finitely stressed body. When these necessary and sufficient conditions are satisfied, the equations for a small static deformation superimposed on the large deformation are strongly elliptic. He showed that the condition of strong ellipticity are sufficient but not necessary for uniqueness of the displacement boundary-value problem. Dey and Addy (1977) studied the reflection and refraction of plane waves at a free surface of an initially stressed elastic half space. They concluded that the reflection coefficients depend on the elastic constants, angle of incidence and on the parameters corresponding to the initial stress present in the medium. Later, in 1979, they have extended their problem at a plane interface between two initially stressed elastic half-spaces. Again, Dey and Addy (1980) studied the influence of the initial stress on the reflection and transmission coefficients due to incident P-waves at the earth’s core-mantle boundary. They showed the variations of these reflection and transmission coefficients with respect to the angle of emergence for a particular value of the initial stress and concluded that the effect of the initial stress is to produce the reflected P- and S-waves with numerically higher amplitudes but a transmitted P-wave with smaller amplitudes.

Chattopadhyay et al. (1982) studied the phenomenon of reflection of elastic waves at a free surface of an initially stressed sandy medium. They obtained the variations
of the reflection coefficients with the angle of incidence at different values of the initial stress parameters, when $P$ and $SV$-waves are made incident at the free surface. Sidhu and Singh (1982) commented their paper that the decoupling of the $P$ and $SV$-motion is not possible in any arbitrary direction as claimed by Chattopadhyay et al. (1982). They also showed that the decoupling of the $P$- and $SV$-motion is possible only if the initial stresses are along the principal axes. They have also pointed out that the displacement potentials assumed by Chattopadhyay et al. (1982) are either irrelevant or incorrect. Pal and Chattopadhyay (1984) considered a problem of reflection of plane waves at a free boundary of a homogeneous prestressed orthotropic half-space and observed that the reflection phenomena of pure mode of $P$ and $SV$-waves in such a medium are different from that of in the classical isotropic elastic medium. Apart from the directions parallel to the coordinate axes in the medium, there exists a specific direction along which pure modes of $P$- and $SV$-waves may propagate through the medium. It has been shown that under certain conditions, the $P$-wave propagating in a specific direction gives rise to a reflected wave which is either the superposition of both $P$ and $SV$-waves or an $SV$-wave only, obeying the simple laws of reflection, but it cannot give rise to a pure $P$ mode only. Similar phenomena are also observed in the case of an incident $SV$-wave. Since initial stress generally induces orthotropy in an isotropic medium, similar reflection phenomena should also be observed in a stress free orthotropic medium. However, the reflection phenomenon of an incident $SH$-wave is similar to that in the classical elastic medium. Under equal normal initial stresses, the medium behaves like a classical isotropic medium. Kar and Kalyani (1987) studied the reflection and transmission of $SH$-waves due to the presence of a sandwiched initially stressed sandy layer. They discussed the effects of the sandiness parameter, angle of incidence and initial stress on the amplitude ratios of reflected and transmitted $SH$-waves.

Dowaikh and Ogden (1990) considered the propagation of infinitesimal surface waves on a half-space of incompressible isotropic elastic material subject to a general pure homogeneous pre-strain. They obtained a secular equation for wave propagation along principal axes of the pre-strain for a general strain-energy function, and derived the conditions which ensure the stability of the underlying pre-strain. The influence of the pre-stress on the existence of surface waves is examined and, in particular, it is found that, under a certain range of hydrostatic pre-stress, a unique wave speed
exists which is bounded above by a limiting speed and corresponds to the shear wave speed in an infinite body. Sharma and Gogna (1990) derived the dispersion equation for Love wave propagation in a slow elastic layer overlying a liquid saturated porous solid half-space when both the media are assumed to be initially stressed. The effect of porosity and initial stress on the phase velocity of Love waves has been discussed. Later, Sharma and Gogna (1993) employed Biot’s theory to study the reflection and refraction of $SH$-waves in an initially stressed medium consisting of a sandy layer overlying a fluid-saturated porous solid half-space. Ogden and Sotiropoulos (1995) discussed interfacial waves along the plane interface between two prestressed incompressible elastic solids; one of the solids is a half-space while the other is a layer of uniform thickness such that the principal axes of the underlying pure homogeneous deformation in the two solids are aligned with one axis normal to the interface. They obtained the dispersion equation and conditions on the material parameters for the existence of a unique interfacial wave speed at low frequency. They have also shown that under special circumstances, non-dispersive waves can exist at low frequency limit. Asymptotic results at the high-frequency limits are also obtained. The effect of prestress on the propagation and reflection of plane waves in an incompressible isotropic elastic half-space has been examined by Ogden and Sotiropoulos (1997). Ogden and Sotiropoulos (1998) studied the reflection of plane waves from the boundary of a pre-stressed compressible elastic half-space. They showed that in a prestressed compressible elastic solid, pure shear waves can propagate only in specific directions in the considered principal plane and, in a general direction, a quasi-shear wave may be accompanied by a quasi-longitudinal wave, as is the case in the anisotropic linear theory. They determined the reflection coefficients of either an incident (quasi-) shear wave or an incident (quasi-) longitudinal wave at the boundary of the half-space.

Degtyar and Rokhlin (1998) studied the stress effect on boundary conditions and on elastic wave transmission through an interface between anisotropic media. They studied elastic wave propagation through a plane interface between two anisotropic stressed solids and between a fluid and a stressed anisotropic stressed solid. They found that the presence of static stresses effects the wave velocities in semi-spaces and boundary conditions at the interface. Acoustic waveforms have been simulated for monopole, dipole, and quadrupole sources in a prestressed formation by Tsili and Xiaoming (2005). They modelled the prestressed medium as an unstressed medium with
the stress effect replaced by effective elastic moduli. They showed that the quadrupole wave in a prestressed formation splits into a fast and a slow wave. The velocities of the waves are similar to those of crossed dipole waves in the same formulation. Both waves are faster than quadrupole waveforms without stress. Sharma and Garg (2006) studied the wave propagation in a pre-stressed anisotropic elastic medium. They derived modified Christoffel equations, phase velocities, group velocities and ray directions of three quasi-waves existed in a three dimensional pre-stressed anisotropic medium and studied the effect of initial stress on them.

Here, we shall derive the dynamical equation for $SH$-wave in an elastic medium under initial stress. The general form of Biot’s (1965) three dimensional dynamical equations, involving incremental stress tensor $s_{ij}$, are given by

$$\frac{\partial}{\partial x_j} (s_{ij} + S_{kj}\omega_{ik} + S_{ij}e - S_{ik}e_{jk}) = \rho \ddot{u}_i \quad (i, j, k = x, y, z). \quad (1.15)$$

where $\omega_{ik}$ are the components of the rotation given by

$$\omega_{ik} = \frac{1}{2}(u_{i,k} - u_{k,i}),$$

$\rho$ is the density of the medium, $e_{jk}$ are the components of the strain tensor defined earlier and $e = e_{ij}\delta_{ij}$. $S_{ij}$ are the components of the initial stress. The incremental stresses $s_{ij}$ are related to the elastic constants $B_{ijkl}$ by the relation

$$s_{ij} = B_{ijkl}u_{k,l}.$$ 

The another property that ensures the existence of density function for the medium is given by

$$B_{ijkl} - B_{kl ij} = S_{kl}\delta_{ij} - S_{ij}\delta_{kl}.$$ 

The relation between strain components and the rotational components is given by (Biot’s 1965, Chapter 1, pp. 49, eq. (7.37))

$$\frac{\partial}{\partial x_j}(e_{jk} + \omega_{jk}) = \frac{\partial e}{\partial x_k}. \quad (1.16)$$

The three equilibrium conditions satisfied by the initial stress field are given by (Biot, 1965)

$$\frac{\partial S_{ij}}{\partial x_j} + \rho \Delta X_i(x_i) = 0, \quad (1.17)$$
where $\Delta X_i$ represents the increment in the body force per unit mass from the initial point to the displaced point.

In the absence of external forces and using the relations (1.16) and (1.17), the equation (1.15) reduces to

$$
\frac{\partial s_{ij}}{\partial x_j} + S_{kj} \frac{\partial \omega_{ik}}{\partial x_j} - S_{ik} \frac{\partial e}{\partial x_k} + S_{ik} \frac{\partial \omega_{jk}}{\partial x_j} = \rho \ddot{u}_i \quad (i, j, k = x, y, z). \tag{1.18}
$$

We take the rectangular Cartesian coordinates $(x, y, z)$ such that $z$-axis is pointing vertically downwards and $x$-$y$ axes on the horizontal plane. Let us choose the principal directions of initial stresses along the directions of elastic symmetry and coordinate axes. The state of initial stress is, therefore, defined by principal components $S_{xx}, S_{yy}$ and $S_{zz}$ of the initial stress and tangential components are considered to be zero.

For the plane $SH$-wave propagating in the $x - z$ plane and causing displacement along $y$-direction, we shall take $u_x = u_z = 0$, $u_y = u_y(x, z, t)$, $\frac{\partial}{\partial y} = 0$ and $S_{yy} = 0$. With these assumptions, the equation (1.18) reduces to

$$
\frac{\partial s_{yk}}{\partial x_j} + S_{kj} \frac{\partial \omega_{yk}}{\partial x_j} + S_{yk} \frac{\partial \omega_{jk}}{\partial x_j} = \rho \ddot{u}_y. \quad (j, k = x, y, z). \tag{1.19}
$$

This is the equation of the plane $SH$-wave motion in an initially stressed medium without body forces.

The incremental stress-strain relations in the initially stressed slow elastic medium in $x$-$z$ plane are written as (Biot, 1965):

$$
s_{yz} = 2\left(\frac{Q_1}{\eta}\right)\epsilon_{yz} \quad s_{yz} = 2\left(\frac{Q_2}{\eta}\right)\epsilon_{yz}
$$

where $Q_1$ and $Q_2$ are the elastic constants which are the functions of the initial stresses.

Here, $\eta > 1$ corresponds to sandy materials and $\eta = 1$ corresponds to elastic solid (Weiskoff, 1945).

Substituting the above values of the incremental stress in equation (1.19), the equation of motion for the plane $SH$-wave, in an initially stressed sandy elastic medium becomes

$$
(Q_1/\eta + S_{zz}/2) \frac{\partial^2 u_y}{\partial z^2} + (Q_2/\eta + S_{xx}/2) \frac{\partial^2 u_y}{\partial x^2} = \rho \frac{\partial^2 u_y}{\partial t^2}. \tag{1.20}
$$

1.3 Waves at a corrugated interface

No natural surface in the earth medium is perfectly plane or perfectly smooth. They are quite undulated in nature. When a wave is incident upon a plane boundary between
two media, one gets a reflected wave in one direction (specularly) whereas the incident wave on a rough surface gives rise the reflected wave in different directions. Thus, a rough surface is a surface which scatters the energy of the incident wave in different directions. The extent to which, surface roughness/discontinuities, affects the wave scattering is therefore of great interest in various fields. Many recent studies based on the body and surface wave analysis have led to the conclusion that there are several types of discontinuities present in the earth. These discontinuities within the earth are irregular in nature and can not be considered as plane interface always. Therefore, the study of reflection and transmission of elastic waves at the rough or a surface which is not plane, (i.e., corrugated interface) is of great practical importance in theoretical as well as in observational seismology. Scattering of elastic waves from a corrugated interface has been studied by many investigators and they adopted different methods for solving such problems. Probably, the first reported work on scattering of waves from rough surfaces was by Rayleigh (1878) who discussed the scattering of a normally incident sound wave from a corrugated (sinusoidal) surface separating two acoustic media. He assumed that the scattered field can be written as a sum of plane waves travelling away from the rough surface. This approach has been extended by many authors to non-normal incidence at random rough surface. Rayleigh (1907) attempted a problem of reflection and transmission of light and sound waves incident normally at irregular boundary surface. He investigated this problem by assuming the amplitude and the slope of corrugation to be very small in comparison with the wavelength of the corrugation and expressed the corrugated interface by means of Fourier series. The unknown coefficients in the solutions of the boundary conditions are determined to any order of approximation in terms of a small parameter characteristic of the boundary. This method is known as Rayleigh’s method after his name. Later, Rayleigh’s method was applied to various other fields to study the reflection and transmission phenomena of waves at irregular boundary surface. Sato (1955) investigated the reflection of elastic waves at a corrugated, free surface by using the Rayleigh’s method of approximation. In order to find the effect of a corrugated interface on reflection and refraction of elastic waves, the case of incidence of $SH$-wave is presented by Asano (1960). Using Rayleigh’s method, he presented the formulae for reflection and refraction coefficients of regularly and irregularly waves for the first and second-order approximation of the corrugation. He concluded that the amplitudes of irregularly reflected and refracted
waves for the first and second-order approximation of the corrugation are proportional to the amplitude of corrugation. Numerical calculations were carried out for the case of normal incidence on a periodic boundary surface. For such a model, he, furthermore, concluded that the effect of amplitude of corrugation on the amplitude of a regularly reflected wave is opposite to and larger than its effect on a regularly refracted wave. That is, the larger the amplitude of corrugation is, the smaller, the amplitude of a regularly reflected wave, while the larger, the amplitude of a regularly refracted wave. Again, in 1961, he discussed a problem involving the reflection and refraction of elastic waves, especially, for the case of incidence of $P$ and $SV$-waves on the corrugated boundary surface between two uniform elastic half spaces. The expressions for reflection and transmission coefficients are obtained in closed form for the first and second-order approximation of the corrugation. He found that, in general, the effect of corrugation on reflection is larger than the effect of corrugation on refraction. In a certain range of wavelength of corrugation, there exist boundary waves of which the amplitude decreases exponentially with the distance from the boundary surface. Its rate of decrease depends on the order of spectrum and the amplitude of corrugation. When the wavelength of corrugated boundary surface is smaller than that of the incident wave, almost all irregular waves become boundary waves. Asano (1966) presented the results of reflection and refraction of elastic waves at a corrugated interface for the case of normal incidence of $P$-waves for three different models and the case of oblique incidence of $P$-waves for one model. He obtained that all irregular waves become boundary waves for the wavelengths of corrugation smaller than the wavelengths of $S$-waves in the incident medium, and the amplitude of the regularly reflected $P$-waves decreases as the amplitude of corrugation increases. He also found that the velocity contrast has pronounced effect on the reflected and refracted waves not only at a plane surface but also at a corrugated interface. Also, there are significant differences in the amplitude of irregular waves for the different angles of incidence with a given ratio of wavelength of corrugation to the wavelength of the incident $P$-wave.

Abubakar (1962a) and Dunkin and Eringen (1962) used perturbation method to study the problem of reflection of body waves from a rough surface of a semi-infinite elastic solid. In perturbation technique, the rough surface is regarded as a perturbation to a smooth plane and the consequent change in the scattering coefficients, due to the presence of roughness, is calculated. This approach requires that the height
deviation of the surface, away from the smooth plane, is everywhere small compared to the wavelength of the incident wave. In addition, the gradient of the surface must be small in comparison to unity. Abubakar (1962a) found that the reflected waves are composed of specularly reflected waves and various diffracted waves, propagating in both horizontal directions, if the wavelength of the incident wave is long compared with that of the corrugated surface. If the wavelength of the incident distortional wave is long compared with that of the surface, the amplitudes of some of the scattered waves decrease exponentially. In general, the phases of the wave changes on reflection and the phase angles of the reflected waves are functions of the wavelength of the corrugation and the angle of incidence. Later, Abubakar (1962b, c), using the perturbation technique, studied the reflection and refraction of $SH$-waves at an irregular interface between two uniform elastic solid half spaces. Abubakar (1963) discussed the effect of an irregular surface with an isolated irregularity like a trough or ditch on incident plane harmonic $P$-and $SV$-waves. The maximum depth of the pitch is assumed small compared to the wavelength of the incident wave. It is found that when either a $P$-or an $SV$-wave is incident on such a boundary, besides the specularly reflected $P$-and $SV$-waves whose amplitudes are independent of the curvature of the surface, there exist scattered waves travelling in various directions. Adams and Chang (1964) applied Weber integral method to solve the wave equation and studied the wave propagation phenomenon at an irregular infinite interface.

Aki and Larner (1970) investigated the surface motion of a layered medium with irregular interface due to an incident plane $SH$-wave. The effect of surface irregularity on the propagation of waves in an elastic plate was studied by Sumner and Deresiewicz (1972). They employed the method of perturbation to determine the scattered field for the first order in a small parameter descriptive of the height of the irregularity. They showed that the roots of the Rayleigh-Lamb frequency equation are symmetrically located about the axes of the reals and imaginaries. Hudson et al. (1973) measured the conversion coefficient for longitudinal to Rayleigh wave scattering experimentally, when the longitudinal elastic wave is incident on a rough section of an otherwise plane free surface. The longitudinal elastic waves are considered harmonic in time, and the roughness is considered in the form of linear symmetric grooves. They used the results to test the predictions of perturbation theory and to find the condition under which the results may be accepted satisfactory. Apparently the theory breaks down when
the surface slope exceeds 25° if the linear extent of the irregularity is small. They also observed that outside the range of application of the theory \(i.e.\) when the extent of the irregularity becomes large), the experimental results are interpreted in terms of energy balance, and approximate conversion coefficients are derived. They concluded that Rayleigh waves are attenuated to about a tenth of their energy in travelling a distance of ten wavelengths across regular grooves, a wavelength in width and with slopes of 10 – 25°. Chattopadhyay and Pal (1982) studied the propagation of \(SH\)-waves in an inhomogeneous medium with an irregular interface overlying an initially stressed elastic half space. Chakraborty \textit{et al.} (1983) discussed the effects of initial stress and irregularity on the propagation of \(SH\)-waves and observed that the presence of either the initial compressive stress or the irregularity in the interface has pronounced effect on the phase velocity of the waves. Kar \textit{et al.} (1986) studied the effect of irregularity and sandiness on the propagation of Love waves in a homogeneous isotropic elastic layer, situated over an infinite isotropic medium. The irregularity is expressed by a parabolic shape at the interface. They found that when the rigidity of the lower substratum is greater than that of the layer, the presence of either sand or irregularity or both at the interface has the effect of lowering the phase velocity of the waves, particularly in the low period range. Gupta (1987) studied the effect of lateral and vertical inhomogeneity on reflection and transmission of \(SH\)-waves at an irregular boundary using Rayleigh’s method of approximation. She has found that reflection and transmission coefficients are strongly influenced by the lateral and vertical inhomogeneity, amplitude of the corrugated boundary, wavelength of the corrugated interface, frequency and the velocity contrast of the half-spaces.

Ogilvy (1987) performed an excellent review of the literature that provide a comprehensive study of the theories in existence for the study of wave scattering from rough surfaces and highlighted the limitations of the perturbation technique and the Kirchhoff approximation. The results from experimental investigations of rough surfaces scattering are compared with the theoretical predictions. A comparison is also made between the assumed profiles, used in the theory, and the measured profiles. Paul and Michel (1988) used numerical modelling to investigate the effect of small-scale irregularities of a reflecting boundary on the elastic wave reflections. The scattered wave field is computed by using a discretized form of boundary integral equations and a plane-wave decomposition of seismic wave fields. They showed that corrugations with mean wave-
length of the order of, or smaller than, the seismic wavelength have little effect on the reflected \( P \)-wave. However, the pattern of \( P \)-to-\( S \) conversion is very different from that with a plane boundary. Scattered \( S \)-waves appear at postcritical angles for any angle of incidence of the \( P \) wave. The amplitude of these non-geometrical shear waves decreases rapidly with decreasing amplitude of the corrugation, or when the mean wavelength of the corrugations becomes larger than the dominant seismic wavelength. The local geometry of the irregularities has negligible effect on the scattered \( S \)-waves. Ding and Dravinski (1996) investigated the scattering of elastic \( SH \)-waves in a multilayered media with irregular interfaces and used an indirect boundary integral equation method. An extensive parametric error analysis is performed in order to assess the convergence of the proposed method. They concluded that the presence of layer irregularities may cause significant change in the surface response when compared with the corresponding flat-layer model response. Away from the irregularities, the amplitude of the scattered waves decreases and the response approaches the free-field one. The surface motion was found to be very sensitive upon the nature of the incident wave (angle of incidence and frequency), location of the observation site and geometry and material properties of the layered medium. The effects of irregular boundaries on the seismic waves and their induced responses qualitatively and quantitatively are investigated by Zhang and Shinozuka (1996) using the first order perturbation approach. It has been found that the effects are strongly dependent upon the relation between the wavenumber spectrum of the irregularity and the seismic waves under consideration or, more specifically, upon the relation of the wavenumber contents between the wavenumber spectrum of the irregularity and the seismic waves.

Tomar et al. (2002) studied the reflection and refraction of \( SH \)-waves at a corrugated interface between transversely isotropic and visco-elastic solid half spaces by using Rayleigh’s method. They showed that the reflection and refraction coefficients are proportional to the corrugation of the interface and are strongly influenced by the transverse isotropy and visco-elastic behavior of the half-spaces. Tarasenko et al. (2003) employed a new approach for the investigation of the effect of roughness on the propagation of the elastic surface waves. The presence of the rough layer is accounted by the modification of the boundary conditions. In the framework of this approach, they have investigated the propagation of Rayleigh waves on an isotropic substrate having rough surface. It is shown that the roughness leads to the linear dispersion of
the phase velocity of the surface waves, which is non-dispersive on the ideal surfaces. Kumar et al. (2003) studied the reflection and refraction of $SH$-waves at a corrugated interface between two different anisotropic and vertically heterogeneous elastic solid half-spaces and found that reflection and transmission coefficients are function of the corrugation of the interface and are strongly influenced by the anisotropy and heterogeneity of the half-spaces. Fu (2005) conducted comparisons of several approximation solutions to rough surface scattering for $SH$-waves. These approximations include Kirchhoff approximation theory, Taylor expansion-based perturbation theory, two-scale model, Rytov phase approximation and Born series method with each valid for a range of roughness scales.

1.4 Rayleigh’s method of approximation

Rayleigh’s method of approximation has been used in this thesis to find the effect of corrugation on the reflection and transmission coefficients due to an incident plane $SH$-wave at a corrugated interface between two elastic half-spaces. In order to describe the basic idea of the Rayleigh’s method, let us assume that a plane harmonic scalar wave of potential $\Phi_0$ be incident on the corrugated boundary of the half-space. The undulation of the boundary surface is only in one direction. Let us assume that the $x$-axis and $y$-axis be on the horizontal surface, while the $z$-axis is taken vertically downwards. We consider the equation of the corrugated boundary interface as

$$z = \zeta(x),$$

(1.21)

and the irregularity has a periodic character, i.e., $\zeta(x) = \zeta(x + L)$, where $\zeta$ is a periodic function of $x$, independent of $y$, whose mean value is zero and $L$ is the period of irregularity. Let the incidence plane of the wave coincides with the plane $x - z$ of chosen cartesian coordinate system. According to the Rayleigh’s method, the total field over the surface defined above can be represented in the form of a superposition of plane waves (Sobczyk, 1985)

$$\Phi(x, z) = \exp[ik(x \sin \theta_0 - z \cos \theta_0)] + \sum_{n=-\infty}^{\infty} A_n \exp[ik(x \sin \theta_n + z \cos \theta_n)].$$

(1.22)

Here, the first term in the equation (1.22) represents the incident wave with unit amplitude and the series on the right hand side represents a scattered field consisting
of a wave reflected specularly \((n = 0)\) and scattered waves of higher orders. The direction of propagation of the \(n^{th}\) component of the spectrum forms with the \(z\)-axis an angle \(\theta_n\), defined by the formula:

\[
\sin \theta_n = \sin \theta_0 + \frac{nq}{k},
\]

where \(\theta_0\) is the angle of incidence, \(k\) is the wavenumber and \(q = \frac{2\pi}{L}\). For large \(n\), from the above result, we see that \(|\sin \theta_n| > 1\) and consequently \(\cos \theta_n\) becomes an imaginary quantity. If \(\cos \theta_n = i|B_n|\), then the second part of equation (1.22) will becomes

\[
\sum_{n=-\infty}^{\infty} A_n \exp(ikx \sin \theta_n) \exp(-k|B_n|z)
\]

This represents the spectrum of waves propagating from the surface into the interior of the medium and of the inhomogeneous plane waves (corresponding to the positive imaginary parts of \(\cos \theta_n\)) propagating along the rough surface and vanishing exponentially with depth. Here, the coefficients \(A_n\) defining the amplitude of the \(n^{th}\) component of the spectrum are unknown. They are determined from the appropriate boundary conditions on the surface. The usual boundary conditions are the continuity of the displacements and traction. Equating to zero the right hand side of the expression (1.22) or the normal derivative of this expression on to an uneven surface, we obtain an infinite system of equations for the unknown coefficients \(A_n\). For a surface whose height is small in comparison with the wavelength of the incident wave and whose gradients are small, an approximate solution of that system of equations can be obtained.

### 1.5 Plan of thesis

In this thesis, we have studied the reflection and transmission of \(SH\)-waves at a corrugated interface between two elastic solid half-spaces with different elastic properties. The reflection and transmission coefficients have been computed for both regularly and irregularly reflected and transmitted \(SH\)-waves. The term "regular" refers to the waves that originate from a plane interface, while the term "irregular" denotes all the wave components that originate from the corrugation of the interface.

Using general analytical expression for the displacements and stresses in both media, as well as Snell’s law (for the regular wave components), the Spectrum relation
(for the irregular wave components) and Rayleigh's method of approximation, the displace- 
ment continuity and stress equilibrium are expressed at the corrugated interface, 
resulting into general expressions for the reflection and transmission coefficients. We 
have assumed that both amplitude and slope of the corrugated interface are small.

Next, in a special case, a simple periodic interface has been considered and solu- 
tions for the first/second order approximation of the corrugation have been given for 
the reflection and transmission coefficients due to an incident plane $SH$-wave. The 
effect of amplitude of the corrugation, the elastic properties and the frequency of the 
incident wave on these coefficients have been studied numerically for a specific model 
and the results are presented graphically. The results of some earlier workers in the 
field have been reduced as particular cases. The summary of the present research work 
under the division of the chapters is given below.

Chapter-I is based on the introduction of the subject. In this chapter the basic 
literature of the elastic waves in different medium (eg anisotropic medium, heteroge- 
neous medium, visco-elastic medium, initially stressed medium) has been reviewed. 
The mathematical description of basic equations of waves in isotropic, visco-elastic 
and initially stressed medium have been presented. Then the Rayleigh's method of 
approximation used in the theses has been explained and the plan of thesis has also 
been given.

Chapter II is concerned with a problem of reflection and transmission of an incident 
plane $SH$-wave at a corrugated interface between two anisotropic heterogeneous elastic 
solid half spaces. Both the half spaces are taken transversely isotropic and laterally 
and vertically heterogeneous. The expressions for the reflection and transmission co- 
efficients have been obtained in closed form for the first-order approximation of the 
corrugation. Numerical computations are performed for the case of a particular cor- 
rugated interface showing the effect of heterogeneity and anisotropy on the reflection 
and transmission coefficients. The variations of these coefficients at a plane and at a 
corrugated interface with respect to the angle of incidence, by taking different values of 
anisotropy and heterogeneity parameters, have been presented graphically. The effect 
of amplitude of corrugated interface and frequency of the incident wave has also been 
observed on the reflection and transmission coefficients.

In Chapter III, a problem of reflection and transmission of shear wave incident upon 
a corrugated interface between two different monoclinic solid half-spaces has been in-
vestigated and the expressions for the reflection and transmission coefficients for first and the second approximation of the corrugation are obtained. The amplitudes of these coefficients corresponding to the plane as well as corresponding to the corrugated interface have been obtained and plotted against the corrugation parameter, elastic parameters of the media, the frequency of the incident wave and the angle of incidence.

Chapter IV deals with the study of reflection and transmission coefficients when a plane $SH$-wave is incident at a corrugated interface between two isotropic, laterally and vertically heterogeneous visco-elastic solid half-spaces. The density and complex rigidity of each medium are considered to vary exponentially along the horizontal and the vertical directions. The effects of the amplitude of corrugation, the heterogeneity, the angle of incidence, the frequency, the angle between propagation and attenuation vectors and the visco-elasticity of the media on these coefficients have been obtained numerically for a specific model and the results obtained are performed graphically. Comparison of these coefficients has been made between those in viscoelastic media and those in uniform elastic media.

In Chapter V, a problem of reflection and transmission of $SH$-waves at a corrugated interface between two laterally and vertically inhomogeneous anisotropic elastic and inhomogeneous isotropic visco-elastic solid half-spaces, has been studied. The expressions of the reflection and transmission coefficients have been derived to study the effects of corrugation of the interface, anisotropy, visco-elasticity, frequency, angle of the incidence wave and heterogeneity of the half-spaces. Comparison of these coefficients has been made between those in viscoelastic media and those in uniform elastic media. From the obtained results, the effect of corrugated interface on the reflection and transmission coefficients has been depicted graphically.

In Chapter VI, we have investigated a problem of reflection and transmission due to a plane $SH$-wave incident at a corrugated interface between a dry sandy half-space and an anisotropic elastic solid half-space. The derived expressions of the reflection and transmission coefficients are utilized to discuss the effect of the amplitude of corrugation, elastic properties of the half-spaces and also of the angle of incidence on these coefficients. The energy partition relation is derived and the variations of the energy ratios of various reflected and refracted wave with the increase of the sandiness parameter have been studied. Numerical computations have been performed for a particular model and the variations of these coefficients is presented graphically for
different parametric values. Many interesting results characterizing the sandiness, the anisotropy, the corrugation of the interface, the frequency and the angle of the incidence are distinctly marked.

Chapter VII deals with a problem of plane $SH$-wave incident at a corrugated interface between an initially stressed slow elastic half-space lying over an anisotropic elastic solid half-space. The expressions of reflection and transmission coefficients and the energy partitioning equation for the first order approximation of the corrugation have been derived. Effects of the sandiness, the amplitude of corrugation of the interface, the initial stress, the anisotropy and the angle of the incidence on these coefficients have been studied for a periodic interface analytically and numerically for a specific model. In all these problems, closed form expressions for the reflection and transmission coefficients are obtained for the case of a sinusoidal corrugated interface.

The following research papers have been published/communicated for publication
1. Reflection and transmission of $SH$-waves at a corrugated interface between two laterally and vertically heterogeneous anisotropic elastic half space, *Earth Planet and Space*, 55, (2003), 531-547.
5. Shear waves at a corrugated interface between anisotropic elastic and viscoelastic solid half-spaces, Accepted in *Journal of Seismology*.
6. Transmission of $SH$-waves at a corrugated interface between an initially stressed sandy elastic half-space and an anisotropic elastic solid half-space, Communicated to *Journal of Sound and Vibration*.