CHAPTER 6
TWO-SAMPLE PROBLEM

6.1 In the last three chapters we discussed the case when the individuals in a population are simultaneously exposed to two risks of failure. Now we consider the problem of comparing two populations in which the same two risks of failure operate simultaneously on the individuals. For example, it would be of great interest to compare people living in Bombay with those living in Bhopal with respect to the risks of dying from, say, road accidents and air pollution. Or, whether the individuals in India or in U.S.A. are more likely to die from cancer and tuberculosis.

One approach would be to compare each risk individually, since they are assumed to operate independently. However, since the risks are competing with each other, it is thought to be more meaningful to consider the problem as that of comparing two bivariate populations.

Let $X$ denote the random failure time due to Risk 1 in population I and $Y$ the random failure time of the same risk in population II. Similarly, let $U$ and $V$ denote the random failure times due to Risk II in populations I and II, respectively.

Let $\{(X_1, U_1), (X_2, U_2), \ldots, (X_n, U_n)\}$ be a random sample of size $n$ from the first population and $\{(Y_1, V_1), (Y_2, V_2), \ldots, (Y_m, V_m)\}$ be an independent random sample of size $m$ from the second population. The two risks are assumed to act independently in each of the two populations. Let $X_1(U_1), X_2(U_2), \ldots, X_n(U_n)$ be i.i.d. observations from the first population with absolutely
continuous distribution function \( F_1(F_2) \). Similarly, let
\( Y_1(V_1), Y_2(V_2), \ldots, Y_m(V_m) \) be i.i.d. observations from the second
population with absolutely continuous distribution function
\( G_1(G_2) \). \( F_1, F_2, G_1 \) and \( G_2 \) are assumed to belong to the class \( \mathcal{F} \)
of all absolutely continuous distribution functions \( H(x) \) with
\( H(x) = 0 \) for \( x < 0 \).

The failure of any individual in any sample is due to
only one cause. Therefore, both \((X_i, U_i)\) and \((Y_j, V_j)\) are not
observable. In practice we observe \((T_1, \delta_1), (T_2, \delta_2), \ldots, (T_n, \delta_n)\)
from the first sample and \((S_1, \epsilon_1), (S_2, \epsilon_2), \ldots, (S_m, \epsilon_m)\)
from the second sample, where,

\[
T_i = \min(X_i, U_i)
\]

\[
= \begin{cases} 
X_i, & \text{if } X_i \leq U_i, \\
U_i, & \text{if } U_i < X_i.
\end{cases}
\]

\[
\delta_i = \begin{cases} 
1, & \text{if } X_i > U_i, \\
0, & \text{if } X_i \leq U_i,
\end{cases}
\]

\( i = 1, 2, \ldots, n. \)

\[
S_j = \min(Y_j, V_j)
\]

\[
= \begin{cases} 
Y_j, & \text{if } Y_j \leq V_j, \\
V_j, & \text{if } V_j < Y_j,
\end{cases}
\]

\[
\epsilon_j = \begin{cases} 
1, & \text{if } Y_j > V_j, \\
0, & \text{if } Y_j \leq V_j,
\end{cases}
\]

\( j = 1, 2, \ldots, m. \)

Thus, the available information is the cause of failure
and the time to failure of all the \( N = n + m \) individuals in the
two samples. On the basis of this information we wish to test
the hypothesis,
\[ H_0 : F_1(x) = G_1(x), F_2(x) = G_2(x), \text{ for every } x \geq 0 \]
against the alternative,
\[ H_A : F_1(x) < G_1(x), F_2(x) < G_2(x), \text{ for every } x > 0 \]
with strict inequality over a set of nonzero probability.

We construct some tests based on heuristic grounds for the above problem, discuss their properties and compare them in the Pitman ARE sense.

6.2 The Statistic \( W_1 \)

The information available consists of times to failure and causes of failure of the \( N \) individuals. First we propose the use of the Mann-Whitney test statistic based on the actual times to failure \( T_1, T_2, \ldots, T_n \) and \( S_1, S_2, \ldots, S_m \) of the individuals in the two samples.

Let \( F_1, F_2, G_1 \) and \( G_2 \) be the survival functions corresponding to the four distribution functions. Let \( F \) and \( G \) denote the survival functions of \( T = \min(X, U) \) and \( S = \min(Y, V) \), respectively. Then, because of the independence of \( X \) and \( U \), and of \( Y \) and \( V \), it follows that
\[ F(x) = F_1(x) F_2(x), \text{ for all } x \geq 0 \]
\[ G(x) = G_1(x) G_2(x), \text{ for all } x \geq 0 \]

Therefore, under \( H_0 \),
\[ F(x) = G(x), \text{ for all } x \geq 0. \]
And, under \( H_A \),
\[ F(x) < G(x), \text{ for all } x \geq 0 \]
with strict inequality over a set of positive probability, where \( F \) and \( G \) are the distribution functions of the times to failure \( T \) and \( S \), respectively.

Thus, under the alternative \( H_A \) we expect \( T \) to be stochastically greater than \( S \). That is, subjects in the second sample will tend to die before the subjects in the first sample. Hence, we propose a test for testing \( H_0 \) against \( H_A \) based on the test statistic

\[
W_1 = \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} \varphi(T_i, S_j)
\]

(6.1)

where,

\[
\varphi(T_i, S_j) = \begin{cases} 
1, & \text{if } T_i > S_j, \\
0, & \text{if } T_i \leq S_j.
\end{cases}
\]

(6.2)

The above kernel can be expressed in terms of the original observations as follows:

\[
\varphi(T_i, S_j) = \begin{cases} 
1, & \text{if } \min(X_i, U_i) > \min(Y_j, V_j), \\
0, & \text{if } \min(X_i, U_i) \leq \min(Y_j, V_j).
\end{cases}
\]

(6.3)

Thus, \( W_1 \) is the Wilcoxon-Mann-Whitney (1945) statistic based on the times to failure of the individuals in the two samples. It is the two sample \( U \)-statistic corresponding to the kernel (6.2) of degree (1,1).

The test for the above problem is to reject \( H_0 \) in favour of \( H_A \) if \( W_1 > w_{1-a,n,m} \), where \( w_{1-a,n,m} \) is the critical point obtained from the null distribution of the statistic \( W_1 \) so that the test has a level of significance equal to \( \alpha \).
Let $R_i$ denote the rank of $T_i$ in the combined arrangement of $T_1, T_2, \ldots, T_n$ and $S_1, S_2, \ldots, S_m$. Then, it is known (Lehmann (1975)) that

$$mn W_1 = \sum_{i=1}^{n} R_i - \frac{n(n+1)}{2} \quad (6.4)$$

**Exact Expectation and Variance of the Statistic $W_1$**

It is well known that,

$$E[W_1] = \int_{0}^{\infty} F(x) \, dG(x) \quad (6.5)$$

Under $H_0$,

$$E(W_1) = \frac{1}{2} \quad (6.6)$$

Under $H_A$,

$$E[W_1] > \frac{1}{2} \quad (6.7)$$

Also under $H_0$,

$$V(W_1) = \frac{n+m+1}{12nm} \quad (6.8)$$

**Asymptotic Distribution of $W_1$**

$W_1$ is a two sample generalized U-statistic based on square integrable kernel. The asymptotic normality of such generalized U-statistics has been discussed by Lehmann (1963), Puri and Sen (1971), etc.

Hence, we have the following well known theorem.

**Theorem 6.1**

The asymptotic distribution of $N^{1/2} [W_1 - E(W_1)]$ as $N \to \infty$ in such a way that $p_N = \frac{n}{N}$ tends to $p$, $0 < p < 1$, is normal with mean zero and variance

$$\sigma^2_{W_1} = \frac{\xi_{10}}{p} + \frac{\xi_{01}}{q}, \quad q = 1-p,$$
where, \( \xi_{10} = \text{Cov}[\phi(T_1, S_1), \phi(T_1, S_2)] \),
\( \xi_{01} = \text{Cov}[\phi(T_1, S_1), \phi(T_2, S_1)] \)

\[
\text{Under } H_0, \sigma^2_{W_1} = \frac{1}{12pq} .
\]  

(6.9)

The unbiasedness and consistency of \( W_1 \) follows immediately from the results proved above. Also see Hettmansperger (1984), Randles and Wolfe (1979).

### 6.3 The Statistic \( W_2 \)

The statistic \( W_1 \) looks at only the times to failure of each of the individuals in the two samples. But the causes of failure of these individuals are also known. We, now, simultaneously look at the causes and times to failure of the individuals in the two samples.

Eight mutually exclusive arrangements of the pairs \((T_i, \delta_i)\) and \((S_j, \varepsilon_j)\) are possible.

Now suppose that \( T_i > S_j \) and \( \delta_i = 1, \varepsilon_j = 1 \). \( \delta_i = 1 \) would mean that \( X_i > U_i, T_i = \min(X_i, U_i) = U_i ; \varepsilon_j = 1 \) would mean that \( Y_j > V_j, S_j = \min(Y_j, V_j) = V_j \) and \( T_i > S_j \) would imply that \( U_i > V_j \). Combining the above information we get the following arrangement of \((X_i, U_i)\) and \((Y_j, V_j)\)

\[
X_i > U_i > V_j \text{ and } Y_j > V_j
\]  

(6.10)

Similarly we analyse the other seven arrangements. The eight possible arrangements are represented in the table that follows.
(III.9)

<table>
<thead>
<tr>
<th>$T_X &lt; T_n$</th>
<th>$T_X &lt; T_n$</th>
<th>$T_n &lt; T_X$</th>
<th>$T_n &lt; T_X$</th>
<th>$f_S &gt; T_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_X &lt; T_A &lt; T_n$</td>
<td>$T_X &lt; T_A &lt; T_n$</td>
<td>$T_n &lt; T_A$</td>
<td>$T_n &lt; T_A$</td>
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<tr>
<td>$T_X &lt; T_A &lt; T_n$</td>
<td>$T_X &lt; T_A &lt; T_n$</td>
<td>$T_n &lt; T_X$</td>
<td>$T_n &lt; T_X$</td>
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</tr>
<tr>
<td>$f_A &lt; f_A &lt; T_n$</td>
<td>$f_A &lt; f_A &lt; T_n$</td>
<td>$f_A &lt; f_A &lt; T_n$</td>
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<td>$0 = 3 : 0 = 9$</td>
<td>$0 = 3 : 1 = 9$</td>
<td>$1 = 3 : 1 = 9$</td>
<td>$1 = 3 : 1 = 9$</td>
</tr>
</tbody>
</table>
Consider the statistic $B = [B_1, B_2]$ (6.12)

where, $B_1 = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_5(X_i, U_i, Y_j, V_j)$ (6.13)

$B_2 = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_6(X_i, U_i, Y_j, V_j)$ (6.14)

\[
\phi_5(X_i, U_i, Y_j, V_j) = \begin{cases} 
1, & \text{if } \min(X_i, U_i) > \min(Y_j, V_j), Y_j > Y_j, \\
-1, & \text{if } \min(X_i, U_i) < \min(Y_j, V_j), U_i > X_i, \\
0, & \text{otherwise.} 
\end{cases}
\] (6.15)

\[
\phi_6(X_i, U_i, Y_j, V_j) = \begin{cases} 
1, & \text{if } \min(X_i, U_i) > \min(Y_j, V_j), Y_j > V_j, \\
-1, & \text{if } \min(X_i, U_i) < \min(Y_j, V_j), U_i > Y_i, \\
0, & \text{otherwise.} 
\end{cases}
\] (6.16)

The statistics $B_1$ and $B_2$ as defined by (6.13) and (6.14) are respectively the generalized U-statistics associated with the two-sample symmetric kernels, $\phi_5(X_i, U_i, Y_j, V_j)$ and $\phi_6(X_i, U_i, Y_j, V_j)$ respectively, each of degree $(1,1)$.

The above kernels can be expressed in terms of observable pairs $(T_i, \delta_i)$ and $(S_j, \varepsilon_j)$ as follows:

\[
\phi_5(X_i, U_i, Y_j, V_j) = \begin{cases} 
1, & \text{if } T_i > S_j, \varepsilon_j = 0, \\
-1, & \text{if } T_i < S_j, \delta_i = 0, \\
0, & \text{otherwise.} 
\end{cases}
\] (6.17)
\[ \varphi_6(X_i, U_i, Y_j, V_j) = \begin{cases} 
1, & \text{if } T_i > S_j, \varepsilon_j = 1, \\
-1, & \text{if } T_i < S_j, \delta_i = 1, \\
0, & \text{otherwise}. 
\end{cases} \] (6.18)

The rationale of the values which the above kernels take is as follows. For \( \varphi_5(X_i, U_i, Y_j, V_j) \), we assign a score 1 if an \( X \) observation is greater than a \( Y \) observation and \(-1\) if the opposite is observed. For example, the arrangement \( T_i > S_j, \delta_i = 1, \varepsilon_j = 0 \) implies that \( X_i > U_i > Y_j, V_j > Y_j \). Thus an \( X \) observation exceeds a \( Y \) observation and a score 1 is assigned to such an arrangement. Similarly, for \( \varphi_6(X_i, U_i, Y_j, V_j) \), we assign a score 1 if a \( U \) observation is greater than a \( V \) observation and \(-1\) in the opposite case.

Gehan (1965) and Gilbert (1963) proposed tests based on statistics of the type \( B^1 \) for testing \( H_0 : F^1(x) = G^1(x) \) against the alternative \( A : F^1(x) \leq G^1(x) \) in the presence of censoring. In these cases \( U_1, U_2, \ldots, U_n \) and \( V_1, V_2, \ldots, V_m \) are censoring random variables and the authors are not interested in testing the equality of censoring distributions \( F_2 \) and \( G_2 \). However, when we have competing risks, \( X_1 \) (\( Y_1 \)) censors \( U_1 \) (\( V_1 \)) and vice-versa. Thus equality or departure from equality of \( F_2 \) and \( G_2 \) is as important as equality or departure from equality of \( F_1 \) and \( G_1 \). Hence, for our testing problem, we look at the statistic \( B = [B_1, B_2] \) instead of \( B_1 \) only.

It becomes difficult to evaluate \( B_1 \) and \( B_2 \) as the sample sizes increase. However, both \( B_1 \) and \( B_2 \) can be expressed as functions of ranks of the observed minima \( T_1, T_2, \ldots, T_n \), and \( S_1, S_2, \ldots, S_m \).
First, take a combined arrangement of all the \( n \) \( T_i \)'s and all those \( S_j \)'s for which corresponding \( \epsilon_j \) is 1. Let \( R_1, R_2, \ldots, R_n \) denote the ranks of \( T_1, T_2, \ldots, T_n \) in this combined arrangement. Similarly, let \( Q_1, Q_2, \ldots, Q_m \) denote the ranks of \( S_1, S_2, \ldots, S_m \) in the combined arrangement of \( S_1, S_2, \ldots, S_m \) and all those \( T_i \)'s for which \( \delta_i = 1 \). Then

\[
nmB_2 = \left[ \sum_{i=1}^{n} R_i - \frac{n(n+1)}{2} \right] - \left[ \sum_{j=1}^{m} Q_j - \frac{m(m+1)}{2} \right]
\]

\[
= \sum_{i=1}^{n} R_i - \sum_{j=1}^{m} Q_j + \frac{(m-n)(m+n+1)}{2} \tag{6.19}
\]

Similarly,

\[
nmB_1 = \sum_{i=1}^{n} R_i^* - \sum_{j=1}^{m} Q_j^* + \frac{(m-n)(m+n+1)}{2} \tag{6.20}
\]

where \( R_i^* \) is the rank of \( T_i \) in the combined arrangement of \( T_1, T_2, \ldots, T_n \) and all those \( S_j \)'s for which corresponding \( \epsilon_j = 0 \) and \( Q_j^* \) is the rank of \( S_j \) in the combined arrangement of \( S_1, S_2, \ldots, S_m \) and all those \( T_i \)'s for which \( \delta_i = 0 \).

**Exact Expectation of the Statistic \( B \)**

\[
E[B] = E[B_1, B_2]'
\]

\[
= [E(B_1), E(B_2)]' \tag{6.21}
\]

where,

\[
E[B_1] = E[\phi_2(X_i, U_i, Y_j, V_j)]
\]

\[
= P[X_i > U_i > Y_j, V_j > Y_j] + P[U_i > X_i > Y_j, V_j > Y_j] - P[Y_j > V_j > X_i, U_i > X_i] - P[V_j > Y_j > X_i, U_i > X_i]
\]
\begin{align*}
&= \iint_{0 < x < y < \infty} [\Phi_1(x) - \Phi_1(y)] \Phi_2(x) \, d\Phi_2(y) \, d\Phi_1(x) \\
&+ \iint_{0 < x < y < \infty} [\Phi_2(x) - \Phi_2(y)] \Phi_1(x) \, d\Phi_1(y) \, d\Phi_2(x) \\
&- \iint_{0 < x < y < \infty} [\Phi_2(x) - \Phi_2(y)] \Phi_1(x) \, d\Phi_1(y) \, d\Phi_2(x) \\
&+ \iint_{0 < x < y < \infty} [\Phi_1(x) - \Phi_1(y)] \Phi_2(x) \, d\Phi_2(y) \, d\Phi_1(x) \\
&= \int \int_{0 < x < y < \infty} [\Phi_2(x) - \Phi_2(y)] \Phi_1(x) \, d\Phi_1(y) \, d\Phi_2(x) \\
&+ \int \int_{0 < x < y < \infty} [\Phi_1(x) - \Phi_1(y)] \Phi_2(x) \, d\Phi_2(y) \, d\Phi_1(x) \\
&- \int \int_{0 < x < y < \infty} [\Phi_2(x) - \Phi_2(y)] \Phi_1(x) \, d\Phi_1(y) \, d\Phi_2(x) \\
&- \int \int_{0 < x < y < \infty} [\Phi_1(x) - \Phi_1(y)] \Phi_2(x) \, d\Phi_2(y) \, d\Phi_1(x) \\
&= \int \int_{0 < x < y < \infty} [\Phi_2(x) - \Phi_2(y)] \Phi_1(x) \, d\Phi_1(y) \, d\Phi_2(x) \\
&+ \int \int_{0 < x < y < \infty} [\Phi_1(x) - \Phi_1(y)] \Phi_2(x) \, d\Phi_2(y) \, d\Phi_1(x) \\
&- \int \int_{0 < x < y < \infty} [\Phi_2(x) - \Phi_2(y)] \Phi_1(x) \, d\Phi_1(y) \, d\Phi_2(x) \\
&- \int \int_{0 < x < y < \infty} [\Phi_1(x) - \Phi_1(y)] \Phi_2(x) \, d\Phi_2(y) \, d\Phi_1(x) \\
&= \int \int_{0 < x < y < \infty} [\Phi_2(x) - \Phi_2(y)] \Phi_1(x) \, d\Phi_1(y) \, d\Phi_2(x) \\
&+ \int \int_{0 < x < y < \infty} [\Phi_1(x) - \Phi_1(y)] \Phi_2(x) \, d\Phi_2(y) \, d\Phi_1(x) \\
&- \int \int_{0 < x < y < \infty} [\Phi_2(x) - \Phi_2(y)] \Phi_1(x) \, d\Phi_1(y) \, d\Phi_2(x) \\
&- \int \int_{0 < x < y < \infty} [\Phi_1(x) - \Phi_1(y)] \Phi_2(x) \, d\Phi_2(y) \, d\Phi_1(x) \\
&\text{Substituting } F_1 = G_1, F_2 = G_2, \text{ we get under } H_0 \\
E[B_1] &= E[B_2] = 0 \\
\Rightarrow E[B] &= 0
\end{align*}
It is very difficult to calculate the exact variance of the statistic $B$ and has not been attempted.

**Asymptotic Distribution of $B$**

The kernels $\phi_1$ and $\phi_2$ are all square integrable. The asymptotic normality of the generalized $U$-statistics based on such kernels has been discussed by Bhapkar (1961), Lehmann (1963) and Puri and Sen (1971), among others. We shall make use of their following result.

Let $\theta_i(F_1, F_2, G_1, G_2)$ be an estimable function with square integrable symmetric kernel $\phi_i((x_1, u_1), \ldots, (x_{r_i}, u_{r_i}), (y_1, v_1), \ldots, (y_{s_i}, v_{s_i}))$ for $i = 1, 2, \ldots, k$. Let

$U_i((x_1, u_1), \ldots, (x_n, u_n); (y_1, v_1), \ldots, (y_m, v_m))$ be the generalized $U$-statistic associated with $\phi_i$, $i = 1, 2, \ldots, k$. The limiting distribution of $N^{1/2}[U((x, u), (x, v)) - \theta(E, \xi)]$ as $N \to \infty$ in such a way that $p_n = n/N$ tends to $p$, $0 < p < 1$, is $k$-variate normal with mean vector $0$ and dispersion matrix,$$
abla = \begin{pmatrix}
\sigma_{ij}
\end{pmatrix}, \quad i, j = 1, 2, \ldots, k.
$$

Here,$$
\sigma_{ij} = \frac{r_i r_j}{p} \xi_{10}(i, j) + \frac{s_is_j}{q} \xi_{01}(i, j),
$$

$$
\xi_{10}(i, j) = E[\psi_{i, 10}(x_1, u_1), \psi_{j, 10}(x_1, u_1)]
$$

$$
\xi_{01}(i, j) = E[\psi_{i, 01}(y_1, v_1), \psi_{j, 01}(y_1, v_1)]
$$

where,$$
\psi_{i, 10}(x_1, u_1) = E[\phi_i((x_1, u_1), (x_2, u_2), \ldots, (x_{r_i}, u_{r_i}); (y_1, v_1), (y_2, v_2), \ldots, (y_{s_i}, v_{s_i})) - \theta_i]
$$
Theorem 6.2  The joint asymptotic distribution of
$N^{1/2} \left[ B - E(B_1), B_2 - E(B_2) \right]$ as $N \to \infty$, in such a way that
$p_N = \frac{p}{N}$ tends to $p$, $0 < p < 1$, is bivariate normal with mean
vector $\mathbf{0}$ and dispersion matrix.

$$
\Sigma = \begin{bmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{bmatrix}
$$

The expressions for $\sigma_{11}$, $\sigma_{22}$ and $\sigma_{12}$, under $H_0$, are given later
by (6.25), (6.26) and (6.27) on pages 171, 172, and 179, respectively.

Proof: The joint asymptotic normality follows from the results
quoted above. We have already obtained the null and non-null
expectations of $B_1$ and $B_2$. Here, we evaluate the asymptotic
null dispersion matrix.

Calculation of $\sigma_{11}$

Here $r_1 = s_1 = 1,$

$$
\Psi_{1,01}(y_1, v_1) = E[\phi_1((X_1, U_1), (X_2, U_2), \ldots, (X_{r_1}, U_{r_1}));
(y_1, v_1), (y_2, v_2), \ldots, (y_{s_1}, v_{s_1}) - \theta_i],
$$

and $q = 1 - p$. 

\[ + I(u_1 > x_1) \int_0^{x_1} \bar{G}_2(y) \, dG_1(y) \]
\[ - I(u_1 > x_1) \int_{x_1}^\infty \bar{G}_1(y) \, dG_2(y) \]
\[ - I(u_1 > x_1) \int_{x_1}^\infty \bar{G}_2(y) \, dG_1(y) \]

where,
\[
I(a > b) = \begin{cases} 
1, & \text{if } a > b \\
0, & \text{otherwise}
\end{cases}
\]

which, after simplification, reduces to
\[
\begin{align*}
& \begin{cases} 
G_1(x_1) - \int_0^{x_1} G_2(y) \, dG_1(y) - \bar{G}_1(x_1) \bar{G}_2(x_1), & \text{if } u_1 > x_1 \\
I_1 \text{ (say)}
\end{cases} \\
& \begin{cases} 
G_1(u_1) - \int_0^{u_1} G_2(y) \, dG_1(y) = I_2 \text{ (say)}, & \text{if } x_1 > u_1 \\
0
\end{cases}
\end{align*}
\]

Therefore, under \( H_0 \),
\[
\xi_{10}(1,1) = \mathbb{E}[\chi^2_{1,10} (X_1, U_1)]
\]
\[
= \int \int_{0 < x_1 < u_1 < \infty} I_1^2 \, dF_1(x_1) \, dF_2(u_1)
\]
\[
+ \int \int_{0 < u_1 < x_1 < \infty} I_2^2 \, dF_1(x_1) \, dF_2(u_1)
\]

Here,
\[
\int \int_{0 < u_1 < x_1 < \infty} I_2^2 \, dF_1(x_1) \, dF_2(u_1)
\]
\[
= \int \int_{0 < u_1 < x_1 < \infty} \{F_2^2(u_1) - 2F_1(u_1) \int_0^{u_1} F_2(y) \, dF_1(y) + (\int_0^{u_1} F_2(y) \, dF_1(y))^2\} \times dF_1(x_1) \, dF_2(u_1)
\]
\[= \int \int F_2^2(u_1) \, dF_1(x_1) \, dF_2(u_1) + \int \int (\int F_2(y) \, dF_1(y))^2 \, x \, dF_1(x_1) \, dF_2(u_1)\]
\[-2 \int \int F_1(u_1) \{ \int F_2(y) \, dF_1(y) \} \, dF_1(x_1) \, dF_2(u_1)\]
\[= A + B - C, \text{ (say)}\]
\[A = \int \int F_2^2(u_1) \, dF_1(x_1) \, dF_2(u_1)\]
\[= \int F_2^2(u_1) \, dF_2(u_1) - \int F_3^3(u_1) \, dF_2(u_1)\]
\[= 3 \int F_2^2(u_1) \, F_2(u_1) \, dF_1(u_1) - 2 \int F_1(u_1) \, F_2(u_1) \, dF_1(u_1)\]
\[C = 2 \int \int F_1(u_1) \{ \int F_2(y) \, dF_1(y) \} \, dF_1(x_1) \, dF_2(u_1)\]
\[= 2 \int F_2^3(u_1) \, F_1(u_1) \{F_1(u_1) \, F_2(u_1) - \int F_1(y) \, dF_2(y)\} \, dF_2(u_1)\]
\[= 2 \int F_2^3(u_1) \, F_2(u_1) \, dF_2(u_1) - 2 \int F_1^3(u_1) \, F_2(u_1) \, dF_2(u_1)\]
\[-2 \int \int F_1(u_1) \, F_1(y) \, dF_2(y) \, dF_2(u_1)\]
\[+ 2 \int \int F_2^2(u_1) \, F_1(y) \, dF_2(y) \, dF_2(u_1)\]

which, after simplification, reduces to
\[= 6 \int F_2^2(y) \, F_2^2(y) \, dF_1(y) - 2 \int F_1(y) \, F_2^2(y) \, dF_1(y)\]
\[-2 \int_0^\infty F_1(y) F_2^2(y) \, dF_1(y)\]

\[+ 2 \int_{0 < y < u_1 < \infty} F_1(y) F_2(u_1) \, dF_1(u_1) \, dF_2(y)\]

\[-4 \int_{0 < y < u_1 < \infty} F_1(u_1) F_2(u_1) F_1(y) \, dF_1(u_1) \, dF_2(y)\]

Therefore,

\[\int_{0 < x < u_1 < \infty} \int_0^u I_2^2 \, dF_1(x_1) \, dF_2(u_1)\]

\[= 3 \int_0^\infty F_1^2(y) F_2(y) \, dF_1(y) - 2 \int_0^\infty F_1(y) F_2(y) \, dF_1(y)\]

\[-6 \int_0^\infty F_1^2(y) F_2^2(y) \, dF_1(y) + 4 \int_0^\infty F_1^2(y) F_2^2(y) \, dF_1(y)\]

\[-2 \int_{0 < y < u_1 < \infty} F_1(y) F_2(u_1) \, dF_1(u_1) \, dF_2(y)\]

\[+ 4 \int_{0 < y < u_1 < \infty} F_1(u_1) F_2(u_1) F_1(y) \, dF_1(u_1) \, dF_2(y)\]

\[+ \int_{0 < u_1 < \infty} \left[ \int_0^u F_2(y) \, dF_1(y) \right]^2 \, dF_1(x_1) \, dF_2(u_1)\]

And,

\[\int_{0 < x_1 < u_1 < \infty} \int_0^u I_1^2 \, dF_1(x_1) \, dF_2(u_1)\]

\[= \int_{0 < x_1 < u_1 < \infty} F_1^2(x_1) \, dF_1(x_1) \, dF_2(u_1)\]

\[+ \int_{0 < x_1 < u_1 < \infty} F_2^2(x_1) \, dF_1(x_1) \, dF_2(u_1)\]
\[ \begin{align*}
&+ \int \int_{0 < x_1 < u_1 < \infty} (\int_0^{x_1} F_2(y) \, dF_1(y))^2 \, dF_1(x_1) \, dF_2(u_1) \\
&- 2 \int \int_{0 < x_1 < u_1 < \infty} F_1(x_1) \, F_1(x_1) \, F_2(x_1) \, dF_1(x_1) \, dF_2(u_1) \\
&- 2 \int \int_{0 < x_1 < u_1 < \infty} F_1(x_1) (\int_0^{x_1} F_2(y) \, dF_1(y)) \, dF_1(x_1) \, dF_2(u_1) \\
&+ 2 \int \int_{0 < x_1 < u_1 < \infty} F_1(x_1) \, F_2(x_1) (\int_0^{x_1} F_2(y) \, dF_1(y)) \, dF_1(x_1) \, dF_2(u_1) \\
&= A_1 + A_2 + A_3 - A_4 - A_5 + A_6, \text{(say)}
\end{align*} \]
Combining the expressions for $A$, $B$, $C$ and $A^1$, $A^2$, ..., $A_6$ and simplifying them we get, under $H_0$,

$$
\mathcal{E}_{10}(1,1) = \frac{1}{3} - 3 \int_0^\infty F_2(y) F_2(y) F_1(y) dF_1(y)
$$

$$
- 2 \int_0^\infty F_1(y) F_1(y) F_2(y) F_2(y) dF_1(y)
$$

$$
- 2 \int_0^\infty F_2(y) F_3(y) dF_1(y)
$$

$$
+ 2 \int_0^\infty F_1(y) F_2(y) F_1(y) F_2(y) dF_1(x) dF_2(y)
$$

$$
+ \int_0^\infty \left[ \int F_2(y) dF_1(y) \right]^2 dF_1(x) dF_2(u)
$$

$$
+ \int_0^\infty \left[ \int F_2(y) dF_1(y) \right]^2 dF_1(x) dF_2(u)
$$

(6.25 a)
Also,

\[ \Psi_{1,01}(y_1, v_1) = \mathbb{E}[\phi_5(x_1, u_1, y_1, v_1)] \]

\[ = P[u_1 > x_1 > y_1, v_1 > y_1] + P[x_1 > u_1 > y_1, v_1 > y_1] \]
\[ - P[y_1 > v_1 > x_1, u_1 > x_1] - P[v_1 > y_1 > x_1, u_1 > x_1] \]

\[ = \int_{y_1}^{\infty} \int_{y_1}^{\infty} \bar{F}_2(x) \, dF_1(x) \]
\[ + \int_{y_1}^{\infty} \int_{y_1}^{\infty} \bar{F}_1(x) \, dF_2(x) \]
\[ - \int_{y_1}^{\infty} \int_{0}^{\infty} \bar{F}_2(x) \, dF_1(x) \]
\[ - \int_{y_1}^{\infty} \int_{0}^{\infty} \bar{F}_2(x) \, dF_1(x) \]

where, \( I (a \geq b) \) is as defined in (6.24)

Hence, \( \Psi_{1,01}(y_1, v_1) \)

\[ = \begin{cases} 
\int_{y_1}^{\infty} \bar{F}_2(x) \, dF_1(x) + \int_{y_1}^{\infty} \bar{F}_1(x) \, dF_2(x) - \int_{0}^{\infty} \bar{F}_2(x) \, dF_1(x), & y_1 < v_1 \\
- \int_{0}^{y_1} \bar{F}_2(x) \, dF_1(x), & v_1 < y_1 
\end{cases} \]

Therefore, under \( H_0 \),

\[ \Psi_{1,10}(x, u) = -\Psi_{1,01}(x, u) \]

Hence, under \( H_0 \),

\[ \xi_{01}(1,1) = \xi_{10}(1,1) \]
and $\sigma_{11} = \frac{1}{pq} \xi_{10} (1,1)$

where, $\xi_{10} (1,1)$ is as given in (6.25 a)

**Calculation of $\sigma_{22}$**

Comparing the expressions for $\Psi_{2,10} (x, u)$ and $\Psi_{1,10} (x,u)$, we find that $\Psi_{2,10} (x, u)$ is the same as $\Psi_{1,10} (x, u)$ with the roles of $G_1$ and $G_2$, $x$ and $u$ interchanged. Therefore, under $H_0$,

$$\xi_{10} (2,2) = \frac{1}{3} - 3 \int_{0}^{\infty} F_1(y) \bar{F}_1(y) \bar{F}_2(y) \ dF_2(y)$$

$$- 2 \int_{0}^{\infty} F_2(y) \bar{F}_2(y) \bar{F}_1(y) \ dF_2(y)$$

$$- \int_{0}^{\infty} \bar{F}_2(y) \bar{F}_1(y) \ dF_1(y)$$

$$+ 2 \int_{0}^{\infty} F_2(y) F_1(x) \bar{F}_2(x) \bar{F}_1(x) \ dF_2(x) \ dF_1(y)$$

$$+ \int_{0}^{\infty} \left( \int_{0}^{x} F_1(y) \ dF_2(y) \right)^2 \ dF_1(x) \ dF_1(u)$$

$$+ \int_{0}^{\infty} \left( \int_{0}^{u} F_1(y) \ dF_2(y) \right)^2 \ dF_1(x) \ dF_2(u)$$

Similarly, comparing the expressions for $\Psi_{2,01} (y_1, v_1)$ and $\Psi_{1,01} (y_1, v_1)$, we find that $\Psi_{2,01} (y_1, v_1)$ is the same as $\Psi_{1,01} (y_1, v_1)$ with the roles of $F_1$ and $F_2$, $y_1$ and $v_1$ interchanged. Therefore, under $H_0$,

$$\Psi_{2,10} (x, u) = - \Psi_{2,01} (x, u)$$
\[ \xi_{01} (2,2) = \xi_{10} (2,2) \]

\[ \sigma_{22} = \frac{1}{pq} \xi_{10} (2,2) \tag{6.26} \]

where \( \xi_{10} (2,2) \) is as given in (6.26 a).

**Calculation of \( \sigma_{12} \)**

Here \( r_1 = r_2 = 1, \quad s_1 = s_2 = 1 \)

Under \( H_0 \),

\[ \xi_{10} (1,2) = E[\Psi_{1,10} (x_1, u_1), \Psi_{2,10} (x_1, u_1)] \]

where,

\[ \Psi_{1,10} (x_1, u_1) = \begin{cases} F_1(x_1) - \bar{F}_1(x_1) \bar{F}_2(x_1) - \int_0^{x_1} F_2(y) \, dF_1(y), & x_1 > u_1 \\ F_1(u_1) - \int_0^{u_1} F_2(y) \, dF_1(y), & x_1 > u_1 \end{cases} \]

\[ \Psi_{2,10} (x_1, u_1) = \begin{cases} F_2(x_1) - \bar{F}_2(x_1) \bar{F}_1(x_1) - \int_0^{u_1} F_1(y) \, dF_2(y), & x_1 > u_1 \\ F_2(u_1) - \bar{F}_1(u_1) \bar{F}_2(u_1) - \int_0^{u_1} F_1(y) \, dF_2(y), & x_1 > u_1 \end{cases} \]

Therefore,

\[ \xi_{10} (1,2) = \int_0^\infty \int_0^\infty \Psi_{1,10} (x_1, u_1) \Psi_{2,10} (x_1, u_1) \, dF_1(x_1) \, dF_2(u_1) \]

\[ = \int_0^\infty \int_0^\infty \Psi_{1,10} (x_1, u_1) \Psi_{2,10} (x_1, u_1) \, dF_1(x_1) \, dF_2(u_1) \]

\[ + \int_0^\infty \int_0^\infty \Psi_{1,10} (x_1, u_1) \Psi_{2,10} (x_1, u_1) \, dF_1(x_1) \, dF_2(u_1) \]

\[ = A_1 + B_1 \]
\[ A_1 = \int \int_{0 < x_1 < u_1 < \infty} F_1(x_1) F_2(x_1) \, dF_1(x_1) \, dF_2(u_1) \]

\[ - \int \int_{0 < x_1 < u_1 < \infty} F_2(x_1) \bar{F}_1(x_1) \bar{F}_2(x_1) \, dF_1(x_1) \, dF_2(u_1) \]

\[ - \int \int_{0 < x_1 < u_1 < \infty} F_1(x_1) (\int F_2(y) \, dF_2(y)) \, dF_1(x_1) \, dF_2(u_1) \]

\[ - \int \int_{0 < x_1 < u_1 < \infty} F_1(x_1) (\int F_1(y) \, dF_1(y)) \, dF_1(x_1) \, dF_2(u_1) \]

\[ + \int \int_{0 < x_1 < u_1 < \infty} \bar{F}_1(x_1) \bar{F}_2(x_1) (\int F_1(y) \, dF_1(y)) \, dF_1(x_1) \, dF_2(u_1) \]

\[ + \int \int_{0 < x_1 < u_1 < \infty} (\int F_2(y) \, dF_2(y)) (\int F_1(y) \, dF_1(y)) \, dF_1(x_1) \, dF_2(u_1) \]

\[ = A_{11} - A_{12} - A_{13} - A_{14} + A_{15} + A_{16} \]

\[ A_{11} = \int \int_{0 < x_1 < u_1 < \infty} F_1(x_1) F_2(x_1) \, dF_1(x_1) \, dF_2(u_1) \]

\[ = \int_{0}^{\infty} F_1(x) F_2(x) \, dF_1(x) - \int_{0}^{\infty} F_1(x) F_2^2(x) \, dF_1(x) \]

\[ A_{12} = \int \int_{0 < x_1 < u_1 < \infty} F_2(x) \bar{F}_1(x) \bar{F}_2(x) \, dF_1(x_1) \, dF_2(u_1) \]

\[ = \int_{0}^{\infty} F_2(x) \bar{F}_1(x) \bar{F}_2(x) \, dF_1(x) \]

\[ A_{13} = \int \int_{0 < x_1 < u_1 < \infty} F_2(x_1) (\int F_2(y) \, dF_2(y)) \, dF_1(x_1) \, dF_2(u_1) \]
\[= \int_0^\infty \frac{1}{x} \int_0^x \frac{1}{y} \left[ \int_0^y \frac{1}{z} \right] \, dz \, dx\]

which after, change of integration, reduces to
\[
\begin{align*}
= 2 \int_0^\infty F_1(y) F_2(y) \, dF_1(y) - \frac{1}{2} \int_0^\infty F_2(y) \, dF_1(y) \\
- \frac{3}{2} \int_0^\infty F_1^2(y) F_2(y) \, dF_1(y) \\
- 2 \iint_{0 < y < x} F_1(y) F_2(x) \, dF_2(y) \, dF_1(x) \\
+ 2 \iint_{0 < y < x} F_1(x) F_2(x) F_1(y) \, dF_2(y) \, dF_1(x) \\
+ \int \int_{0 < y < x} F_1(y) F_2^2(x) \, dF_2(y) \, dF_1(x) \\
- \int \int_{0 < y < x} F_1(x) F_2^2(x) F_1(y) \, dF_2(y) \, dF_1(x)
\end{align*}
\]
\[ A_1 = 4 \int_0^\infty F_1(x) F_2(x) \, dF_1(x) - 4 \int_0^\infty F_1(x) \, dF_2(x) \, dF_1(x) \]
\[- \int_0^\infty F_2(x) \, dF_1(x) + 2 \int_0^\infty F_2^2(x) \, dF_1(x) - \int_0^\infty F_2^3(x) \, dF_1(x) \]
\[ + \int_0^\infty F_1(x) \, dF_2^2(x) \, dF_1(x) - \int_0^\infty F_1(x) \, dF_2^3(x) \, dF_1(x) \]
\[ + 2 \int_0^\infty F_1(x) \, dF_2^2(x) \, dF_1(x) - 3 \int_0^\infty F_1^2(x) \, dF_2(x) \, dF_1(x) \]
\[- \int_0^\infty F_1(y) \, F_2(x) \, dF_1(x) \, dF_2(y) \]
\[ + 2 \int_0^\infty F_1(y) \, F_2(x) \, dF_1(x) \, dF_2(y) \]
\[- \int_0^\infty \left[ \int_0^x F_2(y) \, dF_1(y) \right]^2 \, dF_1(x) \, dF_2(u) \]

\[ B_1 = \int_0^\infty \int_0^{x_1} F_1(u_1) \, F_2(u_1) \, dF_1(x_1) \, dF_2(u_1) \]
\[- \int_0^\infty \int_0^{x_1} F_1(u_1) \, F_1(u_1) \, F_2(u_1) \, dF_1(x_1) \, dF_2(u_1) \]
\[- \int_0^\infty \int_0^{x_1} F_1(u_1) \, \left[ \int_0^{x_1} F_2(y) \, dF_1(y) \right] \, dF_1(x_1) \, dF_2(u_1) \]
\[- \int_0^\infty \int_0^{x_1} F_2(u_1) \, \left[ \int_0^{x_1} F_1(y) \, dF_2(y) \right] \, dF_1(x_1) \, dF_2(u_1) \]
\[ B_1 = 4 \int_{0}^{\infty} F_1(x) F_2(x) \, dF_2(x) - 4 \int_{0}^{\infty} F_1^2(x) F_2(x) \, dF_2(x) \]

\[ - \int_{0}^{\infty} F_1(x) \, dF_2(x) + 2 \int_{0}^{\infty} F_1^2(x) \, dF_2(x) - \int_{0}^{\infty} F_1^3(x) \, dF_2(x) \]

\[ + \int_{0}^{\infty} F_1^2(x) F_2(x) \, dF_2(x) - \int_{0}^{\infty} F_1^3(x) F_2^2(x) \, dF_2(x) \]

\[ + 2 \int_{0}^{\infty} F_1^3(x) F_2(x) \, dF_2(x) - 3 \int_{0}^{\infty} F_1(x) F_2^2(x) \, dF_2(x) \]

\[ - \int_{0}^{\infty} \int_{y}^{\infty} F_2(y) F_1(x) \, dF_1(y) \, dF_2(x) \]

\[ + 2 \int_{0}^{\infty} \int_{y}^{\infty} F_2(u) F_1(u) F_2(y) \, dF_1(y) \, dF_2(u) \]

\[ - \int_{0}^{\infty} \int_{u}^{\infty} [\int_{F_1(y)}^{\infty} F_2(y) \, dF_2(y)]^2 \, dF_1(x) \, dF_2(u) \]

After change of integration and simplification, we get,

\[ A_1 + B_1 = 2 \int_{0}^{\infty} F_1(y) F_2^2(y) \, dF_1(y) - 2 \int_{0}^{\infty} F_1^2(y) F_2^2(y) \, dF_1(y) \]

\[ - \frac{2}{3} \int_{0}^{\infty} F_1(y) F_2^3(y) \, dF_1(y) - 2 \int_{0}^{\infty} F_1(y) F_2(x) \, dF_1(x) \, dF_2(y) \]
\[ + \int \int_{0 < y < x < \infty} F_1(y) F_2^2(x) \, dF_1(x) \, dF_2(y) \]

\[ - \int \int_{0 < x < \infty} \left[ \int_{0}^{x} F_2(y) \, dF_1(y) \right]^2 \, dF_1(x) \, dF_2(u) \]

\[ - \int \int_{0 < x < \infty} \left[ \int_{0}^{u} F_1(y) \, dF_2(y) \right]^2 \, dF_1(x) \, dF_2(u) \]

After simple but lengthy calculations

\[ \int \int_{0 < x < \infty} \left[ \int_{0}^{u} F_2(y) \, dF_1(y) \right]^2 \, dF_1(x) \, dF_2(u) \]

reduces to

\[ \frac{4}{3} \int_{0}^{\infty} F_1(x) F_2^3(x) \, dF_1(x) - 2 \int_{0}^{\infty} F_1^2(x) F_2^3(x) \, dF_1(x) \]

\[ - \int \int_{0 < y < x < \infty} F_2^2(x) F_1(y) \, dF_1(x) \, dF_2(y) \]

\[ + 2 \int \int_{0 < x < \infty} F_1(x) F_2^2(x) F_1(y) \, dF_1(x) \, dF_2(y) \]

\[ + \int \int_{0 < y < x < \infty} \left[ \int_{0}^{u} F_2(y) \, dF_1(y) \right]^2 \, dF_1(x) \, dF_2(u) \]

Further simplifying and combining terms,

\[ \xi_{10} (1,2) = A_1^* + B_1^* \]

\[ = 2 \int_{0}^{\infty} F_1(y) F_1(y) F_2^2(y) \, dF_1(y) \]

\[ - 2 \int \int_{0 < y < x < \infty} F_1(x) F_2(x) F_2(x) \, dF_1(x) \, dF_2(y) \]
By symmetry of kernels we have, under $H_0$,

$$
\xi_{10}(1,2) = \xi_{01}(1,2)
$$

$$
\sigma_{12} = \frac{1}{pq} \xi_{10}(1,2) - \sigma_{21}
$$

where $\xi_{10}(1,2)$ is as given in (6.27 a)

Combining the above steps we get the dispersion matrix $\Sigma$ under $H_0$.

Thus, even under $H_0$, the dispersion matrix $\Sigma$ depends on the unknown distribution function $F_1$ and $F_2$. A simplifying assumption could be to assume that the proportional hazards model or the Lehmann type alternative, $F_2(x) = [F_1(x)]^r$, $r > 0$ holds. Cheng and Chang (1985, 1986) and Hollander, Proschan and Sconing (1985) assumed the proportional hazards competing risks model and obtained estimators for the survival function for the cases when $r$ is known and unknown. When $r$ is unknown, it can be easily estimated.

**Corollary 6.1** If $F_2(x) = [F_1(x)]^r$, $r > 0$, the dispersion matrix is given by

$$
\Sigma = \frac{1}{pq} \begin{bmatrix}
\xi_{10}(1,1) & \xi_{10}(1,2) \\
\xi_{10}(1,2) & \xi_{10}(2,2)
\end{bmatrix}
$$
where,
\[ \xi_{10} (1,1) = \frac{1}{3} - \frac{8}{r+1} \times \frac{6}{r+2} - \frac{3}{r+3} + \frac{3}{2r+1} + \frac{5}{2r+3} \]
\[ - \frac{1}{3r+1} + \frac{4}{3r+2} + \frac{5r}{3(r+1)(2r+3)(3r+2)} \]
\[ \xi_{10} (1,2) = \frac{5(1-r)}{3(r+1)(2r+3)(3r+2)} \]
\[ \xi_{10} (1,1) = \frac{1}{3} + r[ -\frac{8}{r+1} + \frac{3}{r+2} - \frac{1}{r+3} + \frac{6}{2r+1} + \frac{4}{2r+3} \]
\[ - \frac{3}{3r+1} + \frac{5}{3r+2} + \frac{5r^2}{3(r+1)(2r+3)(3r+2)} \]

**Proof:** Substituting \( F_2(x) = [F_1(x)]^T \) in the expressions for \( \xi_{10} (1,1) \), \( \xi_{10} (2,2) \) and \( \xi_{10} (1,2) \) given by (6.25 a), (6.26 a) and (6.27 a), and carrying out simple but lengthy calculations, the above result follows.

**Corollary 6.2** In particular, if \( F = \Sigma \), then, under \( H_0 \),

\[ \Sigma = \frac{1}{pq} \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{bmatrix} \]

**Proof:** Put \( r = 1 \) in the expressions for \( \xi_{10} (1,1) \), \( \xi_{10} (2,2) \) and \( \xi_{10} (1,2) \) in the above corollary and the result follows.

For testing \( H_0 \) against \( H_A \) we propose a test based on the statistic defined by

\[ W_2 = N[B_1, B_2] \Sigma^{-1}_{H_0} [B_1, B_2]' \]

Large values of \( W_2 \) are significant for testing \( H_0 \) against \( H_A \).
Asymptotic Distribution of $W_2$

From Theorem 6.2 it follows that the limiting distribution of $W_2$ as $N \to \infty$ in such a way that $\frac{p_N}{N} = \frac{n}{N}$ tends to $p$, $0 < p < 1$, is non-central chi-square with two degrees of freedom (see, for example, Puri and Sen (1971)).

Here it is remarked that, in general, the asymptotic variance-covariance matrix of the statistic $W_2$ is not free of the unknown distribution functions even under the null hypothesis. Hence, in practice one must studentize the statistic with the help of a suitable consistent estimator of the unknown variance-covariance matrix. One such estimator is suggested later in this chapter. However, replacing $\Sigma$ by its consistent estimator $\hat{\Sigma}$, (say), will not alter the limiting distribution of the statistic $W_2$.

6.4 Efficiency Comparisons

In the preceding chapters we have compared tests in the Pitmann ARE sense when all the competing test statistics have limiting normal distribution. We also know that if two test statistics $Q_1$ and $Q_2$ have limiting non-central chi-square distributions with the same degrees of freedom, then for testing against a sequence of alternatives converging to the null hypothesis, the Pitman ARE of one test with respect to the other is the ratio of two noncentrality parameters (see, for example, Andrews (1954), Hannan (1956), Puri and Sen (1971)).

In our case $W_1$ has limiting normal distribution and $W_2$ has limiting non-central chi-square distribution with two
degrees of freedom. Therefore, above procedures cannot be adopted for calculating the ARE of \( W_1 \) relative to \( W_2 \).

Here we give a recent and more general definition of efficiency due to Rothe (1981). (Also see, Kesar Singh (1984)).

Assume that \( X_1, X_2, \ldots \) are independent and identically distributed observations from a probability space \( (\Omega, \mathcal{S}, P_\theta) \), where \( \theta \) is an unknown parameter taking values in a metric space \( \Lambda \). Consider the problem of testing \( H_0 : \theta = \theta_0 \) against the alternative \( \theta \neq \theta_0 \), where \( \theta_0 \) is an accumulation point in the metric space. Let \( \{T_{1n}\} \) and \( \{T_{2n}\} \) be two competing test statistics for the above problem. The sizes of all the tests are fixed at a level \( \alpha \in (0,1) \). Then, for a \( \theta \neq \theta_0 \) and \( \beta \in (0,1) \), we define

\[
N_1(\theta, \alpha, \beta) = \min \{m_1 : \text{Power of } T_{1n} \text{ at } \theta \text{ is } \geq \beta, \text{ for all } n \geq m_1 \}
\]

\[
N_2(\theta, \alpha, \beta) = \min \{m_2 : \text{Power of } T_{2n} \text{ at } \theta \text{ is } \geq \beta, \text{ for all } n \geq m_2 \}
\]

Thus, \( N_1(\theta, \alpha, \beta) \) \( (N_2(\theta, \alpha, \beta)) \) is the first sample size to generate a power \( \beta \) or more for \( T_{1n} \) \( (T_{2n}) \) for all sample sizes bigger than or equal to it.

The Pitman efficiency of \( \{T_{1n}\} \) relative to \( \{T_{2n}\} \) is then defined as

\[
e_{12}(\alpha, \beta) = \lim_{\theta \to \theta_0} \frac{N_2(\theta, \alpha, \beta)}{N_1(\theta, \alpha, \beta)}, \quad (6.31)
\]

if the limit exists.

If the limit does not exist, then one can consider
the lower and upper efficiencies given by

\[ e_{12}^-(a, \beta) = \lim_{\theta \to \theta_0} \inf_{\theta} \frac{N_2(\theta, a, \beta)}{N_1(\theta, a, \beta)} \]

and,

\[ e_{12}^+(a, \beta) = \lim_{\theta \to \theta_0} \sup_{\theta} \frac{N_2(\theta, a, \beta)}{N_1(\theta, a, \beta)} \]

Further, a sequence \( T_n \) is said to be standard if there are functions \( K : \Lambda \to [0, \infty] \) and \( J : [0, \infty] \to [a, 1] \), such that

(a) \( K \) is continuous and \( K(\theta) = 0 \) if and only if \( \theta = \theta_0 \).

(b) \( J \) is increasing and one to one.

(c) For \( \theta_n \) satisfying \( n K[\theta_n] \to 0 \) as \( n \to \infty \), \( \beta(\theta_n, T_n) = J(\eta) \)

where \( \beta(\theta, T_n) \) is the power of \( T_n \) at the alternative \( \theta \).

Under the conditions (a), (b) and (c), definition (6.31) coincides with several somewhat different versions, for example, those of Noether (1955), Fraser (1957), Olshen (1967) and Wieand (1976).

The following theorem by Rothe (1991) gives a set of conditions under which the ARE exists and simultaneously yields an expression for it.

**Theorem 6.3 (Rothe)** Suppose that \( T_{1n} \) and \( T_{2n} \) are both standard. If \( \lim_{\theta \to \theta_0} \frac{K_1(\theta)}{K_2(\theta)} \) exists, then \( e_{12}(a, \beta) \) exists and is given by

\[ e_{12}(a, \beta) = \frac{\lim_{\theta \to \theta_0} K_1(\theta)}{\lim_{\theta \to \theta_0} K_2(\theta)} \] (6.32)
More generally,

\[ e_{12}^+ (\alpha, \beta) = \frac{J_2^{-1}(\beta)}{J_1^{-1}(\beta)} \limsup_{\theta \to \theta_0} \frac{K_1(\theta)}{K_2(\theta)} \]

and,

\[ e_{12}^- (\alpha, \beta) = \frac{J_2^{-1}(\beta)}{J_1^{-1}(\beta)} \liminf_{\theta \to \theta_0} \frac{K_1(\theta)}{K_2(\theta)} \]

where \( K_i, J_i \) are the functions \( K \) and \( J \) for \( T_i \), \( i = 1, 2 \).

Thus, \( e(\alpha, \beta) \) may depend upon \((\alpha, \beta)\). The problem of calculating ARE reduces to the problem of verifying conditions (a), (b) and (c) and hence finding suitable functions \( K \) and \( J \).

Rothe considered the shape of \( J^{-1} \) if the distribution of \( T_n \) has one of the following asymptotic properties.

\( A_0 \): There is a \( u > 0 \), such that \( n K(\theta_n) - \eta \) implies \( T_n \sim N(\eta, 1) \), for every \( n \geq 0 \).

And for \( r \in \mathbb{N} \),

\( A_r \): There is a \( u > 0 \), such that \( n K(\theta_n) - \eta \) implies \( T_n \sim \chi^2 (r, \eta^2 u) \), where \( \chi^2 (r, \delta^2) \) is a non-central chi-square distribution with \( r \) degrees of freedom and noncentrality parameter \( \delta^2 \).

**Theorem 6.4 (Rothe)** For \( 0 < \alpha < \beta < 1 \), if \( T_n \) satisfies \( A_r \), \( r \geq 0 \), then conditions (a), (b) and (c) hold with

\[ J^{-1}(\beta) = d^{1/u}(\alpha, \beta, r) \tag{6.33} \]

\[ d(\alpha, \beta, 0) = \frac{1}{\Phi^2(\beta) - \Phi^2(\alpha)} \tag{6.34} \]
where, $\Phi$ is the distribution function of the standard normal random variable, and for $r \geq 1$, $d^2 = d^2(\alpha, \beta, r)$ is the uniquely determined non-centrality parameter such that $\beta$-fractile of $\chi^2(r, d^2)$ and the $\alpha$-fractile of $\chi^2(r, 0)$ coincide.

Rothe (1981) has remarked that $u = 1/2$, $K_1(\theta) = \xi_1^2(\theta - \theta_0)^2$ and $K_2(\theta) = \xi_2^2(\theta - \theta_0)^2$ serve the purpose, where

$$
\xi_1 = \lim_{N \to \infty} \frac{N^{-1/2} \frac{d}{d\theta} E[W_1]\theta=0}{\sqrt{V_{H_0}[W_1]}} \quad (6.35)
$$

and,

$$
\xi_2 = [c' \Lambda_{H_0}^{-1} c]^{1/2} \quad (6.36)
$$

where $c = [c_1, c_2]'$.

With $c_j = \lim_{N \to \infty} \frac{N^{-1/2} \frac{d}{d\theta} E[B_j]\theta=0}{\sqrt{V_{H_0}[B_j]}}$, $j = 1, 2.$

and $\Lambda_{H_0}$ is the correlation matrix under $H_0$.

Therefore, from Theorem 6.3 (6.32), (6.33) and (6.34), it follows that

$$
e_{W_1, W_2}(\alpha, \beta) = \frac{d^2(\alpha, \beta, 2)}{d^2(\alpha, \beta, 0)} \lim_{\theta \to \theta_0} \frac{K_1(\theta)}{K_2(\theta)} \quad (6.37)
$$

where $\xi_1$ and $\xi_2$ are as defined in (6.35) and (6.36).
We use this approach to find the ARE of the $W_1$ test with respect to the $W_2$ test for the exponential scale alternative and assume that $F_1 = F_2$, $G_1 = G_2$.

Let,

\[ G_1(x) = F_1[(\theta + 1) x] \]
\[ G_2(x) = F_2[(\theta + 1) x] \]
\[ \bar{F}_1(x) = \bar{F}_2(x) = e^{-x}, \ x > 0 \quad (6.38) \]

From (6.5)

\[ E[W_1] = \int_0^\infty G(x) \, dF(x) \]

where $F$ and $G$ are the distribution functions of $T$ and $S$, respectively.

Assuming that we can interchange the order of differentiation and integration, we get

\[ \frac{d}{d\theta} E[W_1]|_{\theta=0} = \int_0^\infty \frac{d}{d\theta} G_1(x)|_{\theta=0} \, dF(x) \]

Now, $\overline{G}(x) = \overline{G}_1(x) \, \bar{F}_2(x)$

\[ = \overline{F}_1[(\theta + 1) x] \, \bar{F}_2[(\theta + 1) x] \]

Hence,

\[ \frac{d}{d\theta} G(x)|_{\theta=0} = x[\overline{F}_1(x) \, f_2(x) + \bar{F}_2(x) \, f_1(x)] \]

Also, $F(x) = \overline{F}_1(x) \, \bar{F}_2(x)$

Therefore, $\frac{d}{dx} F(x) = f_2(x) \overline{F}_1(x) + f_1(x) \bar{F}_2(x)$
When,
\[ F_1(x) = F_2(x) = e^{-x}, \quad x > 0 \]

\[ \frac{d}{d\theta} E[W_1] \big|_{\theta=0} = \frac{1}{4} \]

From (6.9), \( V(W_1) = \frac{1}{12pq} \)

Substituting above results in (6.35), we obtain

\[ \xi_1^2 = \frac{3}{4} pq \] 

(6.39)

Further \( F_1(x) = F_2(x) \) for every \( x > 0 \). Therefore, from corrolary 6.2, it follows that

\[ \Sigma_{H_0} = \frac{1}{pq} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\theta} \end{bmatrix} \]

Hence the correlation matrix \( \Lambda_{H_0} \) is given by

\[ \Lambda_{H_0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

and,

\[ \Lambda_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Substituting \( F_1(x) = F_2(x) = e^{-x} \), \( \overline{F}_1(x) = \overline{F}_2(x) = e^{-(\theta+1)x} \)
in (6.23), and arranging terms, we get

\[ E[B_1] = 2(\theta+1) \int_0^\infty \int_0^y \left[ e^{-x} - e^{-y} \right] e^{-2(\theta+1)x} e^{-y} \, dx \, dy \]

\[ - 2(\theta+1) \int_0^\infty \int_0^\infty \left[ e^{-(\theta+1)x} - e^{-(\theta+1)y} \right] e^{-2x} e^{-(\theta+1)y} \, dx \, dy \]
After simple calculations, we get

\[
\frac{d}{d\theta} E[B_1] \bigg|_{\theta=0} = \frac{1}{4}
\]

Also, \( N \to \infty \) \( N \cdot V[B_1] = \frac{1}{6pq} \)

Hence, \( c_1 = \frac{1}{4} \left( \frac{1}{6pq} \right)^{-1/2} \)

By considerations of symmetry it is easy to see that \( c_2 = \frac{1}{4} \left( \frac{1}{6pq} \right)^{-1/2} \)

Hence, \( \xi_2^2 = [c_1, c_2] \Lambda_{H_0}^{-1} [c_1, c_2]' \)

\[
= c_1^2 + c_2^2
= \frac{3}{4} \cdot pq
\]

(6.40)

From (6.39) and (6.40), we get that

\[
\frac{\xi_1^2}{\xi_2^2} = 1
\]

Therefore,

\[
\varepsilon_{W_1, W_2}(\alpha, \beta) = \frac{d^2(\alpha, \beta, 2)}{\left[ \bar{F}^{-1}(\alpha) - \bar{F}^{-1}(\beta) \right]^2}
\]

(6.41)

The values of \( d^2(\alpha, \beta, r) \) for \( r > 0 \) have been tabulated by Haynam, Govindaraju and Leone (1962). They are also available in Harter and Owen (1970). The values of \( \bar{F}^{-1}(\alpha) \), \( \bar{F}^{-1}(\beta) \) can be seen from Rao, Mittra, Mathai and Murthy (1975) tables.
<table>
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Table 6.1 gives the ARE of the $W_1$ test with respect to the $W_2$ test. It is clear that the $W_1$ test is more efficient than the $W_2$ test for all values of $(\alpha, \beta)$ considered above.

6.5 Estimation of Variance - Covariance matrix $\Sigma$

In Theorem 6.2 we saw that the variance terms $\sigma_1^2$ and $\sigma_2^2$ and the covariance term $\sigma_{12}$, even under $H_0$, depend upon the unknown distribution functions $F_1$ and $F_2$. Here we consider estimates of the variances and covariances so that we can use the studentized version of the statistic $W_2$.

From (6.25 a) and (6.25) we know that

$$\sigma_1^2 = \frac{\xi_{10}(1,1)}{pq}$$

where, $p = \lim_{N \to \infty} \frac{n}{N}$, $q = 1 - p$

and,

$$\xi_{10}(1,1) = \frac{1}{3} - 3 \int_0^\infty F_2(x) \overline{F}_2(x) \overline{F}_1(x) \, dF_1(x)$$

$$- 2 \int_0^\infty F_1(x) \overline{F}_1(x) \overline{F}_2^2(x) \, dF_1(x)$$

$$- \int_0^\infty \overline{F}_1^2(x) \overline{F}_2^3(x) \, dF_1(x)$$

$$+ 2 \int \int_{\alpha \leq y < x < \infty} F_1(y) \overline{F}_1(y) \overline{F}_2^2(x) \, dF_1(x) \, dF_2(y)$$

$$+ \int_0^\infty \overline{F}_1(u) \left[ \int_0^u \overline{F}_2(y) \, dF_1(y) \right]^2 \, dF_2(u)$$
\[+ \int_{0}^{\infty} F_2(u) \left( \int_{0}^{u} F_2(y) \, dF_1(y) \right)^2 \, dF_1(u)\]

\[= \frac{1}{3} - 3 \, P[X_1 > U_1, U_2 > X_1, X_2 > X_1, X_3 > X_1] \]

\[\quad - 2 \, P[X_1 > X_2, X_3 > X_1, X_4 > U_1, X_5 > U_2, U_3 > X_1] \]

\[\quad - P[X_2 > X_1, X_3 > X_1, X_4 > U_1, X_5 > U_2, X_6 > X_1] \]

\[\quad + 2 \, P[X_1 > U_2, U_1 > X_2, X_3 > X_1, U_3 > X_1, X_4 > U_1] \]

\[\quad + P[U_1 > X_2 > U_2, U_1 > X_3 > U_3, X_4 > U_1] \]

\[\quad + P[X_1 > X_2 > U_2, X_1 > X_3 > U_3, U_1 > X_4] \]

\[= \frac{1}{3} - 3 \, P[U_1 < X_1 < \min(X_2, X_3, U_2)] \]

\[\quad - 2 \, P[\max(U_1, U_2, X_2) < X_1 < \min(U_3, X_3)] \]

\[\quad - P[\max(U_1, U_2, U_3) < X_1 < \min(X_2, X_3)] \]

\[\quad + 2 \, P[\max(U_1, U_2) < X_1 < \min(X_3, U_3), X_2 < U_1] \]

\[\quad + P[U_2 < X_2 < U_1, U_3 < X_3 < U_1 < X_1] \]

\[\quad + P[U_2 < X_2 < X_1, U_3 < X_3 < X_1 < U_1] \]

Each of the terms in the above expression, except \(\frac{1}{3}\),
is an estimable function. Thus we can construct the corresponding U-statistics based on the kernels which are indicator functions of the events for which probabilities are encountered above. Then consistent estimator of $\xi_{10}^{1,1}$ is given by $S_{10}^2$, where $S_{10}^2$ is the same as above expression with probabilities replaced by corresponding U-statistics.

Similarly, from (6.26 a) and (6.26), we get

$$\sigma^2 = \frac{\xi_{10}^{2,2}}{pq}$$

where,

$$\xi_{10}^{2,2} = \frac{1}{3} - 3 \int_0^\infty F_1(x) F_1(x) F_2^2(x) \, dF_2(x)$$

$$\quad - 2 \int_0^\infty F_2(x) F_2(x) F_1^2(x) \, dF_2(x)$$

$$\quad - \int_0^\infty F_2^2(x) F_1^3(x) \, dF_2(x)$$

$$\quad + 2 \int \int_{0 < y < x < \infty} F_2(y) F_1(x) F_2(x) F_1(x) \, dF_2(x) \, dF_1(y)$$

$$\quad + \int F_1(u) \left[ \int_0^u F_1(y) \, dF_2(y) \right]^2 \, dF_2(u)$$

$$\quad + \int F_2(x) \left[ \int_0^x F_1(y) \, dF_2(y) \right]^2 \, dF_1(x)$$

$$= \frac{1}{3} - 3 \left[ X_1 < U_1 < \min(X_2, U_2, U_3) \right]$$

$$- 2 \left[ \max(X_1, U_3, X_2) < U_1 < \min(X_2, U_2) \right]$$
\[ P[\max(X_1, X_2, X_3) < U_1 < \min(U_2, U_3)] \]
\[ + 2 P[\max(X_1, X_2) < U_1 < \min(U_3, X_3), X_1 < U_2] \]
\[ + P[X_2 < U_2 < X_1, X_3 < U_3 < X_1 < U_1] \]
\[ + P[X_2 < U_2 < U_1, X_3 < U_3 < U_1 < X_1] \]

And from (6.27 a) and (6.27), it follows that

\[ \sigma_{12} = \frac{\xi_{10}(1,2)}{pq} \]

where,

\[ \xi_{10}(1,2) = 2 \int_0^\infty F_1(y) \overline{F}_1(y) F_2^2(y) \overline{F}_2(y) dF_1(y) \]
\[ - 2 \int_0^\infty \int_0^\infty F_1(y) \overline{F}_1(x) F_2(x) \overline{F}_2(x) dF_1(x) dF_2(y) \]
\[ - \int_0^\infty \overline{F}_2(x) \left[ \int_0^x F_2(y) dF_1(y) \right]^2 dF_1(x) \]
\[ - \int_0^\infty \overline{F}_1(u) \left[ \int_0^u F_2(y) dF_1(y) \right]^2 dF_2(u) \]
\[ = 2 P[\max(U_1, U_2, X_2) < X_1 < \min(U_3, X_3)] \]
\[ - 2P[\max(U_1, U_2) < X_1 < \min(U_3, X_3), X_2 < U_1] \]
\[ - P[U_2 < X_2 < X_1, U_3 < X_3 < X_1 < U_1] \]
\[ - P[U_2 < X_2 < U_1, U_3 < X_3 < U_1 < X_1] \]
Again, by replacing probabilities, which are estimable functions, by corresponding U-statistics, we get $S^2_{01}$ and $S^2_{12}$ as consistent estimators for $\xi_{10} (2,2)$ and $\xi_{10} (1,2)$.

Besides the technique given above, there are several other ways of estimating the variances of U-statistics. Puri and Sen (1971) have recommended estimators which may not be strictly unbiased but are computationally more convenient. Lee (1984) has suggested estimators of variances of U-statistics using Jackknifing and Bootstrapping (Miller (1990)).