2.1 We consider here the problem of comparing the effect of two risks of failure to which individuals in a particular population are exposed. The actual failure is due to only one of the two risks. Let us assume that these risks act independently on each one of the individuals in the population. Let $X$ and $Y$ denote the hypothetical times to failure of an individual subject to Risk I and Risk II, respectively. Since $X$ and $Y$ denote the lifetimes of an individual, the two are essentially positive valued random variables. Let the corresponding distribution functions be $F(x)$ and $G(x)$, both belonging to $\mathcal{F}$, the class of all absolutely continuous distribution functions $H(x)$, with $H(x) = 0$ for $x < 0$. Let $f(x)$ and $g(x)$ be the probability density functions of $X$ and $Y$, respectively, and let $F = 1-F$ and $G = 1-G$ be the corresponding survival functions.

Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be two independent random samples from $F$ and $G$, respectively. The pair $(X_i, Y_i)$ denotes the hypothetical times to failure of the $i^{th}$ individual. However, in the competing risks framework, the failure is assumed to be due to only one of the two risks. The simultaneous failure of the individual due to the two risks, even if it occurs, has zero probability. Hence, for the $i^{th}$ individual both $X_i$ and $Y_i$ are not observable. What is actually observed is the pair $(T_i, \delta_i)$, where $T_i$ denotes the time to failure and $\delta_i$ indicates the cause of failure of the $i^{th}$
individual. That is,

\[ T_i = \min(X_i, Y_i) \]

\[ = \begin{cases} X_i, & \text{if } X_i \leq Y_i, \\ Y_i, & \text{if } X_i > Y_i \end{cases} \]

and,

\[ \delta_i = \begin{cases} 1, & \text{if } X_i > Y_i, \\ 0, & \text{if } X_i \leq Y_i. \end{cases} \]

Although, hypothetically, there are 2n observations in the form of \(X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n\), the actual information available for the n individuals in the sample is restricted to the times to failure \(T_1, T_2, \ldots, T_n\) and the corresponding causes of failure indicated by \(\delta_1, \delta_2, \ldots, \delta_n\).

On the basis of these observed quantities, we wish to test the following hypothesis concerning the failure time distributions \(F\) and \(G\),

\[ H_0 : F(x) = G(x), \text{ for every } x \geq 0 \]

against various alternatives discussed in the sequel.

First we will look at various methods based on likelihood functions for testing the above hypothesis.

2.2 The Likelihood Function

Here we look at the form of the likelihood function under the competing risks set up (see, for example, Miller (1981)).

Suppose that the individuals in the sample fail at times \(t_1, t_2, \ldots, t_n\). Let us consider the failure of the \(i\)th individual. The failure can be due to any one of the two \textit{risks}. Suppose, \(\delta_i = 1\), that is, the individual survives the first risk and fails due to the second one making a contribution of \(g(t_i)F(t_i)\) to the likelihood function. However, \(\delta_i = 0\) implies that the
individual has survived the second risk and has failed due to the first one. Then the contribution to the likelihood function is \( f(t_i) \bar{G}(t_i) \). Since at any given failure time \( t_i \), \( \delta_i \) is either 1 or 0, and simultaneous failure of the individual due to the two risks occurs with zero probability, the contribution of the \( i^{th} \) individual to the likelihood function is

\[
\left[ g(t_i) \bar{F}(t_i) \right]^{\delta_i} \left[ f(t_i) \bar{G}(t_i) \right]^{1-\delta_i}
\]

Further, all the \( n \) individuals in the sample act independently. Therefore, the likelihood function is given by

\[
L(t, \delta) = L(t_1, t_2, \ldots, t_n, \delta_1, \delta_2, \ldots, \delta_n)
= \prod_{i=1}^{n} \left[ g(t_i) \bar{F}(t_i) \right]^{\delta_i} \left[ f(t_i) \bar{G}(t_i) \right]^{1-\delta_i} \quad (2.1)
\]

2.3 Let us assume that the distribution functions \( F \) and \( G \) are known up to a parameter \( \theta \).

Extensive work has been done as far as the estimation of the parameters is concerned. Moeschberger and David (1971) considered the case when \( n \) individuals in the sample are exposed to \( k \) independently operating risks \( Y_1, Y_2, \ldots, Y_k \) with the distribution function \( P_i(x) = P[Y_i < x] \) and the probability density \( p_i(x) \), \( i = 1, 2, \ldots, k \). \( n_i \) individuals fail from the \( i^{th} \) risk, where \( n = \sum_{i=1}^{k} n_i \). Then, the likelihood function is given by

\[
L = n! \left( \prod_{i=1}^{k} n_i ! \right)^{-1} \prod_{i=1}^{k} \left[ \frac{n_i}{\int_{x_{ij}} p_i(x_{ij})} \right]^{n_i} \left( \prod_{i=1}^{k} \int_{x_{ij}} \theta_i \right)^{-n} \quad (2.2)
\]

where, \( x_{ij} = \) time to failure of the \( i^{th} \) individual from the \( j^{th} \) cause.
Sampford (1952), using a slightly different argument, obtained the same likelihood function for the cases when \( k = 2, 3 \). Moeschberger and David (1971) and Hoel (1972) obtained the maximum likelihood estimators of the various parameters when the underlying distributions are assumed to be exponential and Weibull with equal and unequal shape parameters. All these authors remarked that if each theoretical life distribution has a different set of parameters, then the estimation can be performed individually for each risk by maximizing

\[
L_i = \prod_{j=1}^{n_i} p_i(x_{ij}) \prod_{\ell=1}^{k} \bar{p}_\ell(x_{ij})
\]

with respect to the parameter associated with the probability density function \( p_i(y) \).

Moeschberger and David (1971) modified the likelihood function (2.2) to include Type I and Type II censoring. They also obtained the maximum likelihood estimators of the parameters when the lifetimes are grouped into intervals.

They also exploited the Marshall-Olkin (1967) model to handle the case of dependent risks, when the dependence between the two risks is entirely due to the possibility of simultaneous failure from both the risks. The two dependent risks can be dealt with as if there were three independent risks, the third risk corresponds to the simultaneous failure due to two risks.

We can make use of the asymptotic considerations of the maximum likelihood theory for developing tests for these parameters. Three broad types of asymptotic procedures, based on likelihood, are available for testing the null hypothesis,
$H_0 : \theta = \theta_0$, or for constructing confidence intervals for the unknown parameters (see, for example, Rao (1974), Miller (1981) and Cox and Oakes (1984)).

Suppose that the following set of regularity conditions, $A_1$, are satisfied. Let $X$ be a random variable with probability density function $p(\cdot, \theta)$ with respect to $\sigma$-finite measure $\nu$, where $\theta$ belongs to a non-degenerate interval of the real line. Further, assume that,

(a) The derivatives $\frac{3}{\partial \theta} \log p$, $\frac{2}{\partial \theta^2} \log p$, and $\frac{3}{\partial \theta^3} \log p$ exist for almost all $x$ in an interval $A$ of $\theta$, which includes the value of $\theta$.

(b) At the true value of $\theta$,

$$E\left[\frac{p'(x, \theta)}{p(x, \theta)} \mid \theta\right] = 0, \quad E\left[\frac{p''(x, \theta)}{p(x, \theta)} \mid \theta\right] = 0, \quad E\left[\left(\frac{p'(x, \theta)}{p(x, \theta)}\right)^2 \mid \theta\right] > 0$$

where the derivatives are taken with respect to $\theta$.

(c) For every $\theta$ in $A$

$$\left| \frac{3}{\partial \theta^3} \log p \right| < M(x), \quad E[M(X) \mid \theta] < k$$

where $k$ is independent of $\theta$.

The three procedures are, respectively, based on the following test statistics.

(1) Wilks' likelihood ratio statistic,

$$2[\log L(\hat{\theta}) - \log L(\theta_0)]$$

(2) Wald's statistic,

$$\frac{(\hat{\theta} - \theta_0)^2}{\nu(\hat{\theta})} \bigg|_{\theta = \theta_0}$$
(3) Rao's statistic,

\[
\left[ \frac{\partial}{\partial \theta} \log L(\theta) \right]_{\theta = \theta_0}^2
\]

\[
\frac{v(\hat{\theta})}{v(\theta)} \mid_{\theta = \theta_0}
\]

where \( L(\theta) \) is the likelihood function in \( \theta \) and \( \hat{\theta} \) is the maximum likelihood estimator of \( \theta \).

The above three procedures very often give virtually identical results. Each one of the statistics has asymptotically chi-square distribution with 1 degree of freedom. For a comparison of these procedures, see Cox and Oakes (1984).

We now derive the likelihood ratio statistic when the underlying distributions are exponential. Actually, exponential distributions have often been used to describe the failure times of individuals in reliability studies. Davis (1952) has shown that exponential distribution provides a good fit to some lifetime data. It is also of great practical significance because of its simple mathematical form (Epstein and Sobel (1954)) and its association with Poisson processes (Barlow and Proschan (1965)). It is also the only continuous distribution with a constant hazard rate and the lack of memory property.

Let

\[
F(x) = \begin{cases} 
\exp(-ax), & x \geq 0 \\
0, & \text{otherwise}.
\end{cases}
\]

\[
G(x) = \begin{cases} 
\exp(-ax), & x \geq 0 \\
0, & \text{otherwise}.
\end{cases}
\]

where \( a = \theta \phi \), \( \theta > 0 \).

We wish to test the hypothesis

\[
H_0 : \theta = 1
\]
against the alternative

\[ H^1 : \theta \neq 1 \]

Hence, \( \theta \) is the parameter of interest and \( \phi \) is only a nuisance parameter.

The causes of failure given by \( \delta_1, \delta_2, \ldots, \delta_n \) are Bernoulli random variables with

\[
P(\delta_i = 1) = \int_0^\infty F(x) dG(x) = \frac{\alpha}{\alpha + \phi} \quad (2.4)
\]

From (2.1), the likelihood function is given by

\[
L(t, \delta, \alpha, \phi) = \prod_{i=1}^{n} [g(t_i) F(t_i)]^{\delta_i} [f(t_i) G(t_i)]^{1-\delta_i}
\]

Substituting the expressions for \( F \) and \( G \) from (2.3) in the above likelihood function, we get

\[
L(t, \delta, \alpha, \phi) = \alpha \phi^2 \exp\left[-(\alpha + \phi)T\right] \quad (2.5)
\]

where,

\[
T = \sum_{i=1}^{n} t_i
\]

= sum of the observed lifetimes of all the \( n \) individuals in the sample.

\[
n_1 = \sum_{i=1}^{n} \delta_i
\]

= deaths due to cause 2.

\[
n_2 = n - n_1.
\]

Since \( \delta_i \)'s are Bernoulli random variables with probability of success \( \alpha/(\alpha + \phi) \), it follows that
\[ n_1 \sim B(n, \frac{\alpha}{\alpha + \phi}) \quad (2.6) \]

and,
\[ n_2 \sim B(n, \frac{\phi}{\alpha + \phi}) \quad (2.7) \]

where \( B(n,p) \) denotes the Binomial distribution with parameters \( n \) and \( p \).

Then, using (2.5), it follows that the maximum likelihood estimates of \( \alpha \) and \( \phi \) are given by
\[ \hat{\alpha} = \frac{n_1}{T}, \quad \hat{\phi} = \frac{n_2}{T} \quad (2.8) \]

Thus, \( \hat{\alpha} \) is the ratio of individuals failing from second cause to the sum of lifetimes of all the \( n \) individuals in the sample. Similarly, we can interpret \( \hat{\phi} \).

Therefore,
\[ \log L(t, \delta, \hat{\alpha}, \hat{\phi}) = n_1 \log \frac{n_1}{T} + n_2 \log \frac{n_2}{T} - n \]

Under \( H_0 \), the likelihood function is
\[ L(t, \delta, \phi) = \phi^n \exp(-2\phi T) \]

Hence, maximum likelihood estimate of \( \phi \) is
\[ \hat{\phi} = \frac{n}{2T} \]

Therefore,
\[ \log L(t, \delta, \hat{\phi}) = n \log \frac{n}{2T} - n \]

The Wilks' likelihood ratio statistic is,
\[ W = 2[\log L(t, \delta, \hat{\alpha}, \hat{\phi}) - \log L(t, \delta, \hat{\phi})] \]
\[ = 2[n_1 \log \frac{n_1}{T} + n_2 \log \frac{n_2}{T} - n \log \frac{n}{2T}] \quad (2.9) \]

which has asymptotically chi-square distribution with one degree of freedom.
The test for testing $H_0$ against $A$ is to reject $H_0$ for large values of the test statistic $W$.

2.4 Next, we consider another test based on parametric considerations as given by Neyman (1959). Efron (1967) made use of the same theory to derive tests for the equality of distribution functions in the presence of censoring.

Suppose $f_{\theta}(x)$ is a probability density function indexed by a vector valued parameter $\theta$ with $(k+1)$ components, $\theta = (\theta^0, \theta^1, \ldots, \theta^k)$ and having information matrix $I_{\theta} = [I_{ij}, 0 \leq i, j \leq k]$, where,

\[ I_{ij} = \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log f_{\theta}(x), \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log f_{\theta}(x) \]

It is desired to test, $H_0 : \theta = \theta_0$ on the basis of repeated independent observations $X_1, X_2, \ldots, X_n$ from $f_{\theta}(x)$. Let $C$ be the class of all test statistics $T_n(X_1, X_2, \ldots, X_n)$ such that, under $H_0$, $n^{1/2}(T_n - \mu)$ converges to a normal random variable with mean zero and the variance $\sigma^2(\theta_0)$ for some constant $\mu$ not depending on $\theta_0 \in H_0$. Then, under suitable regularity conditions, the efficacy (see, for example, Puri and Sen (1971)) of any member of $C$ is bounded by $\frac{1}{\Gamma_{00}}$, where $\Gamma_{00}$ is the upper left entry of $\Gamma^{-1}$. That is,

\[ \lim_{n \to \infty} \frac{\frac{3}{\theta} \mathbb{E}(T_n)_{\theta = \theta_0}}{n V_{H_0}(T_n)} \leq \frac{1}{\Gamma_{00}}, \quad (2.10) \]

for all $T_n \in C$.

Moreover, the maximum likelihood estimator $\hat{\theta}_0$ of $\theta_0$ attains the upper bound. The bound in (2.10) can be obtained formally from the multiparameter Cramer-Rao inequality.
Suppose that F and G are given by (2,3). From (2.8), we know that
\[ \hat{\alpha} = \frac{n_1}{n}, \quad \hat{\phi} = \frac{n_2}{n}. \]

Differentiating (2.5) twice and using (2.6) and (2.7), we get
\[ \mathbb{E}\left[-\frac{1}{\alpha^2} \log L(t, \hat{\theta}, a, \phi)\right] = n[a(a+\phi)]^{-1} \]
\[ \mathbb{E}\left[-\frac{1}{\phi^2} \log L(t, \hat{\theta}, a, \phi)\right] = n[\phi(a+\phi)]^{-1} \]

Therefore,
\[ \mathbb{V}(\hat{\alpha}) = \frac{a(a+\phi)}{n}, \quad \mathbb{V}(\hat{\phi}) = \frac{\phi(a+\phi)}{n}. \]

Further,
\[ \frac{\partial^2}{\partial \alpha \partial \phi} \log L(t, \hat{\theta}, a, \phi) = \frac{\partial^2}{\partial \phi \partial \alpha} \log L(t, \hat{\theta}, a, \phi) = 0 \]

Hence, \( \hat{\alpha} \) and \( \hat{\phi} \) are uncorrelated.

The variance - covariance matrix of \( \hat{\mathbf{Z}} = [\hat{\alpha}, \hat{\phi}] \), say, \( \Sigma \), is given by
\[ \Sigma = \begin{bmatrix} \frac{a(a+\phi)}{n} & 0 \\ 0 & \frac{\phi(a+\phi)}{n} \end{bmatrix} \]

Let \( \theta = \frac{\alpha}{\phi} = g(\alpha, \phi) \).

Then, from the invariance property of M.L.E's, it follows that
\[ g(\hat{\alpha}, \hat{\phi}) = \frac{n_1}{n_2} \]
is the M.L.E. of \( \theta \). Further, \( g(\hat{\alpha}, \hat{\phi}) \) has limiting normal distribution (see, for example, Rao (1974)).
\[ \mathbb{V}(\hat{\theta}) = \mathbb{V}[g(\hat{\alpha}, \hat{\phi})] = \hat{\mathcal{C}}' \Sigma \hat{\mathcal{C}} \]

where,
\[ \hat{\mathcal{C}}' = \begin{bmatrix} \frac{\partial}{\partial \alpha} g(\alpha, \phi), \frac{\partial}{\partial \phi} g(\alpha, \phi) \end{bmatrix} \]
\[ = \begin{bmatrix} 1 \phi \alpha \phi^2 \end{bmatrix} \]
Hence,
\[ V(\hat{\theta}) = \frac{\sigma(a + \phi)}{n\phi^2} + \frac{\sigma^2(a + \phi)}{n\phi^3}, \]
Under \( H_0, \theta = 1 \).
\[ V(\hat{\theta}) \bigg|_{\theta=1} = \frac{4}{n} \]
Using the results of Neyman (1959), we get that efficacy of \( \hat{\theta} \), given by \( e(\hat{\theta}) \), is
\[ e(\hat{\theta}) = \frac{1}{4} \] (2.11)

**Locally Most Powerful Tests**

In this section we consider locally most powerful tests (see, for example, Rao (1974)) under the competing risk framework for various one-sided alternatives.

Let \( p(x, \theta) \) be the probability density function of \( X_1, X_2, \ldots, X_n \) with respect to a \( \sigma \)-finite measure \( \psi \).

Let \( \theta = \theta_0 \) be the specified value of the parameter. Then a region \( w \) of the sample space is locally most powerful for a one-sided alternative, if the conditions

i) \[ \int_w p(x, \theta_0) \psi \, (dx) = a, \] where \( a \) is the level of significance.

ii) \[ c \int_w p'(x, \theta_0) \psi \, (dx) \] is maximum,

are satisfied, where \( c = +1 \) or \(-1\) according as the alternatives are \( \theta > \theta_0 \) or \( \theta < \theta_0 \).

Theorem 2.1 (Rao (1974)) gives a technique for determining such a region.

**Theorem 2.1 (Rao)** If whatever may be the region \( w \) in the sample space, the integral \( \int_w p'(x, \theta_0) \psi \, (dx) \) exists, then the
region $w_0$ within which $\phi'(x, \theta_0) > kp(x, \theta_0)$ and outside which $\phi'(x, \theta_0) < kp(x, \theta_0)$, where $k$ is so determined that

$$\int p(x, \theta_0) \phi'(dx) = a,$$

is the locally most powerful (LMP) for alternative $\theta > \theta_0$ and $\theta < \theta_0$ according as $c = +1$ or $-1$.

Now we propose LMP tests for testing $H_0 : \theta = 0$, against the alternative $H_1 : \theta > 0$.

**Theorem 2.2** The LMP test for testing $H_0 : \theta = 0$ against the alternative $H_1 : \theta > 0$, is given by: Reject $H_0$, if

$$\sum_{i=1}^{n} \delta_i \left[ \frac{f(t_i)}{F(t_i)} + \frac{f^*(t_i)}{F(t_i)} \right] - k \sum_{i=1}^{n} \frac{f^*(t_i)}{F(t_i)} > k$$

(2.12)

where $f^*(x) = \frac{\partial}{\partial \theta} F_\theta(x)|_{\theta=0}$, $\delta_i = \frac{\partial}{\partial \theta} F_\theta(x)|_{\theta=0}$, and $k$ is chosen such that $P_{H_0} [\text{Rejecting } H_0] = \alpha$ (2.13)

**Proof** From (2.1), the likelihood function in $\theta, (\theta > 0)$, is given by

$$L(t, \delta, \theta) = \prod_{i=1}^{n} \left[ f_\theta(t_i) F(t_i) \right] \left[ F_\theta(t_i) f(t_i) \right]^{1-\delta_i}$$

where, $F_\theta(x) = G(x)$

and, $f_\theta(x) = \frac{\partial}{\partial \theta} F_\theta(x) = g(x)$

Then,

$$\log L(t, \delta, \theta) = \sum_{i=1}^{n} \delta_i \left[ \log f_\theta(t_i) + \log F(t_i) \right]$$

$$+ \sum_{i=1}^{n} (1- \delta_i) \left[ \log f(t_i) + \log F_\theta(t_i) \right]$$

$$\frac{\partial}{\partial \theta} \log L(t, \delta, \theta) \bigg|_{\theta=0} = \sum_{i=1}^{n} \delta_i \frac{f(t_i)}{f(t_i)} - \sum_{i=1}^{n} (1- \delta_i) \frac{f^*(t_i)}{F(t_i)}$$
where, \[ f^*(x) = f_{\theta}^*(x) \Big|_{\theta=0} = \frac{2}{\theta} f_{\theta}^0(x) \Big|_{\theta=0} \] (2.14a)
and \[ f(x) = f_{\theta}(x) \Big|_{\theta=0} = \frac{2}{\theta} f^0(x) \Big|_{\theta=0} \] (2.14b)

Therefore,
\[
\frac{\partial}{\partial \theta} \log L(t, \delta, \theta) \Big|_{\theta=0} = \sum_{i=1}^{n} \delta_i \left[ \frac{f(t_i)}{F(t_i)} + \frac{f^*(t_i)}{F(t_i)} \right] - \sum_{i=1}^{n} \frac{f^*(t_i)}{F(t_i)} \]

Then, using Theorem 2.1, the LMP test is to reject \( H_0 : \theta = 0 \) against the alternative \( H_{\theta'}^* \), when
\[
\sum_{i=1}^{n} \delta_i \left[ \frac{f(t_i)}{F(t_i)} + \frac{f^*(t_i)}{F(t_i)} \right] - \sum_{i=1}^{n} \frac{f^*(t_i)}{F(t_i)} > k
\]
where \( k \) is chosen to satisfy (2.13).

Asymptotic normality of the LMP test follows from the following theorem (see, for example Rao (1974)).

**Theorem 2.3 (Rao)** Suppose the LMP test for testing \( H_0 \) against \( H_{\theta'}^* \) is as given by Theorem 2.1. Further, let the density \( p(x, \theta) \) satisfy the regularity conditions \( A_1 \) mentioned above, such that
\[
E_0 \left[ \frac{p'(x, \theta_0)}{p(x, \theta_0)} \bigg| \theta_0 \right] = 0
\]
and,
\[
V_0 \left[ \frac{p'(x, \theta_0)}{p(x, \theta_0)} \bigg| \theta_0 \right] = i(\theta_0)
\]
exists, where \( i(\theta_0) \) is the Fisher's information in a single
observation at $\theta = \theta_0$.

Then, for large $n$,

$$\left[n i(\theta)\right]^{-1/2} \sum_{r=1}^{n} \frac{p'(x_r, \theta_0)}{p(x_r, \theta_0)}$$

has limiting standard normal distribution. Under $H_0$, $i(\theta)$ is replaced by $i(\theta_0)$.

Information contained in a random sample of size $n$ is

$$n i(\theta_0) = E \left[- \frac{\partial^2}{\partial \theta^2} \log L(t, \theta, \theta_0) \bigg|_{\theta=\theta_0} \right]$$

(2.15)

Now,

$$\frac{\partial^2}{\partial \theta^2} \log L(t, \theta, \theta_0) \bigg|_{\theta=0} = \sum_{i=1}^{n} \xi_i \frac{\bar{F}(t_i) f(t_i) - (\bar{F}(t_i))^2}{(f(t_i))^2}$$

$$- \sum_{i=1}^{n} (1 - \xi_i) \left[ \frac{\bar{F}^*(t_i) \bar{F}(t_i) + (\bar{F}(t_i))^2}{[\bar{F}(t_i)]^2} \right]$$

where, $\bar{F}(x) = \frac{\partial}{\partial \theta} \bar{F}(x) \bigg|_{\theta=0}$

(2.16a)

and, $\bar{F}^*(x) = \frac{\partial}{\partial \theta} f_{\theta}(x) \bigg|_{\theta=0}$

(2.16b)

Hence,

$$\frac{\partial^2}{\partial \theta^2} \log L(t, \theta, \theta_0) \bigg|_{\theta=0} = \sum_{i=1}^{n} \xi_i \left\{ \frac{\bar{F}^*(t_i) \bar{F}(t_i) + (\bar{F}(t_i))^2}{[\bar{F}(t_i)]^2} \right\}$$

$$- \sum_{i=1}^{n} \left[ \frac{\bar{F}^*(t_i) \bar{F}(t_i) + (\bar{F}(t_i))^2}{[\bar{F}(t_i)]^2} \right]$$

(2.17)
We will derive the LMP tests for scale and location alternatives in subsequent sections.

2.6 Locally Most Powerful Rank Tests

In this section we derive the locally most powerful rank (LMP rank) tests for testing \( H_0 \) against the alternative \( H'_\nu \). (see, for example, Hájek and Sidák (1967), and Puri and Sen (1971)). These results have been exploited by Prentice (1973) to find LMP rank tests for the regression problem in the presence of censoring and by Cox and Oakes (1984) to find LMP rank tests for observations that follow the accelerated life model.

Suppose that the following regularity conditions are satisfied (see, for example, Puri and Sen (1971)).

(a) \( F(x, \theta) \) has a density function \( f(x, \theta) \) which alongwith \( \frac{\partial}{\partial \theta} f(x, \theta) \) is continuous with respect to \( \theta \) for \( \theta_0 - a < \theta < \theta_0 + a \), \( a > 0 \), for almost all \( x \). There exist functions \( M_0(x) \) and \( M_1(x) \) integrable over \((-\infty, \infty)\), such that

\[
f(x, \theta) \leq M_0(x), \quad \left| \frac{\partial}{\partial \theta} f(x, \theta) \right| \leq M_1(x),
\]

for \( \theta_0 - a < \theta < \theta_0 + a \).

(b) \( f(x, \theta) > 0 \), if and only if \( f(x, \theta_0) > 0 \)

(c) \( |J^{(1)}(u)| = \left| \frac{d}{du} J(u) \right| \leq k [u(1-u)]^{-i-\frac{1}{2}} + \delta \),

for \( i = 0, 1 \) and for some \( \delta > 0 \), where \( k \) is a constant and where \( J \) is the score function.

From (2.1) the likelihood function in \( \theta \) is given by

\[
L(t, \mathbf{s}, \theta) = \prod_{i=1}^{n} \left[ f_{\theta}(t_i) \Phi(t_i) \right]^{S_i} \left[ f_{\theta}(t_i) \Phi(t_i) \right]^{1-S_i}.
\]
where $t_1, t_2, \ldots, t_n$ are the observed times to failure and $\delta_i$'s indicate the causes of death. Without loss of generality, let $0 < t_1 < t_2 < \ldots < t_n < \infty$. Let

$$W_j = \begin{cases} 1, & \text{if } T \text{ corresponds to a } Y \text{ observation,} \\ 0, & \text{otherwise.} \end{cases}$$

where, $T(j)$ denotes the $j$th smallest $T$ observation.

The likelihood of ranks $P(t, w, \theta)$ is then obtained by integrating $L(t, w, \theta)$ over the range $0 < t_1 < t_2 < \ldots < t_n < \infty$, where,

$$L(t, w, \theta) = \prod_{i=1}^{n} \left[ f_{\theta}(t_i) \Phi(t_i) \right]^w_i \left[ f(t_i) \Phi(t_i) \right]^{1-w_i}$$

$$P(t, w, \theta) = \int \int \ldots \int \{ \prod_{i=1}^{n} \left[ f_{\theta}(t_i) \Phi(t_i) \right]^w_i \}
\quad \times [f(t_i) \Phi(t_i)]^{1-w_i} \ dt_i \} \quad (2.18)$$

In particular,

$$P(t, w, 0) = \int \int \ldots \int \left[ \prod_{i=1}^{n} f(t_i) \Phi(t_i) \right] dt_i \}
\quad = \int \int \ldots \int \left[ \prod_{i=1}^{n} (1-u_i) \right] du_i \}
\quad = \frac{1}{2^n n!} \quad (2.19)$$

**Theorem 2.4** Under the regularity conditions (a), (b) and (c), the LMP rank test for testing $H_0 : \theta = 0$, against the alternative $H_A$ is based on the test statistic $\sum_{j=1}^{n} [a_j W_j - b_j (1-W_j)]$, where $a_j$ and $b_j$ are defined in (2.22) and (2.23), respectively.
Proof. LMP rank test statistic for the above problem is obtained from the derivative of \( \log P(t,w,\theta) \) with respect to \( \theta \), evaluated at \( \theta = 0 \). Hence,

\[
\frac{\partial}{\partial \theta} \log P(t,w,\theta) \bigg|_{\theta=0} = \frac{p'(t,w,\theta)}{p(t,w,\theta)} \bigg|_{\theta=0}
\]

\[
\left[ p(t,w,0) \right]^{-1} \int \cdots \int_{0 < t_1 < \ldots < t_n < \infty} \frac{\partial}{\partial \theta} \left\{ \prod_{i=1}^{n} \left[ f_\theta(t_i) \right] \right\} \prod_{i=1}^{n} dt_i \quad (2.20)
\]

First, consider the case for \( n = 2 \). In this case the integrand of (2.20) is

\[
\frac{\partial}{\partial \theta} \left\{ \prod_{i=1}^{2} \left[ f_\theta(t_i) \right] \frac{w_i}{f(t_i)} \right\} \prod_{i=1}^{2} dt_i \bigg|_{\theta=0}
\]

\[
= w_1 \frac{f_\theta(t_1)}{f(t_1)} \prod_{i=1}^{2} f(t_i) \bar{F}(t_i)
\]

\[
+ w_2 \frac{f_\theta(t_2)}{f(t_2)} \prod_{i=1}^{2} f(t_i) \bar{F}(t_i)
\]

\[
- (1-w_1) \frac{f_\theta(t_1)}{\bar{F}(t_1)} \prod_{i=1}^{2} f(t_i) \bar{F}(t_i)
\]

\[
- (1-w_2) \frac{f_\theta(t_2)}{\bar{F}(t_2)} \prod_{i=1}^{2} f(t_i) \bar{F}(t_i)
\]

\[
= \sum_{j=1}^{2} \left\{ w_j \frac{f(t_j)}{f(t_j)} - (1-w_j) \frac{f_\theta(t_j)}{\bar{F}(t_j)} \right\} \prod_{i=1}^{2} f(t_i) \bar{F}(t_i)
\]

where \( f(x) \) and \( f_\theta(x) \) are as defined in (2.14a) and (2.14b), respectively.
In general, it can be seen that,
\[
\frac{\partial}{\partial \theta} \left\{ \prod_{i=1}^{n} \left[ f_{\theta}(t_i) \frac{F(t_i)}{f(t_i)} \right]^{w_i} \left[ f_{\theta}(t_i) \frac{1}{F(t_i)} \right]^{-1} \right\} \bigg|_{\theta=0} = \sum_{j=1}^{n} \left( w_j \frac{F(t_j)}{f(t_j)} - (1-w_j) \frac{f(t_j)}{F(t_j)} \right) \times \prod_{i=1}^{n} f(t_i)F(t_i) \tag{2.21}
\]

Using (2.19) and (2.21) in (2.20), we get
\[
\frac{\partial}{\partial \theta} \log F(t_w, \theta) \bigg|_{\theta=0} = 2^n n! \int \cdots \int_{0<t_1<\cdots<t_n<\infty} \left\{ \sum_{j=1}^{n} w_j \frac{F(t_j)}{f(t_j)} \right\} \prod_{i=1}^{n} f(t_i)F(t_i) \, dt_i
\]
\[
= \sum_{j=1}^{n} \left[ a_j w_j - b_j (1-w_j) \right]
\]
where,
\[
a_j = n! 2^n \int \cdots \int_{0<t_1<\cdots<t_n<\infty} \left\{ \frac{F(t_j)}{f(t_j)} \prod_{i=1}^{n} f(t_i)F(t_i) \, dt_i \right\} \tag{2.22}
\]
and
\[
b_j = n! 2^n \int \cdots \int_{0<t_1<\cdots<t_n<\infty} \left\{ \frac{f(t_j)}{F(t_j)} \prod_{i=1}^{n} [f(t_i)F(t_i) \, dt_i] \right\} \tag{2.23}
\]

Thus, the test statistic is of the form \( \sum_{j=1}^{n} [a_j w_j - b_j (1-w_j)] \), where \( a_j \) and \( b_j \) are as defined above.

The test is: Reject \( H_0 \) in favour of \( H_A \) for large values of the test statistic.

2.7 LMP and LMP Rank Tests for Scale Alternatives

Now, we will make use of the results of the preceding two sections to derive LMP and LMP rank tests for testing
H₀ : F(x) = G(x), for every x > 0 against the scale alternative, 
H₁ : G(x) = F((θ + l)x), for every x > 0, θ > 0,
that is, the two distribution functions are the same but for
a change in the scale parameter.

**Locally Most Powerful Tests**

**Theorem 2.5**  The LMP test for testing H₀ against H₁ is given by, reject H₀, if

\[
\sum_{i=1}^{n} \left\{ \delta_i \left[ \frac{f(t_i) + t_i f'(t_i)}{f(t_i)} + \frac{t_i f(t_i)}{F(t_i)} - \frac{t_i f(t_i)}{G(t_i)} \right] \right\} > k
\]

(2.24)

where k is so chosen to satisfy (2.13).

**Proof**  The general expression for the test statistic was derived in Theorem 2.2. We just have to find the expression for \( T(x) \) and \( f^*(x) \) under H₁.

When

\[
G(x) = F((θ + l)x)
\]

\[
f_θ(x) = (θ + l) f((θ + l)x)
\]

\[
T(x) = \frac{\partial}{\partial θ} f_θ(x) \bigg|_{θ=0}
\]

\[
= f[(θ+1)x] + x(θ+1)f'[((θ+1)x)]\bigg|_{θ=0}
\]

\[
= f(x) + x f(x)
\]

(2.25)

\[
f^*(x) = \frac{\partial}{\partial θ} F[(θ + l)x] \bigg|_{θ=0}
\]

\[
= x f[(θ + l)x] \bigg|_{θ=0}
\]

\[
= x f(x)
\]

(2.26)

Hence, from (2.12), it follows that the LMP test for testing H₀ against H₁ is to reject H₀, when
Now we shall derive an expression for the test statistic when the underlying distribution is standard exponential. (See 41a)

\[ f(x) = \exp(-x), \quad x > 0 \]

\[ F(x) = \exp(-x), \]

\[ f'(x) = -\exp(-x). \] (2.27)

Then,

\[
\sum_{i=1}^{n} \left\{ \delta_{i} \left[ \frac{f(t_{i})+t_{i}f'(t_{i})}{f(t_{i})} + \frac{t_{i}f(t_{i})}{F(t_{i})} - \frac{f(t_{i})}{F(t_{i})} \right] \right\} > k,
\]

where \( k \) is chosen to satisfy (2.13).

Therefore, when the underlying distribution is standard exponential, the test statistic for the LMP test given by (2.25) and (2.13) is

\[
V_{1} = \sum_{i=1}^{n} \delta_{i} - \sum_{i=1}^{n} t_{i}, \quad (2.28)
\]

Asymptotic Distribution of the \( V_{1} \)-Statistic

Asymptotic normality of \( V_{1} \) is proved in Theorem 2.6 below.

**Theorem 2.6** Under suitable regularity conditions \([n_{i}(\theta)]^{-1/2}v_{l}\) has limiting standard normal distribution where \( n_{i}(\theta) \) is Fisher's information in the sample at \( \theta \). In particular, under \( H_{0} \), if \( f(x) = \exp(-x), \quad x > 0, \quad n_{i}(\theta) = \frac{1}{2} n, \) so that \([n_{i}]^{-1/2}v_{l}, \) has asymptotically standard normal distribution.

**Proof** The asymptotic normality of \( V_{1} \) follows from Theorem 2.3. We just have to find an expression for \( i(\theta_{0}) \), when the underlying
The exponential distribution satisfies the conditions $A_1$ for the existence of the LMP test. The conditions (a) and (b) hold for every absolutely continuous and bounded density function whose range is independent of the parameter. Also,

$$p(x, \theta) = (\theta+1) \exp[-(\theta+1)x], \quad x \geq 0, \quad \theta > 0.$$  

$$\log p(x, \theta) = \log(\theta+1) - (\theta+1)x$$

$$\frac{\partial}{\partial \theta} \log p(x, \theta) = \frac{1}{\theta+1} - x$$

$$\frac{\partial^2}{\partial \theta^2} \log p(x, \theta) = -\frac{1}{(\theta+1)^2}$$

$$\frac{\partial^3}{\partial \theta^3} \log p(x, \theta) = \frac{2}{(\theta+1)^3}$$

Therefore,

$$\left| \frac{\partial^3}{\partial \theta^3} \log p(x, \theta) \right| = \frac{2}{(\theta+1)^3} < 3 = M(x) \quad \text{(say)}.$$  

And,

$$E[M(X) | \theta] < 3 \quad (=k),\quad \text{where } k \text{ is independent of } \theta.$$
of success, \( p = P[X > Y] = \int_0^\infty G(x) dF(x) \).

However, under \( H_0 \), \( p = 1/2 \).

Thus,

\[
n_i(0) = E_{H_0} \left[ - \frac{\partial^2}{\partial \theta^2} \log L(t, S, \theta) \right]_{\theta=0} = \frac{1}{2} n \quad (2.29)
\]

**Locally Most Powerful Rank Tests**

**Theorem 2.7** Under the regularity conditions (a), (b) and (c) of Section 2.6, the LMP rank test for testing \( H_0 \) against \( H_1 \) is based on the test statistic \( \sum_{j=1}^n [a_j W_j - b_j (1-W_j)] \), where \( a_j \) and \( b_j \) are as given by (2.30) and (2.31), respectively.

**Proof** The general expression for the test statistic was derived in Theorem 2.4. We just have to evaluate \( a_j \) and \( b_j \) for the given problem.

From (2.25) and (2.26'), it follows that when \( G(x) = F[(\theta + 1)x] \),

\[
\mathbb{P}(x) = f(x) + xf'(x)
\]

and,

\[
f^*(x) = xf(x).
\]

Therefore, from (2.22) and (2.23), we get that,

\[
a_j = n! 2^n \int \cdots \int_{0 < t_1 < \ldots < t_n < \infty} \left[ \frac{\Phi(t_j)}{f(t_j)} \prod_{i=1}^n f(t_i) \Phi(t_i) dt_i \right]
\]

\[
= n! 2^n \int \cdots \int_{0 < t_1 < \ldots < t_n < \infty} \left\{ \frac{f(t_j) + t_j f'(t_j)}{f(t_j)} \prod_{i=1}^n f(t_i) \Phi(t_i) dt_i \right\}
\]

\[
= n! 2^n \int \cdots \int_{0 < t_1 < \ldots < t_n < \infty} \left\{ 1 + \frac{t_j f'(t_j)}{f(t_j)} \prod_{i=1}^n f(t_i) \Phi(t_i) dt_i \right\}
\]

\[
= 1 + n! 2^n \int \cdots \int_{0 < t_1 < \ldots < t_n < \infty} \left[ \frac{t_j f'(t_j)}{f(t_j)} \prod_{i=1}^n f(t_i) \Phi(t_i) dt_i \right]
\]
and 

\[ b_j = n!2^n \int_{0}^{t_n} \int_{t_{j-1}}^{t_j} \prod_{i=1}^{n} f(t_i) \frac{f(t_i)}{F(t_i)} dt_i \]

Let 

\[ u_i = F(t_i), \quad C(u) = F^{-1}(u) \]

Then, 

\[ a_j = 1 + n!2^n \int_{0}^{t_n} \int_{u_{j-1}}^{u_j} \prod_{i=1}^{n} (1-u_i) du_i \] \hspace{1cm} (2.30)

where, 

\[ \Psi(u) = \frac{C(u)f'(C(u))}{f(C(u))} \]

and, 

\[ b_j = n!2^n \int_{0}^{t_n} \int_{u_{j-1}}^{u_j} \prod_{i=1}^{n} (1-u_i) du_i \] \hspace{1cm} (2.31)

where, 

\[ \Psi(u) = \frac{C(u)f'(C(u))}{1-u} \]

Thus, the test statistic for the above testing problem is

\[ \sum_{j=1}^{n} \left[ a_j W_j - b_j (1-W_j) \right] = \sum_{j=1}^{n} (a_j + b_j) W_j - \sum_{j=1}^{n} b_j \]

where \( a_j \) and \( b_j \) are as defined by (2.30) and (2.31), respectively.

The LMP rank test for testing \( H_0 \) against \( H_1 \) is to reject \( H_0 \) in favour of \( H_1 \) for large values of

\[ \sum_{j=1}^{n} (a_j + b_j) W_j - \sum_{j=1}^{n} b_j, \]

or equivalently, for large values of

\[ \sum_{j=1}^{n} (a_j + b_j) W_j. \]

The expressions for \( a_j \) and \( b_j \) are not easy to evaluate for most of the distributions. Explicit expressions for \( a_j \)'s and \( b_j \)'s when the underlying distribution is standard exponential are given below.

\[ f(x) = e^{-x}, \quad x \geq 0. \]
implies, $$C(u) = - \log(1-u)$$

$$\Psi(u) = \frac{C(u)f(C(u))}{f(C(u))} = \log(1-u)$$

and $$\Psi'(u) = \frac{C(u)f(C(u))}{(1-u)} = - \log(1-u).$$

Thus, from (2.30) and (2.31), it follows that

$$a_j = 1 - b_j, \quad \text{for } j = 1, 2, \ldots, n \quad (2.32)$$

Therefore, the test statistic for the LMP rank test for testing $$H_0$$ against $$H_1$$ is

$$V_2 = \sum_{j=1}^{n} (a_j + b_j)W_j = \sum_{j=1}^{n} W_j = \sum_{j=1}^{n} \delta_j \quad (2.33)$$

The asymptotic properties of $$V_2$$ have been discussed in Chapter 3.

2.8 LMP and LMP Rank Tests for Location Alternatives

In this section, we will make use of the available information in the form of times to failure and causes of failure for deriving LMP and LMP rank tests for testing

$$H_0 : F(x) = G(x), \quad \text{for every } x \geq 0,$$

against the location alternative

$$H_A : G(x) = F(x+\theta), \quad \text{for every } x \geq 0, \theta > 0.$$ 

By changing $$x$$ to $$x + \theta$$, we shift the support to $$(\theta, \infty)$$. Thus, $$\theta > 0$$ indicates a threshold value for the lifetime such that smaller values of the lifetime are not possible. If the lifetime is positive valued, then the log of the lifetime or some other monotonic transformation can be constructed by which the transformed random variable has support $$(-\infty, \infty)$$. For such transformed variables the LMP and the LMP rank test to be derived for the location alternative will be relevant.
Here, we will slightly deviate from our usual set up in the following way. We will assume that the lifetimes may no longer be positive valued random variables, but may actually be spread over the entire real line. The normal distribution or the logistic distribution are seldom used to describe the lifelength of an individual since life distributions are frequently skewed. However, this form of life distributions is appropriate in certain cases (see, for example, Daniels (1945), Davis (1952), Sampford (1952), Nadas (1971), Moeschberger (1974), and Basu and Ghosh (1978)).

Davis (1952) compared the cumulative distribution function, probability density function and hazard rate function when the underlying distributions are normal, exponential, and human mortality (distribution fitted to mortality data of life insurances). They found that the failure rate function of human mortality is similar in general characteristics to that of the normal theory, except that in early life human mortality exhibits a nonzero failure rate. However, this is to be expected, since in early years human beings exhibit a small failure rate and as they grow older, they exhibit an increasing failure rate. Johnson and Johnson (1980) have also used normal theory for mortality data.

The failure rate function of a system which follows normal theory of failure has zero value in the early phases but increases at an accelerated rate throughout its life. This too is intuitively correct for some common objects like shoes and automobiles - a new system is reliable, an old one less reliable.

Locally Most Powerful Tests

Theorem 2.8 The LMP test for testing $H_0$ against $H_1$ is
given by, reject $H_0$, if

$$\sum_{i=1}^{n} b_i \left[ \frac{f(t_i)}{f(t_i)} + \frac{f(t_i)}{F(t_i)} \right] - \sum_{i=1}^{n} \frac{f(t_i)}{F(t_i)} \geq k \quad (2.34)$$

where $k$ is chosen to satisfy (2.13).

**Proof** The general expression for the test statistic was derived in Theorem 2.2. We just have to find the expressions for $\overline{I}(x)$ and $f'(x)$ under $H_{A_1}$.

When, $G(x) = F(x+\theta)$

$$f_\theta(x) = \frac{\partial}{\partial x} F(x+\theta) = f(x+\theta)$$

$$\overline{I}(x) = \frac{\partial}{\partial \theta} f_\theta(x) \bigg|_{\theta=0} = f'(x+\theta) \bigg|_{\theta=0} = f'(x) \quad (2.35a)$$

$$f'(x) = \frac{\partial}{\partial \theta} F(x+\theta) \bigg|_{\theta=0} = f(x+\theta) \bigg|_{\theta=0} = f(x) \quad (2.35b)$$

Hence, from (2.12), it follows that the LMF test for testing $H_0$ against $H_{A_1}$ is to reject $H_0$ when,

$$\sum_{i=1}^{n} b_i \left[ \frac{f'(t_i)}{f(t_i)} + \frac{f(t_i)}{F(t_i)} \right] - \sum_{i=1}^{n} \frac{f(t_i)}{F(t_i)} \geq k \text{.}$$

where $k$ is chosen to satisfy (2.13).

Now we will derive an expression for the test statistic when the underlying distribution is logistic, which is very much like normal distribution with somewhat heavier tails.
The logistic distribution also satisfied the conditions $A_1$ for the existence of the IMP tests. The conditions (a) and (b) hold for every absolutely continuous and bounded density function whose range is independent of the parameter. Also,

$$p(x, \theta) = \frac{\exp(-(x+\theta))}{[1+\exp(x+\theta)]^2}, \quad -\infty < x < \infty, \theta > 0.$$

$$\log p(x, \theta) = x+\theta - 2\log[1+\exp-(x+\theta)].$$

$$\frac{\partial}{\partial \theta} \log p(x, \theta) = 1 - \frac{2\exp(x+\theta)}{1+\exp(x+\theta)}$$

$$\frac{\partial^2}{\partial \theta^2} \log p(x, \theta) = -2 \frac{\exp(x+\theta)}{[1+\exp(x+\theta)]^2}$$

$$\frac{\partial^3}{\partial \theta^3} \log p(x, \theta) = 2 \frac{\exp2(x+\theta) - 2\exp(x+\theta)}{[1+\exp(x+\theta)]^3} + \frac{2\exp(x+\theta)}{[1+\exp(x+\theta)]^3}$$

$$< 3 = M(x) \text{ (say).}$$

And, $E[M(X)|\theta] < 3$ (say) where $k$ is independent of $\theta$.

$$f(x) = \frac{e^x}{(1+e^x)^2}, \quad -\infty < x < \infty$$

$$F(x) = \frac{1}{1+e^x}$$
$2n^{-1/2} V_3$ has asymptotically standard normal distribution under $H_0$.

**Proof** The asymptotic normality of $V_3$ follows from Theorem 2.3. We just have to find an expression for $i(\theta_0)$ when the underlying distribution is logistic.

From (2.17), we know that

$$\left. \frac{\partial^2}{\partial \theta^2} \log L(t, \delta, \theta) \right|_{\theta = 0} = n \sum_{i=1}^{n} \delta_i \left[ \frac{\bar{T}(t_i) f(t_i) - (\bar{T}(t_i))^2}{(f(t_i))^2} \right]$$

$$- \sum_{i=1}^{n} (1 - \delta_i) \left[ \frac{\bar{T}'(t_i) F(t_i) + (\bar{T}'(t_i))^2}{(F(t_i))^2} \right]$$

where $\bar{T}'(x)$ and $\bar{T}(x)$ are as defined in (2.16a) and (2.16b).

In particular, if

$$F_\theta(x) = F(x + \theta)$$

$$\bar{T}(x) = \left. \frac{\partial}{\partial \theta} T_\theta(x) \right|_{\theta = 0} = \left. \frac{\partial}{\partial \theta} f(x + \theta) \right|_{\theta = 0} = f'(x)$$

$$\bar{f}(x) = \left. \frac{\partial}{\partial \theta} f_\theta(x) \right|_{\theta = 0} = \left. \frac{\partial}{\partial \theta} f(x + \theta) \right|_{\theta = 0} = f'(x)$$

Hence, it follows that

$$\left. \frac{\partial^2}{\partial \theta^2} \log L(t, \delta, \theta) \right|_{\theta = 0} = n \sum_{i=1}^{n} \delta_i \left[ \frac{f''(t_i) f(t_i) - (f'(t_i))^2}{f(t_i)^2} \right]$$

$$- \sum_{i=1}^{n} (1 - \delta_i) \left[ \frac{f'(t_i) F(t_i) + (f(t_i))^2}{F(t_i)^2} \right]$$

$$= n \sum_{i=1}^{n} \delta_i \left[ -2e^{t_i(1+e^{t_i})} - 1 \right] - \sum_{i=1}^{n} (1 - \delta_i) \left[ e^{t_i(1+e^{t_i})} - 1 \right].$$
Under \( H_0 \), \( S_i \) and \( T_i \) are independent. Therefore,

\[
ni(e_0) = E_{H_0}[t_i \frac{e_i}{1 + e_i}]
\]

Under \( H_0 \), \( S_i \) and \( T_i \) are independent. Therefore,

\[
ni(e_0) = E_{H_0}[t_i \frac{e_i}{1 + e_i}]
\]

\[
= \frac{n}{i=1} \left[ 1 + E_{H_0}(S_i) \right] E_{H_0}[e_i (1 + e_i)^{-2}].
\]

\[
E_{H_0}(S_i) = P_{H_0}(X_i > Y_i) = \frac{1}{2}.
\]

Let \( \overline{\Phi}(t) \) be the survival function of \( T = \min(X, Y) \). Let \( h(t) \) be the corresponding probability density function. Then,

\[
\overline{\Phi}(t) = \overline{F}(t) \overline{G}(t)
\]

\[
h(t) = \overline{F}(t) g(t) + f(t) \overline{G}(t)
\]

Under \( H_0 \), \( h(t) = 2f(t) \overline{F}(t) = 2e^t (1 + e^t)^{-2} \)

Therefore,

\[
E_{H_0}[\frac{e_i}{(1 + e_i)^2}] = \frac{1}{6}.
\]

Hence,

\[
ni(e_0) = \frac{1}{4} n. \quad (2.38)
\]

**Locally Most Powerful Rank Tests**

**Theorem 2.10** Under the regularity conditions (a), (b) and (c) of Section 2.6, the LMP rank test for testing \( H_0 \) against \( H_A \) is based on the test statistic, \( \frac{n}{j=1} \left[ a_j W_j - b_j (1 - W_j) \right] \), where \( a_j \) and \( b_j \) are as given in (2.39) and (2.40).
Proof. The general expression for the test statistic was derived in Theorem 2.4. We just have to derive expressions for \( a_j \) and \( b_j \), where \( \theta \) is the location parameter of the distribution of interest.

From (2.35a) and (2.35b), it follows that, when

\[
G(x) = F(x + \theta) \quad \Rightarrow \quad F(x) = f'(x), \quad \text{and} \quad f^*(x) = f(x).
\]

Therefore, from (2.22) and (2.23), we get that

\[
a_j = n! 2^n \int_0^{t_1} \cdots \int_0^{t_n} \left[ \frac{f(t_i)}{F(t_j)} \prod_{i=1}^{n} f(t_i) F(t_i) dt_i \right]
\]

Let \( u_i = F(t_i) \), \( \zeta(u) = F^{-1}(u) \). Then,

\[
a_j = n! 2^n \int_0^{\zeta(u)} \cdots \int_0^{\zeta(u)} \left\{ \Psi(u_j) \prod_{i=1}^{n} [(1-u_i) du_i] \right\} \quad (2.39)
\]

where, \( \Psi(u) = \frac{f'(C(u))}{f(C(u))} \)

and, \( b_j = n! 2^n \int_0^{\zeta(u)} \cdots \int_0^{\zeta(u)} \left\{ \Psi^*(u_j) \prod_{i=1}^{n} [(1-u_i) du_i] \right\} \quad (2.40)
\]

where, \( \Psi^*(u) = \frac{f'(C(u))}{F(C(u))} \)

Thus, the test statistic for the above testing problem is

\[
\Sigma [a_j W_j - b_j (1 - W_j)] = \Sigma (a_j + b_j) W_j - \Sigma b_j.
\]

where \( a_j \) and \( b_j \) are as defined in (2.39) and (2.40).

The LMP rank test for testing \( H_0 \) against \( H_{A_1} \) is to reject \( H_0 \) in favour of \( H_{A_1} \) for large values of
Explicit expressions for \( a_j \)'s and \( b_j \)'s, when the underlying distribution is logistic, are given below.

\[
f(x) = \frac{e^x}{(1+e^x)^2}, \quad -\infty < x < \infty
\]

implies, \( \psi(u) = \log \frac{u}{1-u} \)

\[
\psi(u) = \frac{f(C(u))}{f(C(u))} = 1 - 2u
\]

and, \( \psi''(u) = \frac{f(C(u))}{1-u} = u. \)

Therefore, from (2.39) and (2.40), it follows that

\[
a_j = 1 - 2b_j, \quad \text{for } j = 1, 2, \ldots, n
\]

where, \( b_j = n!2^n \int \int \cdots \int \{u \prod_{i=1}^{n} (1-u_i) \, du_1 \} \)

In particular,

\[
b_1 = n!2^n \int \int \cdots \int \{u_1 \prod_{i=1}^{n} (1-u_i) \, du_1 \} = \frac{n!2^n}{2 \cdot 4 \cdot 6 \cdots (2n-2)} \int_0^1 [u_1(1-u_1)^{2n-1} \, du_1]
\]

\[
= \frac{n!2^n}{(n-1)! \cdot 2^{n-1}} \, \beta(2, 2n)
\]

where \( \beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \).
Therefore, \( b_1 = \frac{1}{2n + 1} \)

\[
b_2 = n \sqrt[2n]{\int \cdots \int 0 < u_1 < \cdots < u_n < 1 u_2 \prod_{i=1}^{n} [(1-u_i)du_i]}
\]

\[
= \frac{n! 2^n}{2.4.6\ldots(2n-4)} \int \int_{0 u_1} u_2 (1-u_2)^{2n-3}(1-u_1)du_1du_2 \]

which, after simplification, reduces to

\[
= \frac{1}{2n+1} + \frac{2n}{(2n+1)(2n-1)}
\]

Similarly, we get that

\[
b_3 = \frac{1}{2n+1} + \frac{2n}{(2n+1)(2n-1)} + \frac{2n(2n-2)}{(2n+1)(2n-1)(2n-3)}
\]

In general,

\[
b_j = \frac{1}{2n+1} + \sum_{k=2}^{j} \frac{2n(2n-2)\ldots(2n-2k+6)(2n-2k+4)}{(2n+1)(2n-1)\ldots(2n-2k+5)(2n-2k+3)} \quad (2.42)
\]

for \( j = 2, 3, \ldots, n \).

The test statistic for the LMP rank test for testing \( H_0 \) against \( H_{A_1} \) when the underlying distribution is logistic is given by,

\[
V_4 = \sum_{j=1}^{n} (a_j + b_j)W_j
\]

and, using (2.41),

\[
V_4 = \sum_{j=1}^{n} (1-b_j)W_j = \sum_{j=1}^{n} W_j - \sum_{j=1}^{n} b_jW_j
\]

\[
= n_1 - \sum_{j=1}^{n} b_jW_j \quad (2.43)
\]

where \( b_j \) is as in (2.42).
In the chapter that follows we consider tests based on U-statistics for testing $H_0$ against stochastic ordering and location alternatives under the competing risks set up.