Chapter 3

Estimation in Replicated Measurement
Error Model under Stochastic Linear Restrictions

“The essence of human development lies in exploring the unknown using what you already know about it” - Anonymous

3.1 Introduction

In many practical situations, in addition to the sample information, some prior knowledge about the regression coefficients is available in the form of stochastic linear restrictions. Stochastic restrictions arise from prior statistical information, usually in the form of point or interval estimates of the parameters and are structured as an additional linear model. Such information could be available from previous studies involving same variables or past experience of the experimenter etc. (ref. Toutenburg (1982) and Rao et al. (2008)). The methodology of using stochastic prior information provides a framework for acquiring new knowledge about the phenomenon under study, in the light of what is known. Durbin (1953) was the first one to use sample and prior information simultaneously in parameter estimation. Thereafter, Theil and Goldberger (1961) and Theil (1963) introduced the mixed regression estimator using stochastic linear restrictions. The mixed estimator is more efficient than OLSE. In without ME case, the use of stochastic restrictions has been studied in many situations. For example, Shalabh and Toutenburg (2005) explored the role of stochastic linear restrictions when there are missing observations. The
stochastic response restrictions were considered by Haupt and Oberhofer (2005). Jianwen and Yang (2007) and Revan (2009) discussed mixed estimation in singular linear model and use of stochastic restrictions with multicollinearity respectively. In ME regression model, Shalabh, Garg and Misra (2010) provided the consistent estimators that make use of such prior information. This was done under the assumption of known variance-covariance matrix of ME and the reliability matrix associated with the predictors. However, in case of replicated ultrastructural measurement error (RUME) regression model, the problem of finding consistent estimators which also use prior stochastic information has not been studied so far. In this chapter, we provide the stochastically restricted (SR) consistent estimators for RUME model by simultaneously using sample and prior information. A two stage procedure for obtaining a feasible version of the SR estimators is also discussed.

We consider a RUME multiple regression model as given by (2.2.5)-(2.2.7), under the assumption of stochastic linear restrictions on the regression coefficients. The problem of finding estimators that are consistent as well as make use of stochastic linear restrictions is dealt with. No distributional assumption is imposed on any random component in the model. The methodology is illustrated using an empirical economic study.

In this chapter, Section 3.2 specifies the framework of stochastic linear restrictions. In Section 3.3, we propose the consistent estimators satisfying these stochastic linear restrictions. Section 3.4 investigates the asymptotic properties of the proposed estimators. Section 3.5 contains the results from Monte Carlo simulations study performed to explore the finite sample properties of the estimators and the effect of departure from normality. Section 3.6 deals with the empirical study.

### 3.2 Model Specification

We consider the RUME multiple regression model given by (2.2.5)-(2.2.7) along with assumptions A1-A6. It is assumed that the prior information regarding the regression coefficients is available in the form of stochastic linear restrictions as

$$
\theta_{q \times 1} = R_{q \times p} \beta_{p \times 1} + \varphi_{q \times 1}.
$$

(3.2.1)

Here \( R \) and \( \theta \) are known such that \( \text{rank}(R) = q \leq p \) and \( \varphi \) is a vector of random disturbances with mean zero and known variance-covariance matrix \( \Sigma_{\varphi} \). It is assumed
that the random vector $\varphi$ is independent of $U, V$ and $W$ defined in chapter 2. This is an
essential assumption which ensures the external character of the stochastic prior
information. The vector $\theta$ may be interpreted as a random variable with expectation
$E(\theta) = R\beta$ and hence the stochastic restrictions do not hold exactly but in mean.

There are many instances in real life where the prior information could be
available in the form of stochastic linear restrictions. For example,

(i) an estimate $b^*$ of $\beta$ might exist from an earlier study. Then

$$b^* = \beta + \varphi$$

where $\varphi$ is random in nature.

(ii) the experimenter has reason to believe that the certain component of $\beta$ is
known to lie between two constants $a$ and $b$ ($a < b$). In such a situation

$$\frac{1}{2} (a + b) = \beta_i + \varphi$$

where $\varphi$ may follow $U\left(\frac{1}{2} (a - b), \frac{1}{2} (b - a)\right)$.

The following section deals with consistent estimation of the regression coefficients of
RUME model under stochastic prior information.

### 3.3 Incorporating Stochastic Prior Information in Estimation

The inconsistent estimators $b_A, b_D$ and the consistent estimators $b_{0s} : s = 1, 2, 3$
discussed in Chapter 2, are based only on the sample information. The prior
information in the form of stochastic linear restrictions can be incorporated in the estimators using the methodology of mixed estimation. As discussed in Section 1.4,
the mixed estimator for observable $\eta$ and $\xi$ can be obtained by minimizing $Q$ given by
(1.4.5). We rewrite $Q$ as

$$Q = (\eta - \xi \beta)'(\eta - \xi \beta) + \sigma^2_\varepsilon (\theta - R\beta)' \Sigma^{-1}_\varepsilon (\theta - R\beta),$$

where $\sigma^2_\varepsilon$ is the equation error variance. In the second component of (3.3.1), the
equation error variance $\sigma^2_\varepsilon$ is unknown and needs to be replaced by some suitable estimate. For RUME model, without loss of generality it is assumed in Section 2.2 that the equation error is submerged with the measurement error of response variable i.e.
Thus, we cannot estimate \( \sigma^2_e \) separately from \( \sigma^2_u \). In fact, any estimate of \( \sigma^2_u \) actually provides estimate of the sum of equation error variance and variance of measurement error associated with response variable. So in the absence of any estimate of \( \sigma^2_e \), we have no better choice than to use some estimate of \( \sigma^2_u \). Thus now onwards, we use \( \sigma^2_u \) in place of \( \sigma^2_e \). We first assume that either \( \sigma^2_u \) is known or at least some pre-estimate of \( \sigma^2_u \) is available.

**Remark 3.3.1:** The above problem does not arise if the covariance matrix of \( \varphi \), the random component in (3.2.1), is parameterized as \( \sigma^2_K \) for known \( K \). In this case, \( \varphi \) does not appear in the expression of \( Q \) given by (3.3.1) (refer Rao et al. (2008)). But this parameterization may not be valid for all the situations. □

The first component in \( Q \) captures the information regarding regression coefficients in the current sample. The second component contains the prior information regarding \( \beta \). In our case, since \( \eta \) and \( \xi \) are unknown, thus \( Q \) cannot be minimized. So in order to obtain mixed estimator for regression coefficients of RUME model, we need to find some usable version of \( Q \).

From Remark 2.3.1, we see that the unrestricted estimator \( b_A \) can be obtained by minimizing \( Q_A \) which uses only sample information. Since the first component of \( Q \) i.e. \( (\eta - \xi\beta)'(\eta - \xi\beta) \) deals with only sample information, it can be replaced by \( Q_A \). Therefore, in order to use \( Q \), we replace first component in (3.1) by \( Q_A \) and replace \( \sigma^2_e \) in the second component by \( \sigma^2_u \). The resulting function is

\[
Q_{AR} = (Y - X\beta)'A(Y - X\beta) + \sigma^2_u(\theta - R\beta)'\Sigma^{-1}_\varphi(\theta - R\beta) .
\] (3.3.2)

We still have sum of two functions, the first providing sample information and the second providing prior information. Taking derivative of \( Q_{AR} \) with respect to \( \beta \), we get

\[
\frac{dQ_{AR}}{d\beta} = -2X'AY + 2X'AX\beta - 2\sigma^2_uR'S^{-1}_\varphi\theta + 2\sigma^2_uR'S^{-1}_\varphi R\beta .
\] (3.3.3)

Equating (3.3.3) to zero provides the following estimator

\[
b_{AR}^* = (X'AX + \sigma^2_uR'S^{-1}_\varphi R)^{-1}(X'AY + \sigma^2_uR'S^{-1}_\varphi \theta) .
\] (3.3.4)

The estimator \( b_{AR}^* \) utilizes the stochastic prior information.
Similarly, replacing \((\eta - \xi \beta)'(\eta - \xi \beta)\) in (3.3.1) by \(Q_D\) (defined in Remark 2.3.1), we get

\[
Q^*_A = (Y - X\beta)'D(Y - X\beta) + \sigma^2_0(\theta - R\beta)'\Sigma^{-1}_\phi(\theta - R\beta).
\]  

(3.3.5)

Equating the derivative of \(Q^*_A\) with respect to \(\beta\) to zero provides the following estimator

\[
b^*_A = (X'DX + \sigma^2_0R'S_\phi^{-1}R)^{-1}(X'DY + \sigma^2_0R'S_\phi^{-1}\theta).
\]  

(3.3.6)

Using (2.2.5)-(2.2.7) and Lemma 2.2.2,

\[
\text{plim } b^*_A = \lim_{n \to \infty} \left( \frac{1}{n}XAX + \frac{1}{n}\sigma^2_0 R'S_\phi^{-1}R \right)^{-1} \left( \frac{1}{n}XAY + \frac{1}{n}\sigma^2_0 R'S_\phi^{-1}\theta \right)
\]

\[
= (\Sigma + \sigma^2_0 I_p + \sigma^2_0 I_p)^{-1} (\Sigma + \sigma^2_0 I_p)\beta
\]  

(3.3.7)

and

\[
\text{plim } b^*_D = (\Sigma + \sigma^2_0 I_p + \sigma^2_0 I_p)^{-1} (\Sigma + \sigma^2_0 I_p)\beta.
\]  

(3.3.8)

Hence although the estimators \(b^*_A\) and \(b^*_D\) incorporate stochastic prior information but they are not consistent estimators of \(\beta\).

In the following subsections, we provide consistent estimators of regression coefficients which also incorporate stochastic prior information.

### 3.3.1 Consistent Estimation

In Section 2.3.1, it was observed that minimizing \(Q_A;\text{corrected}\) given by (2.3.18) results in an estimator same as \(b_01\) which is a consistent estimator. This observation motivates us to use \(Q_A;\text{corrected}\) in (3.3.1) as a replacement for \((\eta - \xi \beta)'(\eta - \xi \beta)\). Thus in order to obtain consistent stochastically restricted estimator, we use (2.3.18) in (3.3.1) to get

\[
Q^*_{AR;\text{corrected}} = Q_A - \left(\frac{r}{r-1}\right)\beta'X'(A - D)\beta + \sigma^2_0(\theta - R\beta)'\Sigma^{-1}_\phi(\theta - R\beta).
\]

Taking derivative of \(Q^*_{AR;\text{corrected}}\) with respect to \(\beta\), we get

\[
\frac{dQ^*_{AR;\text{corrected}}}{d\beta} = -2X'AY + 2X'AX\beta - 2\sigma^2_0 R'S_\phi^{-1}\theta + 2\sigma^2_0 R'S_\phi^{-1}R\beta
\]
Equating the derivative in (3.3.9) to zero and simplifying, we get

\[ b_{11}^* = \left[ X'AX - \left( \frac{r}{r-1} \right) X'(A-D)X + \sigma_u^2 R'R_S^{-1}R \right]^{-1} \left( X'AY + \sigma_u^2 R'R_S^{-1} \theta \right) \]

Equating the derivative in (3.3.9) to zero and simplifying, we get

\[ b_{11}^* = \left[ X'AX - \left( \frac{r}{r-1} \right) X'(A-D)X + \sigma_u^2 R'R_S^{-1}R \right]^{-1} \left( X'AY + \sigma_u^2 R'R_S^{-1} \theta \right). \]  

(3.3.10)

Using (2.2.5)-(2.2.7) and Lemma 2.2.2, the above estimator can be easily shown to be consistent since \( \text{plim} \ b_{11}^* = \beta \).

From (3.3.10), it is seen that after adding the stochastic linear restrictions, we only need to bring in \( \sigma_u^2 R'R_S^{-1}R \) and \( \sigma_u^2 R'R_S^{-1} \theta \), into the expression of unrestricted consistent estimator \( b_{01} \). These two terms may be interpreted as the adjustments brought in by the stochastic linear restrictions.

Writing \( S_F = X'FX \) for some matrix \( F \) and applying Lemma 1.2.1 to the first factor on the RHS of (3.3.10), we get

\[
\left[ \frac{S_{(rD-A)}}{r-1} + \sigma_u^2 R'R_S^{-1}R \right]^{-1} = \frac{S_{(rD-A)}}{(r-1)^{-1}} - \frac{S_{(rD-A)}}{(r-1)^{-1}} R \left[ \sigma_u^{-2} \Sigma_\phi + \frac{R \Sigma_\phi^{-1} R}{(r-1)^{-1}} \right]^{-1} R S_{(rD-A)}^{-1}.
\]

(3.3.11)

Substituting (3.3.11) in (3.3.10), we write

\[
b_{11}^* = (r - 1) S_{(rD-A)}^{-1} X'AY + (r - 1) \sigma_u^2 S_{(rD-A)}^{-1} R'R_S^{-1} \theta
\]

\[
- \frac{S_{(rD-A)}}{(r-1)^{-1}} R \left[ \sigma_u^{-2} \Sigma_\phi + \frac{R \Sigma_\phi^{-1} R}{(r-1)^{-1}} \right]^{-1} R S_{(rD-A)}^{-1} X'AY
\]

\[
- \sigma_u^2 \frac{S_{(rD-A)}}{(r-1)^{-1}} R \left[ \sigma_u^{-2} \Sigma_\phi + \frac{R \Sigma_\phi^{-1} R}{(r-1)^{-1}} \right]^{-1} R S_{(rD-A)}^{-1} R'R_S^{-1} \theta
\]

\[
= b_{01} + (r - 1) \sigma_u^2 S_{(rD-A)}^{-1} R'R_S^{-1} \theta - \frac{S_{(rD-A)}}{(r-1)^{-1}} R \left[ \sigma_u^{-2} \Sigma_\phi + \frac{R \Sigma_\phi^{-1} R}{(r-1)^{-1}} \right]^{-1} R b_{01}
\]

\[
- \sigma_u^2 \frac{S_{(rD-A)}}{(r-1)^{-1}} R \left[ \sigma_u^{-2} \Sigma_\phi + \frac{R \Sigma_\phi^{-1} R}{(r-1)^{-1}} \right]^{-1} R S_{(rD-A)}^{-1} R'R_S^{-1} \theta
\]

\[
= b_{01} - \frac{S_{(rD-A)}}{(r-1)^{-1}} R \left[ \sigma_u^{-2} \Sigma_\phi + \frac{R \Sigma_\phi^{-1} R}{(r-1)^{-1}} \right]^{-1} R b_{01}
\]

\[
+ \frac{S_{(rD-A)}}{(r-1)^{-1}} R \left[ \sigma_u^{-2} \Sigma_\phi^{-1} - \sigma_u^2 \left( \sigma_u^{-2} \Sigma_\phi + \frac{R \Sigma_\phi^{-1} R}{(r-1)^{-1}} \right)^{-1} R \frac{S_{(rD-A)}}{(r-1)^{-1}} R'R_S^{-1} \right] \theta.
\]
On simplification, the above expression gives another form of the estimator $b_{11}^*$ as

$$b_{11}^* = b_{01} + (r - 1) S_{(rD-A)}^{-1} R \left[ \sigma_u^2 \Sigma_\varphi + (r - 1) R S_{(rD-A)}^{-1} R \right]^{-1} (\theta - R b_{01}). \quad (3.3.12)$$

**Remark 3.3.2:** In contrast to (3.3.10), the modified form of the estimator $b_{11}^*$ given by (3.3.12) no longer requires the matrix $\Sigma_\varphi$ to be non-singular. Thus the modified form allows the simultaneous use of exact and stochastic prior information. For $\Sigma_\varphi$ a null matrix, (3.3.12) provides the estimator using only the exact linear restrictions which is same as given by (2.3.19). When $\Sigma_\varphi$ is a full rank matrix, the estimator uses only stochastic information. In case $\Sigma_\varphi$ is singular, we get the estimator which uses both exact and stochastic prior information. \(\square\)

Proceeding on similar lines, it is observed that minimization of $Q_{D, \text{corrected}}$ given by (2.3.20) yields the consistent estimator $b_{02}$. Thus, we can also use $Q_{D, \text{corrected}}$ for finding another stochastically restricted estimator. Replacement of $(\eta - \xi \beta)'(\eta - \xi \beta)$ in (3.3.1) by $Q_{D, \text{corrected}}$ provides the function

$$Q_{D,R, \text{corrected}} = Q_D - \left( \frac{1}{r-1} \right) \beta' X'(A - D) X \beta + \sigma_u^2 (\theta - R \beta)' \Sigma_\varphi^{-1} (\theta - R \beta).$$

(3.3.13)
Minimizing (3.3.13) with respect to $\beta$ and applying Lemma 1.2.1 to the resulting form of the estimator, we get the modified estimator as

$$b^*_1 = b_0 + (r - 1)S_{(r - A)}^{-1}R \left[ \sigma_u^{-2} \Sigma_{\psi} + (r - 1)RS_{(r - A)}^{-1}R \right]^{-1} (\theta - Rb).

(3.3.14)$$

Using (2.3.9) and Lemma 2.2.2, it is easily proved that $b^*_1$ is a consistent estimator.

Another restricted estimator can be obtained by using the consistent estimator $b_0$. We observe that the same estimator is obtained if we minimize the function $Q_{AD}$ given by (2.3.22). This provides the necessary motivation to use $Q_{AD}$ in (3.3.1) for obtaining another estimator utilizing the stochastic prior information. Minimizing (3.3.1) after replacing $(\eta - \xi \beta)'(\eta - \xi \beta)$ by $Q_{AD}$ and applying Lemma 1.2.1, we get the following estimator

$$b^*_3 = b_0 + S_{(r - A)}^{-1}R \left[ \sigma_u^{-2} \Sigma_{\psi} + RRS_{(r - A)}^{-1}R \right]^{-1} (\theta - Rb).

(3.3.15)$$

Using (2.3.9) and Lemma 2.2.2, this estimator can be easily shown to be consistent.

Although $b^*_A$ and $b^*_D$ given by (3.3.4) and (3.3.5) are inconsistent, but they incorporate stochastic linear restrictions. We now use these estimators to provide few more stochastically restricted estimators. Using Lemma 1.2.1, the modified forms of $b^*_A$ and $b^*_D$ are obtained as

$$b^*_A = b_A + S_{(r - A)}^{-1}R \left[ \sigma_u^{-2} \Sigma_{\psi} + RRS_{(r - A)}^{-1}R \right]^{-1} (\theta - Rb).

(3.3.16)$$

and

$$b^*_D = b_D + S_{(r - A)}^{-1}R \left[ \sigma_u^{-2} \Sigma_{\psi} + RRS_{(r - A)}^{-1}R \right]^{-1} (\theta - Rb).

(3.3.17)$$

The inconsistency of $b^*_A$ and $b^*_D$ is caused by the inconsistency of $b_A$ and $b_D$. For eliminating inconsistency, we replace $b_A$ and $b_D$ in (3.3.16) and (3.3.17) by their consistent counterparts $b_0$ for $s = 1,2,3$ and obtain the following estimators

$$b^*_s = b_0 + S_{(r - A)}^{-1}R \left[ \sigma_u^{-2} \Sigma_{\psi} + RRS_{(r - A)}^{-1}R \right]^{-1} (\theta - Rb_0).

(3.3.18)$$

and

$$b^*_s = b_0 + S_{(r - A)}^{-1}R \left[ \sigma_u^{-2} \Sigma_{\psi} + RRS_{(r - A)}^{-1}R \right]^{-1} (\theta - Rb_0).

(3.3.19)$$
(2.3.9) and Lemma 2.2.2 lead us to the conclusion that

\[ \text{plim } b_{2s} = \beta \quad \text{and} \quad \text{plim } b_{3s}^* = \beta. \]

In the following discussion, we use a weighted function that helps us in deriving estimators for different forms of the weight matrix.

**Remark 3.3.3:** For \( s = 1, 2, 3 \), we consider the weighted function

\[ Q_{w^*} = (b_{0s} - \beta)'W^*(b_{0s} - \beta) + \sigma^2_\theta (\theta - R\beta)'\Sigma^-1\theta (\theta - R\beta), \quad (3.3.20) \]

where \( W^* \) is a weight matrix. Minimizing \( Q_{w^*} \) with respect to \( \beta \) leads to an estimator of \( \beta \), which utilizes stochastic prior information.

- For \( W^* = (r - 1)^{-1}X'(rD - A)X \), the estimator obtained is same as \( b_{11}^* \) and \( b_{12}^* \).
- When \( W^* = X'(rD - A)X \), we get the estimator \( b_{13}^* \).
- Similarly, taking weight matrices as \( X'AX \) and \( X'DX \), the estimators obtained are same as \( b_{2s}^* \) and \( b_{3s}^* \) respectively. \( \square \)

The motivation for Remark 3.3.3 emanates from the following discussion. Taking \( W^* = (r - 1)^{-1}X'(rD - A)X \),

\[ Q_{w^*} = (r - 1)^{-1}(b_{0s} - \beta)'[X'(rD - A)X](b_{0s} - \beta) + \sigma^2_\theta \beta'R'\Sigma^{-1}\theta - \sigma^2_\theta R'\Sigma^{-1}\theta \]

\[ = (r - 1)^{-1}(b_{0s} [X'(rD - A)X]b_{0s} - b_{0s} [X'(rD - A)X]\beta - \beta' [X'(rD - A)X]b_{0s} \]

\[ + \beta' [X'(rD - A)X]\beta] + \sigma^2_\theta R'\Sigma^{-1}\theta - \sigma^2_\theta R'\Sigma^{-1}\theta \]

\[ = \sigma^2_\theta R'\Sigma^{-1}\theta \]

Minimizing \( Q_{w^*} \) with respect to \( \beta \), we get

\[ \frac{dQ_{w^*}}{d\beta} = (r - 1)^{-1}(-[X'(rD - A)X]b_{0s} + [X'(rD - A)X]\beta) - \sigma^2_\theta R'\Sigma^{-1}\theta \]

\[ + \sigma^2_\theta R'\Sigma^{-1}\theta = 0 \]

\[ \Rightarrow \hat{\beta} = \left[(r - 1)^{-1}X'(rD - A)X + \sigma^2_\theta R'\Sigma^{-1}\theta \right]^{-1} \]

\[ \times \left[(r - 1)^{-1}X'(rD - A)Xb_{0s} + \sigma^2_\theta R'\Sigma^{-1}\theta \right]. \]

Using (2.3.5), it can be easily shown that for \( s = 1 \), this estimator same as \( b_{11}^* \) given by (3.3.10). Similarly, the other forms of the weight matrix can be considered.
The observations in Remark 3.3.3 suggest that the proposed stochastically restricted estimators can be obtained from weighted function $Q^*_W$ by using some appropriate weight matrices. This motivates us to propose one more consistent estimator of $\beta$. On minimizing the unweighted function

$$(b_{0s} - \beta)'(b_{0s} - \beta) + \sigma^2_x(\theta - R\beta)'\Sigma^{-1}_\varphi(\theta - R\beta),$$  \hspace{1cm} (3.3.21)

we get the estimator

$$b^*_4s = b_{0s} + R\left[\sigma^2_x\Sigma^{-1}_\varphi + R R^\prime\right]^{-1}(\theta - Rb_{0s}).$$  \hspace{1cm} (3.3.22)

Using (2.3.9) and Lemma 2.2.2, this estimator can be easily shown to be consistent.

Hence using $b_{0s}, s = 1, 2, 3,$ we provide three classes of four estimators each $(b^1_{fs}; f = 1, 2, 3, 4)$, which are consistent as well as utilize prior information in the form of stochastic linear restrictions. These estimators are termed as Stochastically Restricted (SR) Estimators.

### 3.3.2 Two-Stage Feasible Stochastically Restricted (TSFSR) Estimators

In the previous subsection, the proposed estimators are based on the assumption that $\sigma^2_x$ is known which in general may not be true. In this subsection, we propose to replace $\sigma^2_x$ by some suitable estimator to obtain a feasible version of the estimators proposed earlier. We propose to use the estimator

$$\hat{\sigma}^2_u = \frac{1}{n(r-1)}\left[(Y - X\hat{\beta})'(A - D)X\hat{\beta} - \hat{\beta}'X'(A - D)X\hat{\beta}\right]$$  \hspace{1cm} (3.3.23)

where $\hat{\beta}$ is some estimator of $\beta$.

Using (2.2.5)-(2.2.7),

$$\text{plim } \hat{\sigma}^2_u = \text{plim } \frac{1}{n(r-1)}\left[Y'(A - D)Y - Y'(A - D)X\hat{\beta} - \hat{\beta}'X'(A - D)Y\right]$$

$$= \text{plim } \frac{1}{n(r-1)}\left[\left((ae_{nr} + X\beta + U - V\beta)'(A - D)(ae_{nr} + X\beta + U - V\beta)ight)

- (ae_{nr} + X\beta + U - V\beta)'(A - D)X\hat{\beta} - \hat{\beta}'X'(A - D)

\times (ae_{nr} + X\beta + U - V\beta)\right].$$  \hspace{1cm} (3.3.24)

If $\hat{\beta}$ is consistent, then using Lemma 2.2.2 with (3.3.24), we get
\[
\text{plim } \sigma_u^2 = \text{plim } \frac{1}{n(n-1)} \{U'(A - D)U\} \\
\overset{n \rightarrow \infty}{=} \frac{1}{(r-1)(r-1)} \{r - 1\} \sigma_u^2 \\
\overset{}{=} \sigma_u^2.
\]

Hence \( \sigma_u^2 \) is consistent provided \( \beta \) is consistent.

The algorithm for obtaining the Two-Stage Feasible Stochastically Restricted (TSFSR) Estimators is described below

**Stage 1:** Obtain the unrestricted estimator \( \hat{\beta} \) and compute \( \hat{\sigma}_u^2 \).

**Stage 2:** Use \( \hat{\sigma}_u^2 \) in the place of \( \sigma_u^2 \) in the expressions of \( b_{fs}^*; f = 1, 2, 3, 4 \) and \( s = 1, 2, 3 \), to obtain TSFSR estimators.

We denote these TSFSR estimators by \( \hat{b}_{fs}^*; f = 1, 2, 3, 4 \) and \( s = 1, 2, 3 \).

A natural choice for \( \hat{\beta} \) is among the consistent estimators \( b_{01}, b_{02} \) and \( b_{03} \). Under the condition of Normally distributed measurement errors, \( b_{02} \) dominates both \( b_{01} \) and \( b_{03} \) according to the mean square error criterion (refer Shalabh (2003)). Thus \( b_{02} \) can be used in (3.3.23) to obtain \( \hat{\sigma}_u^2 \). It is to be noted that, in the present work, we have not imposed any distributional assumption on measurement errors. It was observed in chapter 2 that \( b_{02} \) dominates other estimators even in the case of non-normality.

There is another important point which needs to be discussed here. For \( \hat{\sigma}_u^2 \) to be a reasonable estimator of \( \sigma_u^2 \), it must be non-negative. Unfortunately, for certain values of \( Y \) and \( X \), \( \hat{\sigma}_u^2 \) may be negative. One may take \( \hat{\sigma}_u^2 = 0 \) in such a situation but use of this estimate in second stage of algorithm provides no better estimator of \( \beta \) than the unrestricted estimators (2.3.5)-(2.3.7). Hence the estimate \( \hat{\sigma}_u^2 = 0 \) is of no use for utilizing the stochastic information. Thus for negative \( \hat{\sigma}_u^2 \), it is better to use some pre-estimate (obtained from earlier studies) of \( \sigma_u^2 \) in SR estimators.

### 3.4 Large Sample Properties of Estimators

The derivation of exact distribution of the estimators proposed in the last section is difficult. Even if derived, the complexity of the expressions may not serve
any analytical purpose. Hence, in this section, we explore the large sample properties of the SR and TSFSR estimators. For this purpose, we first derive few results.

**Lemma 3.4.1:** As $n \to \infty$, we have

(i) \[ \frac{1}{nr} \left[ \sigma_u^{-2} \Phi + \frac{R S_{(rD, -A)}^T R}{(r-1)^{-1}} \right]^{-1} = O_p(n^{-1}) ; \]

(ii) \[ \frac{1}{nr} \left[ \sigma_u^{-2} \Phi + R S_{(rD, -A)}^T R \right]^{-1} = O_p(n^{-1}) ; \]

(iii) \[ \frac{1}{nr} \left[ \sigma_u^{-2} \Phi + R S_{(rD, -A)}^T \right]^{-1} = O_p(n^{-1}) ; \]

(iv) \[ \frac{1}{nr} \left[ \sigma_u^{-2} \Phi + R S_{(rD, -A)}^T \right]^{-1} = O_p(n^{-1}) . \]

**Proof:** Using the results derived in Section 2.4 of Chapter 2, and Lemmas of Chapters 1 and 2, the proofs of above results are given below.

(i) Consider

\[ \frac{1}{nr} \left[ \sigma_u^{-2} \Phi + \frac{R S_{(rD, -A)}^T R}{(r-1)^{-1}} \right]^{-1} = \left[ I_q + \frac{\sigma_u^{-2} \Phi}{nr} \left( I_p - \frac{\Sigma_{(A)}^{-1} H}{n^{1/2} (r-1)} \right) \Sigma_{(A)}^{-1} + O_p(n^{-1}) \right] \left[ \frac{\sigma_u^{-2} \Phi}{nr} \right]^{-1} . \]

Using (2.4.5), Lemma 1.2.10 and Lemma 2.4.1, we can write

\[ \frac{1}{nr} \left[ \sigma_u^{-2} \Phi + \frac{R S_{(rD, -A)}^T R}{(r-1)^{-1}} \right]^{-1} = \left[ I_q + \frac{\sigma_u^{-2} \Phi}{nr} R \left( I_p - \frac{\Sigma_{(A)}^{-1} H}{n^{1/2} (r-1)} \right) \Sigma_{(A)}^{-1} + O_p(n^{-1}) \right] \left[ \frac{\sigma_u^{-2} \Phi}{nr} \right]^{-1} . \]

(ii) Using (2.4.5), Lemma 1.2.10 and Lemma 2.4.1, the result can be proved.

(iii) The result is proved by using (2.4.1), Lemma 1.2.10 and Lemma 2.4.1.
Using (2.4.2), Lemma 1.2.10 and Lemma 2.4.1, the result is proved.

Using the above results, we now state the result relating the asymptotic distributions of SR and TSFSR estimators.

**Theorem 3.4.1:** If $\hat{\sigma}^2_u$ is consistent, then

\[
\frac{1}{n_i^2}(\hat{\beta}'^s - \beta) = \frac{1}{n_i^2}(\hat{b}'^s - \beta) + O_p(n^{-1}), \quad (3.4.2)
\]

for $f = 1, 2, 3, 4$ and $s = 1, 2, 3$. This implies that the SR and TSFSR estimators have same asymptotic distribution.

**Proof:** Using (2.4.5) and Lemmas 2.4.1 and 2.4.2, we can write

\[
\frac{1}{n_i^2} X'(rD - A)X \overset{1}{\sim} \frac{1}{(r-1)} \Sigma^{-1} + O_P \left(n^{-\frac{1}{2}}\right)
\]

\[
= \frac{1}{(r-1)} \left[ \frac{1}{n} M'CM + \sigma^2_u l_p \right]^{-1} + O_P \left(n^{-\frac{1}{2}}\right) \quad (3.4.3)
\]

and

\[
\frac{1}{n_i} X'DY = \Sigma \beta + O_P \left(n^{-\frac{1}{2}}\right). \quad (3.4.4)
\]

On using (3.4.3) and (3.4.4) in (2.3.6),

\[
b_{02} = \frac{(r-1)}{n_i} \left[ X'(rD - A)X \right]^{-1}[X'DY]
\]

\[
= \frac{(r-1)}{n_i} \left[ X'(rD - A)X \right]^{-1} \left[ X'DY \right]
\]

\[
= \frac{(r-1)}{n_i} \left[ \Sigma^{-1} + O_P \left(n^{-\frac{1}{2}}\right) \right]\left[ \Sigma \beta + O_P \left(n^{-\frac{1}{2}}\right) \right]
\]

\[
= \frac{(r-1)}{n_i} \left[ \beta + O_P \left(n^{-\frac{1}{2}}\right) \right]. \quad (3.4.5)
\]

Taking $\hat{\beta} = b_{02}$ in (3.3.23) and using (2.2.5)-(2.2.7), (3.4.5) and Lemma 2.2.1, we write

\[
\hat{\sigma}^2_u = \frac{1}{n_i(r-1)} \left[ Y'(A - D)Y - Y'(A - D)Xb_{02} - b_{02}'X'(A - D)Y \right]
\]
\begin{align*}
&= \frac{1}{n^{(r-1)}} \left[ (\alpha e_{nr} + X\beta + U - V\beta)'(A - D)(\alpha e_{nr} + X\beta + U - V\beta) \\
&\quad - (\alpha e_{nr} + X\beta + U - V\beta)'(A - D)Xb_{02} - b_{02}'X'(A - D) \\
&\quad \times (\alpha e_{nr} + X\beta + U - V\beta) \right] \\
&= \frac{1}{n^{(r-1)}} \left[ (\alpha e_{nr} + X\beta + U - V\beta)'(A - D)(\alpha e_{nr} + X\beta + U - V\beta) \\
&\quad - (\alpha e_{nr} + X\beta + U - V\beta)'(A - D)X(\beta + O_p(n^{-1})) \\
&\quad - (\beta' + O_p(n^{-1}))X'(A - D)(\alpha e_{nr} + X\beta + U - V\beta) \right] \\
&= \frac{1}{n^{(r-1)}} \left[ U'(A - D)U + O_p(n^{-1}) \right] \\
&= \frac{1}{n^{1/2}(r^{-1})} \left[ n^{1/2}r\sigma_u^2 + O_p(1) - n^{1/2}\sigma_u^2 + O_p(1) + O_p(n^{-1}) \right] \\
&= \sigma_u^2 + O_p \left( n^{-\frac{1}{2}} \right). \quad (3.4.6)
\end{align*}

It can be easily verified that the above result is true even if \( b_{02} \) gets replaced by \( b_{01} \) or \( b_{03} \).

On using Lemma 1.2.10 and (3.4.6), we can write

\begin{align*}
\hat{\sigma}_u^{-2} &= \left[ \sigma_u^2 + O_p \left( n^{-\frac{1}{2}} \right) \right]^{-1} \\
&= \frac{1}{\sigma_u^2} \left[ 1 + \frac{1}{\sigma_u^2} O_p \left( n^{-\frac{1}{2}} \right) \right]^{-1} \\
&= \frac{1}{\sigma_u^2} \left[ 1 + O_p \left( n^{-\frac{1}{2}} \right) \right] \\
&= \sigma_u^{-2} + O_p \left( n^{-\frac{1}{2}} \right). \quad (3.4.7)
\end{align*}

Since TSFSR estimators are distinguished from SR estimators by using hat, we have

\begin{equation}
\tilde{b}_{11} = b_{01} + (r - 1)S_{(r-D-A)}^{-1}R \left[ \sigma_u^{-2} \Sigma_{\varphi} + (r - 1)RS_{(r-D-A)}^{-1} \right]^{-1} (\theta - Rb_{01}). \quad (3.4.8)
\end{equation}

Using (3.4.7), Lemmas 1.2.10 and 3.4.1, (3.4.8) can be written as

\begin{align*}
\tilde{b}_{11} &= b_{01} + \frac{S_{(r-D-A)}^{-1}}{(r-1)^{-1}} R \left[ \sigma_u^{-2} \Sigma_{\varphi} + O_p \left( n^{-\frac{1}{2}} \right) + R \frac{S_{(r-D-A)}^{-1}}{(r-1)^{-1}} R \right]^{-1} (\theta - Rb_{01}) \\
&= b_{01} + \frac{S_{(r-D-A)}^{-1}}{(r-1)^{-1}} R \left[ I_q + \left( \sigma_u^{-2} \Sigma_{\varphi} + R \frac{S_{(r-D-A)}^{-1}}{(r-1)^{-1}} R \right) \right]^{-1} O_p \left( n^{-\frac{1}{2}} \right) \right]^{-1}
\end{align*}
Using Lemma 3.4.1 in (3.4.9), we get
\[ \hat{\beta}_{11} = b_{01} + \frac{S_{(r \theta - D - A)}^1 R'}{(r-1)^{-1}} \left[ l_q + O_p \left( n^{-\frac{1}{2}} \right) \right] \left( \sigma_u^{-2} \Sigma_\varphi + \frac{RS_{(r \theta - D - A)}^1 R}{(r-1)^{-1}} \right)^{-1} (\theta - Rb_{01}). \]  
(3.4.10)

Applying Lemma 1.2.10 to (3.4.10), it is observed that
\[ \hat{\beta}_{11} = b_{01} + \frac{S_{(r \theta - D - A)}^1 R'}{(r-1)^{-1}} \left[ l_q + O_p \left( n^{-\frac{1}{2}} \right) \right] \left( \sigma_u^{-2} \Sigma_\varphi + \frac{RS_{(r \theta - D - A)}^1 R}{(r-1)^{-1}} \right)^{-1} (\theta - Rb_{01}). \]  
(3.4.11)

Using assumption A5, (3.4.3) and Lemma 2.4.1
\[ n \sigma_{\hat{\beta}_{11}}^2 = \left[ \frac{1}{n} X'(r \theta - D)X \right]^{-1} \]
\[ = O(1) + O_p \left( n^{-\frac{1}{2}} \right). \]  
(3.4.12)

Therefore, Lemma 3.4.1 and (3.4.12) gives
\[ S_{(r \theta - D - A)}^{-1} \left( \sigma_u^{-2} \Sigma_\varphi + \frac{RS_{(r \theta - D - A)}^1 R}{(r-1)^{-1}} \right)^{-1} = O(1) + O_p \left( n^{-\frac{1}{2}} \right) O_p(n^{-1}) \]
\[ = O_p(n^{-1}). \]  
(3.4.13)

On using (3.4.5) and (3.4.13) in (3.4.11), we get
\[ \hat{\beta}_{11} = b_{01} + \left\{ \frac{S_{(r \theta - D - A)}^1 R'}{(r-1)^{-1}} \left( \sigma_u^{-2} \Sigma_\varphi + \frac{RS_{(r \theta - D - A)}^1 R}{(r-1)^{-1}} \right)^{-1} + O_p \left( n^{-\frac{1}{2}} \right) \right\} (\theta - Rb_{01}) \]
\[ = b_{01} + \left\{ \frac{S_{(r \theta - D - A)}^1 R'}{(r-1)^{-1}} \left( \sigma_u^{-2} \Sigma_\varphi + \frac{RS_{(r \theta - D - A)}^1 R}{(r-1)^{-1}} \right)^{-1} (\theta - Rb_{01}) \right\} \]
\[ + O_p \left( n^{-\frac{1}{2}} \right) \left( \theta - R \left[ \beta + O_p \left( n^{-\frac{1}{2}} \right) \right] \right) \]
\[ = b_{01} + \frac{S_{(r \theta - D - A)}^1 R'}{(r-1)^{-1}} \left( \sigma_u^{-2} \Sigma_\varphi + \frac{RS_{(r \theta - D - A)}^1 R}{(r-1)^{-1}} \right)^{-1} (\theta - Rb_{01}) + \left( n^{-\frac{3}{2}} \right). \]  
(3.4.14)

Hence, using (3.3.12) and (3.4.14) we get the desired result that
\[ n^{\frac{1}{2}} (\hat{\beta}_{11} - \beta) = n^{\frac{1}{2}} (b_{11} - \beta) + O_p(n^{-\frac{1}{2}}). \]  
(3.4.15)
In a similar way, the result can be shown for other values of \( f \) and \( s \).

The result of Theorem 3.4.1 has an intuitive appeal. Since \( \sigma_0^2 \) is consistent, it approaches \( \sigma^2 \) as sample size increases. This merges different identities of \( b_{fs}^* \) and \( b_{fs}^* \). Hence, we only need to evaluate the asymptotic properties of SR estimators.

In the following discussion, we obtain the expression for SR estimators in the form of order in probability.

**Theorem 3.4.2:** For \( f = 1,2,3,4 \), we have the following result

\[
n^{1/2}(b_{0s} - \beta) = n^{1/2}(b_{0s} - \beta) + O_P\left(n^{-1/2}\right) \tag{3.4.16}
\]

where \( s = 1,2,3 \). This means that the SR estimators in each class have the same asymptotic distributions as those of the unrestricted estimators in that class.

**Proof:** From Lemma 2.4.3 of Chapter 2, we have the result

\[
n^{1/2}(b_{0s} - \beta) = \Sigma^{-1} \left[ h - \frac{1}{r-1} H \beta + d_s h^* \right] + O_P\left(n^{-1/2}\right) \tag{3.4.17}
\]

where for \( s = 1,2,3 \), \( d_1 = 0, d_2 = 1 \) and \( d_3 = \frac{r}{r-1} \) characterize different classes of estimators.

Consider the following SR estimator

\[
b_{11}^* = b_{01} + (r - 1)S_{(r-a)}^{-1}R\left[\sigma_0^{-2} \Sigma_\varphi + (r - 1)RS_{(r-a)}^{-1}R\right]^{-1}(\theta - Rb_{01}). \tag{3.4.18}
\]

(3.4.5) gives

\[
(\theta - Rb_{01}) = \left(\theta - R\left[\beta + O_P\left(n^{-1/2}\right)\right]\right)
\]

\[
= (\theta - R\beta) + O_P\left(n^{-1/2}\right). \tag{3.4.19}
\]

Using (3.4.13), (3.4.17) and (3.4.19) in (3.4.18), we get

\[
b_{11}^* - \beta = b_{01} - \beta + O_P(n^{-1}) .
\]

This leads to
\[ \frac{1}{n^2}(b_{s1}^* - \beta) = \frac{1}{n^2}(b_{01} - \beta) + O_p\left(n^{-\frac{1}{2}}\right). \] (3.4.20)

Proceeding on similar lines as in done for obtaining (3.4.20), we observe that for \( s = 1,2,3 \)
\[ \frac{1}{n^2}(b_{sf}^* - \beta) = \frac{1}{n^2}(b_{0s} - \beta) + O_p\left(n^{-\frac{1}{2}}\right). \]

Using the above result, we prove the theorem giving the asymptotic distribution of the proposed estimators.

**Theorem 3.4.3:** \( \frac{1}{n^2}(b_{sf}^* - \beta) \) for \( f = 0,1,2,3,4 \) and \( s = 1,2,3 \) asymptotically follow Multivariate Normal distribution, that is
\[ \frac{1}{n^2}(b_{sf}^* - \beta) \xrightarrow{d} N_p(0_{p \times 1}, \Sigma^{-1}\Omega_s\Sigma^{-1}) \] (3.4.23)

where \( 0_{p \times 1} \) is the mean vector with all elements zero and \( \Omega_s \).

\[ \Omega_1 = \Theta + \frac{1}{r} \sigma_u^2 \sigma_l^2 I_p + N_{\gamma \mu}; \] (3.4.24)

\[ \Omega_2 = \Theta + \frac{1}{r^2} \sigma_u^2 \sigma_l^2 I_p + N_{\gamma \mu}; \] (3.4.25)

\[ \Omega_3 = \Theta + \frac{1}{r(r-1)} \sigma_u^2 \sigma_l^2 I_p + N_{\gamma \mu}; \] (3.4.26)

\[ \Theta = \frac{1}{r} (\sigma_u^2 + \sigma_l^2 \beta \gamma) (\Sigma_M + \sigma_l^2 I_p) + \frac{1}{r(r-1)} \sigma_u^2 (\beta \gamma' + \text{tr}(\beta \gamma' I_p - (\beta \beta' \gamma I_p)); \]

\[ N_{\gamma \mu} = \frac{1}{(r-1)^2} \gamma_1 \sigma_u^2 [G(\beta \gamma', (rD - A))(\sigma_M \otimes e_r) + (\sigma_M \otimes e_r') G(\beta \gamma', (rD - A))]; \]

\[ \Sigma^{-1} = \lim_{n \to \infty} \Sigma^{-1}. \]

**Proof:** The proof can be easily obtained using Theorems 2.4.1 and 3.4.2.

From (3.4.16), it can be easily observed that for \( s = 1,2,3 \) the asymptotic distribution of \( \frac{1}{n^2}(b_{sf}^* - \beta); f = 1,2,3,4 \) is same as that of \( \frac{1}{n^2}(b_{0s} - \beta) \). This means that in each class, the SR estimators have same asymptotic distribution as that of the unrestricted estimator of that class. Thus, the results obtained for the asymptotic distribution of unrestricted estimators in Chapter 2 will also hold for SR estimators. This indicates that the effect of using additional information in the form of stochastic linear restrictions vanishes with an increase in the sample size. This happens because a
bigger sample provides more information regarding the parameter of interest and the contribution of additional information becomes negligible.

We explore whether the SR and TSFSR estimators, satisfy the stochastic restrictions or not. As mentioned in Section 3.2, the stochastic information is not satisfied exactly but in the expectation. So it needs to be checked whether $E(R\beta^*_f) = R\beta$ and $E(R\beta^*_s) = R\beta$ or not. The evaluation of these expectations is very complex for small sample sizes, but some asymptotic expressions can be obtained. The following theorem assists in this.

**Theorem 3.4.4:** The asymptotic distribution of $n^{\frac{1}{2}}(Rb^*_f - R\beta)$ for $f = 1,2,3,4$ and $s = 1,2,3$ is $q$-variate Normal, that is

$$n^{\frac{1}{2}}(Rb^*_f - R\beta) \xrightarrow{d} N_q(0_{q\times 1}, R\Sigma^{-1}\Omega_s\Sigma^{-1}R') \quad (3.4.27)$$

where $0_{q\times 1}$ is the mean vector with all elements zero.

**Proof:** The proof can be easily obtained using the result of Theorem 3.4.3.

The above result indicates that the stochastic restrictions are satisfied by SR estimators, at least in large samples. Since from Theorem 3.4.1, SR and TSFSR have same asymptotic distribution, the above result is true for TSFSR estimators as well.

In the next section, we investigate the small sample properties of the estimators.
3.5 Simulations

In the last section, we discussed large sample properties of the estimators. In this section, small sample properties of the estimators are assessed using Monte-Carlo simulations. Coding has been done in MATLAB. The stochastic restriction imposed is of the form given by (3.2.1) with \( R = \begin{bmatrix} 0.3 & 0.5 & 0.8 \\ -0.45 & 0.57 & 0.33 \end{bmatrix} \). The random term \( \varphi \) is assumed to follow Multivariate Normal distribution with mean \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) and variance-covariance matrix given as \( \Sigma_\varphi = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.3 \end{bmatrix} \). The vector \( \theta \) is computed at each iteration using (3.2.1). Other simulation settings are same as in Section 2.5 of Chapter 2.

Simulations are performed for both SR as well as TSFSR estimators. 20000 iterations are performed for each parametric combination and MedSEM and MedBias are computed empirically for the unrestricted and SR estimators. For TSFSR estimators, only those iterations are used in computing MedSEM and MedBias where \( \hat{\sigma}_u^2 > 0 \). The trace of MedSEM and MedAB are used for comparison purpose. The simulations results are tabulated in Tables 3.1-3.8. We also performed the graphical analysis of the simulations results. The graphs are reported in Figures 3.1-3.5.
Figure 3.1: Trace of MedSEM of TSFSR estimators when \((\sigma_0^2, \sigma_1^2, \sigma_2^2) = (0.5, 0.5, 1.0)\) and \(r = 2\), Normal distribution case.

Figure 3.2: MedAB of TSFSR estimators when \((\sigma_0^2, \sigma_1^2, \sigma_2^2) = (0.5, 0.5, 1.0)\) and \(r = 2\), Normal distribution case.
Table 3.1
MedAB of estimators for $\gamma_2, \gamma_2, \gamma_2$ = (0.5, 0.5, 0.5) and $r = 2$

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<th>$b_{01}$</th>
<th>$b_{11}$</th>
<th>$b_{21}$</th>
<th>$b_{31}$</th>
<th>$b_{41}$</th>
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<th>$b_{22}$</th>
<th>$b_{32}$</th>
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<tbody>
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<tr>
<td>Gamma</td>
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Table 3.2
MedAB of estimators for $(\gamma_2^2, \gamma_2^2, \gamma_2^2) = (0.5, 0.5, 1.0)$ and $r = 2$

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<th>$b_{31}$</th>
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<td></td>
<td>0.2368</td>
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<td>Gamma</td>
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### Table 3.3
MedAB of estimators for \((\sigma_1^2, \sigma_2^2, \sigma_3^2) = (0.5, 1.0, 0.5)\) and \(r = 2\)

<table>
<thead>
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<th>(n=15)</th>
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<tbody>
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</tr>
<tr>
<td>SR</td>
<td>0.0984</td>
</tr>
<tr>
<td>TSFSR</td>
<td>0.0618</td>
</tr>
<tr>
<td><strong>t</strong></td>
<td>0.1004</td>
</tr>
<tr>
<td>SR</td>
<td>0.0988</td>
</tr>
<tr>
<td>TSFSR</td>
<td>0.0731</td>
</tr>
<tr>
<td><strong>Gamma</strong></td>
<td>0.0971</td>
</tr>
<tr>
<td>SR</td>
<td>0.0647</td>
</tr>
</tbody>
</table>

### Table 3.4
MedAB of estimators for \((\sigma_1^2, \sigma_2^2, \sigma_3^2) = (1.0, 0.5, 0.5)\) and \(r = 2\)

<table>
<thead>
<tr>
<th>(n=15)</th>
<th>(n=45)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Normal</strong></td>
<td><strong>Normal</strong></td>
</tr>
<tr>
<td>SR</td>
<td>0.1169</td>
</tr>
<tr>
<td>TSFSR</td>
<td>0.0687</td>
</tr>
<tr>
<td><strong>t</strong></td>
<td>0.1158</td>
</tr>
<tr>
<td>SR</td>
<td>0.1103</td>
</tr>
<tr>
<td>TSFSR</td>
<td>0.0720</td>
</tr>
<tr>
<td><strong>Gamma</strong></td>
<td>0.1035</td>
</tr>
<tr>
<td>SR</td>
<td>0.0625</td>
</tr>
</tbody>
</table>

---

114 Replicated Ultrastructural Model under Stochastic Linear Restrictions
### Table 3.5
Trace of MedSEM for \((c_1^*, c_2^*, \sigma^2) = (0.5,0.5,0.5)\) and \(r = 2\)

<table>
<thead>
<tr>
<th></th>
<th>Normal SR</th>
<th>TSFSR</th>
<th>t SR</th>
<th>TSFSR</th>
<th>Gamma SR</th>
<th>TSFSR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(b_{01})</td>
<td>(b_{11})</td>
<td>(b_{21})</td>
<td>(b_{31})</td>
<td>(b_{41})</td>
<td>(b_{02})</td>
</tr>
<tr>
<td>(n=15)</td>
<td>0.2168</td>
<td>0.2108</td>
<td>0.2124</td>
<td>0.2118</td>
<td>0.2414</td>
<td>0.2091</td>
</tr>
<tr>
<td>(n=45)</td>
<td>0.0521</td>
<td>0.0517</td>
<td>0.0517</td>
<td>0.0517</td>
<td>0.1397</td>
<td>0.0518</td>
</tr>
</tbody>
</table>

### Table 3.6
Trace of MedSEM for \((c_1^*, c_2^*, \sigma^2) = (0.5,0.5,1.0)\) and \(r = 2\)

<table>
<thead>
<tr>
<th></th>
<th>Normal SR</th>
<th>TSFSR</th>
<th>t SR</th>
<th>TSFSR</th>
<th>Gamma SR</th>
<th>TSFSR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(b_{01})</td>
<td>(b_{11})</td>
<td>(b_{21})</td>
<td>(b_{31})</td>
<td>(b_{41})</td>
<td>(b_{02})</td>
</tr>
<tr>
<td>(n=15)</td>
<td>0.6254</td>
<td>0.6027</td>
<td>0.6140</td>
<td>0.6111</td>
<td>0.4973</td>
<td>0.6026</td>
</tr>
<tr>
<td>(n=45)</td>
<td>0.1266</td>
<td>0.1291</td>
<td>0.1291</td>
<td>0.1291</td>
<td>0.1863</td>
<td>0.1256</td>
</tr>
</tbody>
</table>

### Table 3.7
Trace of MedSEM for \((c_1^*, c_2^*, \sigma^2) = (0.5,0.5,1.0)\) and \(r = 2\)

<table>
<thead>
<tr>
<th></th>
<th>Normal SR</th>
<th>TSFSR</th>
<th>t SR</th>
<th>TSFSR</th>
<th>Gamma SR</th>
<th>TSFSR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(b_{01})</td>
<td>(b_{11})</td>
<td>(b_{21})</td>
<td>(b_{31})</td>
<td>(b_{41})</td>
<td>(b_{02})</td>
</tr>
<tr>
<td>(n=15)</td>
<td>0.6198</td>
<td>0.5973</td>
<td>0.6079</td>
<td>0.6049</td>
<td>0.4911</td>
<td>0.6032</td>
</tr>
<tr>
<td>(n=45)</td>
<td>0.1281</td>
<td>0.1270</td>
<td>0.1274</td>
<td>0.1272</td>
<td>0.1858</td>
<td>0.1259</td>
</tr>
</tbody>
</table>

---

Replicated Ultrastructural Model under Stochastic Linear Restrictions
Table 3.7
Trace of MedSEM for \((\sigma^2_1, \sigma^2_2, \sigma^2_3) = (0.5, 1.0, 0.5)\) and \(r = 2\)

<table>
<thead>
<tr>
<th></th>
<th>Normal SR</th>
<th>TFSFR</th>
<th>t SR</th>
<th>TFSFR</th>
<th>Gamma SR</th>
<th>TFSFR</th>
<th>Normal SR</th>
<th>TFSFR</th>
<th>t SR</th>
<th>TFSFR</th>
<th>Gamma SR</th>
<th>TFSFR</th>
<th>Normal SR</th>
<th>TFSFR</th>
<th>t SR</th>
<th>TFSFR</th>
<th>Gamma SR</th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(b_{01})</td>
<td>(b_{11})</td>
<td>(b_{21})</td>
<td>(b_{31})</td>
<td>(b_{41})</td>
<td>(b_{02})</td>
<td>(b_{12})</td>
<td>(b_{22})</td>
<td>(b_{32})</td>
<td>(b_{42})</td>
<td>(b_{03})</td>
<td>(b_{13})</td>
<td>(b_{23})</td>
<td>(b_{33})</td>
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<td>(b_{24})</td>
</tr>
<tr>
<td></td>
<td>0.1716</td>
<td>0.1673</td>
<td>0.1687</td>
<td>0.1680</td>
<td>0.2143</td>
<td>0.1709</td>
<td>0.1665</td>
<td>0.1677</td>
<td>0.1671</td>
<td>0.2128</td>
<td>0.1719</td>
<td>0.1684</td>
<td>0.1695</td>
<td>0.1690</td>
<td>0.2130</td>
<td>0.2297</td>
<td>0.2176</td>
<td>0.2215</td>
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<tr>
<td></td>
<td>0.1537</td>
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<td>0.1547</td>
<td>0.2449</td>
<td></td>
<td>0.1568</td>
<td>0.1587</td>
<td>0.1580</td>
<td>0.2444</td>
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<td>0.1561</td>
<td>0.1585</td>
<td>0.1577</td>
<td>0.2466</td>
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<td>0.2223</td>
<td>0.2018</td>
<td>0.2058</td>
</tr>
<tr>
<td></td>
<td>0.1675</td>
<td>0.1634</td>
<td>0.1647</td>
<td>0.1643</td>
<td>0.2141</td>
<td>0.1558</td>
<td>0.1573</td>
<td>0.1568</td>
<td>0.2397</td>
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<tr>
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<td>0.2407</td>
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<td>0.1573</td>
<td>0.1568</td>
<td>0.2397</td>
<td></td>
<td>0.1639</td>
<td>0.1649</td>
<td>0.1644</td>
<td>0.2176</td>
<td></td>
<td>0.2276</td>
<td>0.2059</td>
<td>0.2087</td>
</tr>
</tbody>
</table>

Table 3.8
Trace of MedSEM for \((\sigma^2_1, \sigma^2_2, \sigma^2_3) = (1.0, 0.5, 0.5)\) and \(r = 2\)

<table>
<thead>
<tr>
<th></th>
<th>Normal SR</th>
<th>TFSFR</th>
<th>t SR</th>
<th>TFSFR</th>
<th>Gamma SR</th>
<th>TFSFR</th>
<th>Normal SR</th>
<th>TFSFR</th>
<th>t SR</th>
<th>TFSFR</th>
<th>Gamma SR</th>
<th>TFSFR</th>
<th>Normal SR</th>
<th>TFSFR</th>
<th>t SR</th>
<th>TFSFR</th>
<th>Gamma SR</th>
<th>TFSFR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(b_{01})</td>
<td>(b_{11})</td>
<td>(b_{21})</td>
<td>(b_{31})</td>
<td>(b_{41})</td>
<td>(b_{02})</td>
<td>(b_{12})</td>
<td>(b_{22})</td>
<td>(b_{32})</td>
<td>(b_{42})</td>
<td>(b_{03})</td>
<td>(b_{13})</td>
<td>(b_{23})</td>
<td>(b_{33})</td>
<td>(b_{43})</td>
<td>(b_{04})</td>
<td>(b_{14})</td>
<td>(b_{24})</td>
</tr>
<tr>
<td></td>
<td>0.2297</td>
<td>0.2176</td>
<td>0.2215</td>
<td>0.2202</td>
<td>0.3305</td>
<td>0.2223</td>
<td>0.2101</td>
<td>0.2134</td>
<td>0.2120</td>
<td>0.2390</td>
<td>0.2221</td>
<td>0.2184</td>
<td>0.1929</td>
<td>0.1911</td>
<td>0.3232</td>
<td>0.2247</td>
<td>0.2132</td>
<td>0.2161</td>
</tr>
<tr>
<td></td>
<td>0.1987</td>
<td>0.2027</td>
<td>0.2010</td>
<td>0.3323</td>
<td></td>
<td>0.1917</td>
<td>0.1958</td>
<td>0.1945</td>
<td>0.2322</td>
<td></td>
<td>0.2106</td>
<td>0.2139</td>
<td>0.2127</td>
<td>0.3256</td>
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<td>0.2210</td>
<td>0.2085</td>
<td>0.2118</td>
</tr>
<tr>
<td></td>
<td>0.2018</td>
<td>0.2058</td>
<td>0.2040</td>
<td>0.3305</td>
<td></td>
<td>0.1948</td>
<td>0.1997</td>
<td>0.1978</td>
<td>0.3233</td>
<td></td>
<td>0.1924</td>
<td>0.1963</td>
<td>0.1950</td>
<td>0.3234</td>
<td></td>
<td>0.2149</td>
<td>0.2059</td>
<td>0.2087</td>
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<tr>
<td></td>
<td>0.2286</td>
<td>0.2162</td>
<td>0.2202</td>
<td>0.2185</td>
<td>0.3302</td>
<td>0.2211</td>
<td>0.2117</td>
<td>0.2146</td>
<td>0.2137</td>
<td>0.3259</td>
<td>0.2221</td>
<td>0.2089</td>
<td>0.2120</td>
<td>0.2105</td>
<td>0.3247</td>
<td>0.2203</td>
<td>0.2089</td>
<td>0.2120</td>
</tr>
</tbody>
</table>

\[ n=15 \quad n=45 \]
From Figures 3.1, 3.2 and Tables 3.1-3.8, it is observed that the MedAB and Trace of MedSEM move towards zero as the sample size increases. This validates the theoretical findings that estimators are asymptotically unbiased. For $f = 1,2,3$ and each $s = 1,2,3$

$$tr\text{MedSEM}(\hat{b}_{f3}) < tr\text{MedSEM}(b^{*}_{f3}) < tr\text{MedSEM}(b_{0s}), \quad (3.5.1)$$

and

$$\text{MedAB}(\hat{b}_{f3}) < \text{MedAB}(b^{*}_{f3}) < \text{MedAB}(b_{0s}). \quad (3.5.2)$$

This gives that the Trace of MedSEM and MedAB for both SR and TSFSR estimators are less as compared to those of the unrestricted estimators. Thus, the use of stochastic information provides improved estimators in terms of both bias and variability. The only exceptions are the estimators $b_{4s}^{*}$ and $\hat{b}_{4s}$ for $s = 1,2,3$. Although, they provide the largest reduction in bias as compared to other restricted estimators in their respective class, they do not provide reduction in variability except when sample size is small and $\sigma_{y}^{2}$ is large. Looking at the values in Tables 3.1-3.8 for SR estimators, we observe that

$$tr\text{MedSEM}(b^{*}_{13s}) < tr\text{MedSEM}(b^{*}_{23s}) < tr\text{MedSEM}(b^{*}_{33s}) \quad (3.5.3)$$

and

$$\text{MedAB}(b^{*}_{13s}) < \text{MedAB}(b^{*}_{23s}) < \text{MedAB}(b^{*}_{33s}). \quad (3.5.4)$$

However, the differences between $b_{1s}^{*}$, $b_{2s}^{*}$ and $b_{3s}^{*}$ with respect to the trace of MedSEM and MedAB are not very large. The ordering for TSFSR estimators is similar to the ordering for SR estimators given by (3.5.3) and (3.5.4). The fact that stochastic restricted estimators are more accurate than ordinary estimators is due to the addition of the stochastic linear restrictions which helps in using more information about the unknown regression parameters.

Now, we compare SR and TSFSR estimators using values in Tables 3.1-3.8. It is observed that the MedAB and Traces of MedSEM for TSFSR estimators are lower than corresponding values for SR estimators. This suggests that if possible (i.e. when $\sigma_{u}^{2} > 0$), the TSFSR estimators should be preferred over SR estimators.
Figure 3.3: Trace of MedSEM for TSFSR estimators for Normal (N), t and Gamma (g) distributions when $(\sigma_0^2, \sigma_1^2, \sigma_2^2) = (0.5, 0.5, 1.0)$ and $r = 2.$
Figure 3.4: MedAB for TSFSR estimators for Normal (N), t and Gamma (g) distributions when 
$(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (0.5, 0.5, 1.0)$ and $r = 2$. 
From Figures 3.3-3.4 and Tables 3.1-3.8, lucid conclusions can’t be drawn about the effect of non-normality on the properties of the estimators. But the differences in MedAB and Trace of MedSEM of the estimators for Normal, t and Gamma distributions are not very large. This suggests that to some extent, the estimators are robust to the assumption of normality of measurement error and other random components in the model. It is further observed that the bias and variance increase or decrease as $\sigma^2$ or $\sigma^2_v$ increases.

We now try to determine the extent to which the stochastic restrictions are satisfied by the proposed estimators. Since the estimators under study may not have finite expectations, MedBias vector is used to verify whether restrictions are satisfied at least in the central part of the distribution of the estimators. We compute the vector $(R \times \text{MedBias})$ and plot its norm in Figure 3.5. It is clear that the TSFSR estimators satisfy stochastic restrictions more precisely than the SR estimators. This gives further impetus to preference of TSFSR estimators over SR estimators.

![Figure 3.5: Norm of $R \times \text{MedBias}$ vs sample size when $(\sigma^2, \sigma^2_v, \sigma^2_\nu) = (0.5, 0.5, 1.0)$ and $r = 2$.](image)

### 3.6 Empirical Study

In this section, we illustrate the proposed estimators using the same situation as discussed in Section 2.6 but under stochastic information. We consider the data set of countries having low GDP (<50 billion USD) and small Population size (<50 millions). The effect of exchange rate averaged over whole year ($x_1$), GDP ($x_2$) and Population size ($x_3$) on TB ($y$) is explored for these countries. For this, we randomly
select a group of 30 countries from the data given in Appendix (indicated by *).
Moderate sample size is considered since Theorem 3.4.2 suggests that effect of the
stochastic restrictions vanishes for large sample sizes. The model under consideration
is given by (2.6.1) for $i = 1, \ldots, 30$ and $j = 1, 2$.

For developing the stochastic information framework, we consider the study of
Ullah et al. (2001). Our model setup is similar to the one used by them except that two
additional variables $x_2$ and $x_3$ are included and more recent data set for years 1992
and 2002 is used. Using the data for years 1977 and 1987, they provided consistent
estimate of the regression coefficient relating exchange rate with TB (Estimate=9.4,
SE=4.5). We use the results reported by them, in the form of stochastic linear
restrictions by taking $R = (1 \ 0 \ 0), \theta = 9.4$ and $\text{var}(\varphi) = 4.5 \times 4.5 = 20.25$.
Since $\sigma_2^2$ is unknown, we use (3.3.23) and $\hat{\varphi} = b_{02}$ to get $\hat{\theta}_0^2 = 3.6173 \ e + 005$.
Tables 3.9 and 3.10 provide the unrestricted and TSFSR estimates of the regression
coefficients respectively along with estimated SEs and 80% CIs. The probability of
type-I error is taken to be large (20%) so that if the results of unrestricted estimates are
formulated in the form of exact linear restrictions, the chances of misspecification are
quite low (ref. Section 2.6). The bootstrapping is done to estimate the standard error of
the estimates.

From Table 3.9, it can be easily observed that exchange rate and TB have a
significant positive relation. The estimates of regression coefficients corresponding to
GDP and Population size are insignificant since their CI's contain zero. From Tables
3.9 and 3.10, it is observed that SEs of the TSFSR estimates are less than those of the
unrestricted estimates. This suggests that the use of stochastic information in
estimation improves the estimates. Although this reduction is not very large, but this
could be due to the fact that the stochastic information used is highly variable due to
large $\text{var}(\varphi)$. 


### Table 3.9
Unrestricted estimates of regression coefficients, their SEs and 80% confidence intervals

<table>
<thead>
<tr>
<th></th>
<th>$b_{01}$ Estimate (SE)</th>
<th>$b_{02}$ Estimate (SE)</th>
<th>$b_{03}$ Estimate (SE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exchange Rate</td>
<td>4.5234 (1.8203)</td>
<td>4.3529 (1.5982)</td>
<td>4.1824 (1.4458)</td>
</tr>
<tr>
<td>GDP</td>
<td>13.5169 (21.0420)</td>
<td>15.4055 (19.6236)</td>
<td>17.2942 (18.8365)</td>
</tr>
<tr>
<td>Population</td>
<td>1.4121 (25.0385)</td>
<td>-2.9242 (24.1750)</td>
<td>-7.2604 (23.5444)</td>
</tr>
</tbody>
</table>

### Table 3.10
TSFSR estimates of regression coefficients, their SEs and 80% confidence intervals

<table>
<thead>
<tr>
<th></th>
<th>$b_{11}$ Estimate (SE)</th>
<th>$b_{12}$ Estimate (SE)</th>
<th>$b_{13}$ Estimate (SE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exchange Rate</td>
<td>4.6060 (1.5033)</td>
<td>4.4328 (1.3524)</td>
<td>4.2650 (1.2900)</td>
</tr>
<tr>
<td>GDP</td>
<td>13.9829 (18.5823)</td>
<td>15.8878 (17.8277)</td>
<td>17.7927 (17.5376)</td>
</tr>
<tr>
<td>Population</td>
<td>1.4991 (24.2341)</td>
<td>-2.8341 (23.5449)</td>
<td>-7.1673 (23.0767)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$b_{21}$ Estimate (SE)</th>
<th>$b_{22}$ Estimate (SE)</th>
<th>$b_{23}$ Estimate (SE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exchange Rate</td>
<td>4.5498 (1.3521)</td>
<td>4.3803 (1.3670)</td>
<td>4.2107 (1.2985)</td>
</tr>
<tr>
<td>GDP</td>
<td>13.6665 (18.6077)</td>
<td>15.5603 (17.8330)</td>
<td>17.4542 (17.5321)</td>
</tr>
<tr>
<td>Population</td>
<td>1.3666 (24.2993)</td>
<td>-2.9713 (23.6290)</td>
<td>-7.3091 (23.1484)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$b_{31}$ Estimate (SE)</th>
<th>$b_{32}$ Estimate (SE)</th>
<th>$b_{33}$ Estimate (SE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exchange Rate</td>
<td>4.5626 (1.5190)</td>
<td>4.3935 (1.3623)</td>
<td>4.2244 (1.2920)</td>
</tr>
<tr>
<td>GDP</td>
<td>13.7457 (18.5981)</td>
<td>15.6423 (17.8283)</td>
<td>17.5190 (17.5293)</td>
</tr>
<tr>
<td>Population</td>
<td>1.3987 (24.2900)</td>
<td>-2.9380 (23.6220)</td>
<td>-7.2748 (23.1446)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$b_{41}$ Estimate (SE)</th>
<th>$b_{42}$ Estimate (SE)</th>
<th>$b_{43}$ Estimate (SE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exchange Rate</td>
<td>9.3997 (0.0001)</td>
<td>9.3997 (0.0001)</td>
<td>9.3997 (0.0002)</td>
</tr>
<tr>
<td>GDP</td>
<td>13.5169 (18.6272)</td>
<td>15.4055 (17.8455)</td>
<td>17.2942 (17.5408)</td>
</tr>
<tr>
<td>Population</td>
<td>1.4121 (24.3085)</td>
<td>-2.9242 (23.6356)</td>
<td>-7.2604 (23.1543)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$b_{51}$ Estimate (SE)</th>
<th>$b_{52}$ Estimate (SE)</th>
<th>$b_{53}$ Estimate (SE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exchange Rate</td>
<td>9.3997 (0.0001)</td>
<td>9.3997 (0.0001)</td>
<td>9.3997 (0.0002)</td>
</tr>
<tr>
<td>GDP</td>
<td>13.5169 (18.6272)</td>
<td>15.4055 (17.8455)</td>
<td>17.2942 (17.5408)</td>
</tr>
<tr>
<td>Population</td>
<td>1.4121 (24.3085)</td>
<td>-2.9242 (23.6356)</td>
<td>-7.2604 (23.1543)</td>
</tr>
</tbody>
</table>
3.7 Conclusions

A replicated ultrastructural measurement error (RUME) multiple regression model is considered where replicated observations on study and predictor variables are available. Some prior information regarding the regression coefficients is assumed to be available in the form of stochastic linear restrictions. Three classes of SR consistent estimators are proposed under additional information. The SR estimators cannot be used in case $\sigma^2$ is unknown. To overcome this problem, a two stage procedure for obtaining restricted estimators is discussed. The resulting estimators are called TSFSR estimators. No distributional assumption is imposed on any random component in the model. The asymptotic properties of the unrestricted and restricted consistent estimators are reported. It is observed that asymptotically, the estimators follow Multivariate Normal distribution and are unbiased. Monte Carlo simulations are performed to explore the small sample properties of the estimators. It is observed that inclusion of prior information improves the estimators in terms of both bias and variability. The effect of stochastic information vanishes with increasing sample size. In small samples, the TSFSR estimators dominate the SR estimators in terms of both bias and variability. To some extent, the proposed estimators are robust to the assumption of normality. The utility of the proposed estimators is illustrated using a real data set on trade balance, exchange rate and GDP.